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GRAPHS AND MULTI-GRAPHS IN HOMOGENEOUS 3-MANIFOLDS WITH ISOMETRY GROUPS OF DIMENSION 4

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ABSTRACT. We study the existence of multi-graphs which are immersed in $E^3(\kappa,\tau)$, having constant mean curvature H, where $E^3(\kappa,\tau)$ is a homogeneous, simply connected 3-manifold whose isometry group has dimension 4.

1. INTRODUCTION

The classification of complete simply connected homogeneous manifolds of dimension 3 is well known (see [14, Chap. The eight geometries]). Such a manifold has an isometry group of dimension 3, 4 or 6. When the dimension of the isometry group is 6, we have a space form. When the dimension of the isometry group is 3, the manifold has the geometry of the Lie group Sol_3.

In this paper we will consider the complete homogeneous manifolds $E^3(\kappa,\tau)$ whose isometry group has dimension 4. Such a manifold is a Riemannian fibration over a 2-*dimensional* space form $M^2(\kappa)$ (having constant Gauss curvature κ). There is a Riemannian submersion $\pi : E^3(\kappa,\tau) \longrightarrow M^2(\kappa)$, which also is a Killing submersion [see Definition 3.1]. If $E^3(\kappa,\tau)$ is not compact, then $E^3(\kappa,\tau)$ is topologically $M^2(\kappa)\times\mathbb{R}$, each fiber is diffeomorphic to $\mathbb{R}$ (the real line), and the bundle curvature is τ. If $E^3(\kappa,\tau)$ is compact, with $\kappa > 0$ and $\tau \neq 0$, then $E^3(\kappa,\tau)$ are the Berger spheres. Here each fiber is diffeomorphic to S^1 (the unit circle) and the tangent unit vector field to the fiber is a Killing vector field. We will denote by E_3 this vector field. Here E_3 will be called the vertical vector field. These manifolds are classified, up to isometry, by the curvature κ of the base surface of the submersion and the bundle curvature τ, where κ and τ can be any real numbers satisfying $\kappa \neq 4\tau^2$. Namely, these manifolds are:

- $E^3(\kappa,\tau) = \mathbb{H}^2(\kappa)\times\mathbb{R}$ if $\kappa < 0$ and $\tau = 0$,
- $E^3(\kappa,\tau) = \mathbb{S}^2(\kappa)\times\mathbb{R}$ if $\kappa > 0$ and $\tau = 0$,
- $E^3(\kappa,\tau) = Nil_3$ (Heisenberg space) if $\kappa = 0$ and $\tau \neq 0$,
- $E^3(\kappa,\tau) = \widetilde{PSL}_2(\mathbb{R})$ if $\kappa < 0$ and $\tau \neq 0$,
- $E^3(\kappa,\tau) = \mathbb{S}^3_\tau$ (spheres of Berger) if $\kappa > 0$ and $\tau \neq 0$.

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In a recent paper Harold Rosenberg and Jose Espinar have proved that if Σ is a completely immersed H-surface (that is, surfaces having constant mean curvature H) in $\mathbb{H}^2 \times \mathbb{R}$ whose angle function does not change sign and $H > 1/2$, then Σ is a vertical cylinder over a complete curve $\gamma \subset \mathbb{H}^2$ of constant geodesic curvature $2H$ (see [10, Theorem 4.1]). They were inspired by the ideas presented in the paper by Laurent Hauswirth, Harold Rosenberg and Joel Spruck [11]. In that paper the authors prove that if Σ is a completely immersed surface in $\mathbb{H}^2 \times \mathbb{R}$ having constant mean curvature $H = 1/2$ and transverse to the vertical field E_3, then Σ is an entire vertical graph over $\mathbb{H}^2$ (see [11, Theorem 1.2]). We use similar ideas to prove an analogous theorem in $E^3(\kappa, \tau)$ (see Theorem 5.1).

Now, we briefly describe the contents and organization of this paper. In section 2, we establish the notation. In section 3, we deal with entire graphs in $E(\kappa, \tau)$. In section 4, we study the H-*cylinder*. In section 5, we study multi-graphs in $E^3(\kappa, \tau)$.

2. Notation

We denote by Σ an oriented Riemannian H-surface immersed in $E^3(\kappa, \tau)$ with Riemannian metric g, by N its unit normal vector field, and by $\overrightarrow{H}$ its mean curvature vector. Here $H = g(\overrightarrow{H}, N)$ is the length of the mean curvature vector with respect to the normal N and ν is its angle function, that is, $\nu = g(N, E_3)$.

Let M^n denote a Riemannian manifold of dimension n, and $\mathfrak{X}(M^n)$ will denote the set of tangent vector fields on M.

Definition 2.1. Let M^n be a Riemannian manifold of dimension n and $\Omega \subset M^n$ be an open domain in M^n such that $\overline{\Omega}$ is compact and $\partial\Omega$ is of class C^∞. The Cheeger constant, which is denoted by $C(M^n)$, is defined by

$$C(M^n) = \inf_{\Omega}\{\frac{A(\partial\Omega)}{V(\Omega)}; \Omega \subset M^n, \overline{\Omega} \text{ compact}\}.$$

where A is the area function and V the volume function on M^n.

We have denoted by $M^2(\kappa)$ the 2-dimensional space form having constant Gauss curvature κ. In this paper we are considering $\kappa \in \{-1, 0, 1\}$. Thus, depending on the value of κ we have $M^2(-1) \equiv \mathbb{D}^2$, the hyperbolic disk; $M^2(0) \equiv \mathbb{R}^2$, the Euclidean space; and $M^2(1) \equiv \mathbb{S}^2$, the Euclidean sphere.

The space $M^2(\kappa)$ is endowed with the metric

$$ds^2 = \lambda^2(\kappa)(dx^2 + dy^2),$$

where $\lambda(0) = 1$ and $\lambda(\kappa \neq 0) = \dfrac{2}{1 + \kappa(x^2 + y^2)}$. The natural orthonormal frame on $M^2(\kappa)$ is given by $\{e_1 = \lambda^{-1}\partial_x, e_2 = \lambda^{-1}\partial_y\}$.

3. The space $E^3(\kappa, \tau)$

We begin this section by giving the definition of a Killing submersion which is due to H. Rosenberg, R. Souam, and E. Toubiana; see [4].

Consider a Riemannian 3-manifold (M^3, g) which fibers over a Riemannian surface (M^2, h), where g and h denote the Riemannian metrics respectively.

Definition 3.1. A Riemannian submersion $\pi : (M^3, g) \longrightarrow (M^2, h)$ such that:

(1) each fiber is a complete geodesic,

(2) the fibers of the fibration are the integral curves of a unit Killing vector field ξ on M^3

will be called a Killing submersion.

Definition 3.2. Let $\pi : (M^3, g) \longrightarrow (M^2, h)$ be a Killing submersion.

(1) Let $\Omega \subset M^2$ be a domain. An *H-section* over Ω is an *H-surface* which is the image of a section.

(2) Let $\gamma \subset M^2$ be a smooth curve having constant geodesic curvature $2H$. Observe that the surface $\pi^{-1}(\gamma) \subset M^3$ has mean curvature H [see Lemma 4.1]. We call such surface a *vertical H-cylinder*.

When the space $E^3(\kappa,\tau)$ is not compact, it is given by (topologically)

$$E^3(\kappa,\tau) = \{(x,y,t); (x,y) \in M^2(\kappa), t \in \mathbb{R}\}. \tag{3.1}$$

The projection $\pi : E^3(\kappa,\tau) \longrightarrow M^2(\kappa)$ given by $\pi(x,y,t) = (x,y)$ is a Killing submersion and the translations along the fibers are isometries, so E_3 is a unit Killing vector field.

Let $\{E_1, E_2\}$ be the horizontal lift of $\{e_1, e_2\}$; thus $\{E_1, E_2, E_3\}$ is an orthonormal frame on $E^3(\kappa,\tau)$.

Let $X \in \mathfrak{X}(M^2(\kappa))$ be a vector field on $M^2(\kappa)$ and $\overline{X}$ its horizontal lift to $E^3(\kappa,\tau)$.

The natural frame for the space form $M^2(\kappa)$ is given by $\{\partial_x, \partial_y\}$. Thus, the natural frame for $E^3(\kappa,\tau)$ will be $\{\partial_x, \partial_y, \partial_t\}$, where ∂_t is tangent to the fibers. In this case $E_3 = \partial_t$.

Lemma 3.1 (see [14]). *The fields E_1, E_2, E_3 in the frame $\{\partial_x, \partial_y, \partial_t\}$ are given by*

$$\begin{array}{ll} k \neq 0, & \kappa = 0, \\ E_1 = \dfrac{1}{\lambda}\partial_x - 2\tau\dfrac{\lambda_y}{\lambda^2}\partial_t, & E_1 = \partial_x - \tau y \partial_t, \\ E_2 = \dfrac{1}{\lambda}\partial_x - 2\tau\dfrac{\lambda_x}{\lambda^2}\partial_t, & E_2 = \partial_y - \tau y \partial_t, \\ E_3 = \partial_t, & E_3 = \partial_t. \end{array}$$

Furthermore, the space $E^3(\kappa,\tau)$ is endowed with the metric

$$g = \begin{cases} \lambda^2(dx^2+dy^2) + \left(2\tau\left(\dfrac{\lambda_y}{\lambda}dx - \dfrac{\lambda_x}{\lambda}dy\right) + dt\right)^2 & \textit{if } \kappa \neq 0, \\ \\ \lambda^2(dx^2+dy^2) + (\tau(ydx - xdy) + dt)^2 & \textit{if } \kappa = 0, \end{cases}$$

and the Riemannian connection is given by

$$\begin{array}{lll} \overline{\nabla}_{E_1}E_1 = -\dfrac{\lambda_y}{\lambda^2}E_2, & \overline{\nabla}_{E_1}E_2 = \dfrac{\lambda_y}{\lambda^2}E_1 + \tau E_3, & \overline{\nabla}_{E_1}E_3 = -\tau E_2, \\ \overline{\nabla}_{E_2}E_1 = \dfrac{\lambda_x}{\lambda^2}E_2 - \tau E_3, & \overline{\nabla}_{E_2}E_2 = -\dfrac{\lambda_x}{\lambda^2}E_1, & \overline{\nabla}_{E_2}E_3 = \tau E_1, \\ \overline{\nabla}_{E_3}E_1 = -\tau E_2, & \overline{\nabla}_{E_3}E_2 = \tau E_1, & \overline{\nabla}_{E_3}E_3 = 0. \end{array}$$

We also have

$$[E_1, E_2] = \frac{\lambda_y}{\lambda^2}E_1 - \frac{\lambda_x}{\lambda^2}E_2 + 2\tau E_3, \quad [E_1, E_3] = 0, \quad [E_2, E_3] = 0.$$

3.1. **Graphs in $E^3(\kappa,\tau)$.** Since $E^3(\kappa,\tau)$ is a Riemannian submersion over a 2-dimensional space form $M^2(\kappa)$, it is possible to extend the notion of graphs as follows:

Definition 3.3. A graph in $E^3(\kappa,\tau)$ is the image of a section of the Killing submersion $\pi : E^3(\kappa,\tau) \longrightarrow M^2(k)$. When the section is defined over the entire $M^2(\kappa)$, we will say that the graph is entire.

Remark 3.1. There is no entire graph in $\mathbb{S}^3_\tau$, since the fibration is not trivial; that is, $\mathbb{S}^3_\tau$ is not the product $\mathbb{S}^2 \times \mathbb{S}^1$.

We will prove that Remark 3.1 holds for the other spaces when the graph has constant mean curvature H, $2H > C(M^2(\kappa))$, where $C(M^2(\kappa))$ is Cheeger's constant (see Proposition 3.1).

Given a domain $\Omega \subset M^2(\kappa)$, let

$$s : \Omega \subset M^2(\kappa) \longrightarrow \{(x, y, u(x, y)) \in E^3(\kappa,\tau)\}$$

be a section, where $u \in C^0(\partial\Omega) \cap C^\infty(\Omega)$. We will identify Ω with its lift to $M^2 \times \{0\}$. Then the graph $\Sigma(u)$ of $u \in C^0(\partial\Omega) \cap C^\infty(\Omega)$ is written as

$$\Sigma(u) = \{(x, y, u(x, y)) \in E^3(\kappa,\tau); (x, y) \in \Omega\}.$$

With the above notation, we have the following lemma:

Lemma 3.2. *Let $\Sigma(u)$ be the graph of a function $u : \Omega \subset M^2 \longrightarrow \mathbb{R}$ having constant mean curvature H. Then the function u satisfies the equation*

$$2H = div_{M^2(\kappa)}\left(\frac{\alpha}{W}e_1 + \frac{\beta}{W}e_2\right),$$

where $W = \sqrt{1+\alpha^2+\beta^2}$ and

$$\begin{aligned} &\kappa \neq 0, && \kappa = 0,\\ &\alpha = \frac{u_x}{\lambda} + 2\tau\frac{\lambda_y}{\lambda_2}, && \alpha = u_x - \tau y,\\ &\beta = \frac{u_y}{\lambda} + 2\tau\frac{\lambda_x}{\lambda_2}, && \beta = u_y + \tau x. \end{aligned}$$

Proof. We consider the smooth function $u^* : E^3(\kappa,\tau) \longrightarrow \mathbb{R}$ defined by $u^*(x, y, t) = u(x, y)$. Set $F(x, y, t) = t - u^*(x, y, t)$ (observe that with this choice of F, we are fixing the unit normal vector of $\Sigma(u)$ which points up in $E^3(\kappa,\tau)$, that is, $g(N, E_3) > 0$, where N is the unit normal vector along $\Sigma(u)$). If the mean curvature vector points up, then $H > 0$; if it points down, then $H < 0$. Since 0 is a regular value of F and $\Sigma(u) = F^{-1}(0)$, we have

$$2H = -\overline{div}\left(\frac{\overline{\nabla}F}{|\overline{\nabla}F|}\right),$$

where $\overline{div}$ and $\overline{\nabla}$ denote the divergence and gradient in $E^3(\kappa,\tau)$.

We consider the case $\kappa \neq 0$; for $\kappa = 0$ the proof is analogous. We will calculate $\overline{\nabla}F$. Since $\{E_1, E_2, E_3\}$ is an orthonormal frame, we have

$$\overline{\nabla}F = aE_1 + bE_2 + cE_3,$$

with

$$\begin{aligned} a &= g(\overline{\nabla}F, E_1) = dF(E_1) = dF\left(\frac{1}{\lambda}\partial_x - 2\tau\frac{\lambda_y}{\lambda^2}\partial_t\right) = -\frac{u_x}{\lambda} - 2\tau\frac{\lambda_y}{\lambda^2}, \\ b &= g(\overline{\nabla}F, E_2) = dF(E_2) = dF\left(\frac{1}{\lambda}\partial_y + 2\tau\frac{\lambda_x}{\lambda^2}\partial_t\right) = -\frac{u_y}{\lambda} + 2\tau\frac{\lambda_x}{\lambda^2}, \\ c &= g(\overline{\nabla}F, E_3) = dF(E_3) = dF(\partial_t) = 1. \end{aligned}$$

Hence

$$\overline{\nabla}F = -\alpha E_1 - \beta E_2 + E_3,$$

where

$$\begin{aligned} \alpha &= \frac{u_x}{\lambda} + 2\tau\frac{\lambda_y}{\lambda^2}, \\ \beta &= \frac{u_y}{\lambda} - 2\tau\frac{\lambda_x}{\lambda^2}. \end{aligned}$$

We denote this by $W = |\overline{\nabla}F| = \sqrt{1+\alpha^2+\beta^2}$ and by $\overline{X} = -\dfrac{\overline{\nabla}F}{|\overline{\nabla}F|}$.

Following the above notation, we obtain

$$\overline{X} = \left(\frac{1}{W}(\alpha E_1 + \beta E_2)\right) - \left(\frac{1}{W}E_3\right) = A - B,$$

where A is a horizontal and B is a vertical vector field. Now, by computing the divergence of X, we obtain

$$2H = div_{M^2(\kappa)}\left(\frac{1}{W}(\alpha e_1 + \beta e_2)\right).$$

□

Recall that for the space form $M^2(\kappa)$, we have denoted by $C(M^2(\kappa))$ its Cheeger's constant. The next proposition was proved by I. Salavessa in [5] and generalized by Barbosa, Bessa and Montenegro in [7].

Proposition 3.1. *There is no entire H-graph in $E^3(\kappa,\tau)$ such that $2H > C(M^2(\kappa))$.*

Proof. Let us suppose that such an entire graph exists. Let

$$\Sigma(u) = \{(x, y, u(x, y)) \in E^3(\kappa, \tau); (x, y) \in M^2(\kappa)\},$$

where $u : M^2(\kappa) \longrightarrow \mathbb{R}$ is a solution of

$$2H = div_{M^2(\kappa)}\left(\frac{\alpha}{W}e_1 + \frac{\beta}{W}e_2\right). \tag{3.2}$$

Let $\Omega \subset M^2(\kappa)$ be an open domain with compact closure and smooth boundary $\partial\Omega$. Thus, from equation (3.2) and the divergence theorem, we obtain

$$2HV(\Omega) = \int_\Omega div_{M^2(\kappa)}\left(\frac{\alpha}{W}e_1 + \frac{\beta}{W}e_2\right)dV \le A(\partial\Omega).$$

Taking into account Cheeger's constant, we obtain

$$C(M^2(\kappa)) < 2H \le C(M^2(\kappa)).$$

This contradiction completes the proof. □

Remark 3.2. In the case $2H = C(M^2(\kappa))$ there are many examples of entire graphs in $E^3(\kappa,\tau)$. For instance, in the case $\tau \equiv 0$ and $\kappa \equiv -1$, that is, $E^3(\kappa,\tau) \equiv \mathbb{H}^2 \times \mathbb{R}$, we have the next beautiful example due to Ricardo Sa Earp [13, page 24]:

$$t = \frac{\sqrt{x^2+y^2}}{y}, \qquad y > 0.$$

This entire $H = \frac{1}{2}$-graph is invariant by hyperbolic translation.

Another interesting example for an $H = \frac{1}{2}$-surface in the space $\mathbb{H}^2 \times \mathbb{R}$ due to I. Salavessa is given in [5, Theorem 2].

We have found one example of an entire graph having constant mean curvature H with $2H = C(M^2(\kappa))$ in the case $\tau \equiv -\frac{1}{2}$ and $\kappa \equiv -1$, that is, when $E^3(\kappa,\tau) \equiv \widetilde{PSL}_2(\tau,\mathbb{R})$; see [2, example 6.2.2]. In this example we have an entire $H = \frac{1}{2}$-graph, which is invariant by rotational isometries. This graph is given by

$$u(x,y) = 2\sqrt{\cosh(2\tanh^{-1}(\sqrt{x^2+y^2}))} - 2\arctan(\sqrt{\cosh(2\tanh^{-1}(\sqrt{x^2+y^2}))}).$$

4. Cylinders in $E^3(\kappa,\tau)$

Let $\gamma \subset M^2(k)$ be a curve with $k_g = 2H$ (where k_g denotes the geodesic curvature), and we consider $Q = \pi^{-1}(\gamma)$ as the vertical H-*cylinder*. Observe that topologically (not metrically), $Q = \gamma \times \mathbb{R}$.

If $X \in TM^2$, we denote by $\overline{X}$ its horizontal lift to $TE^3(\kappa,\tau)$. From now on, we will always consider horizontal lifts unless explicitly stated otherwise.

Lemma 4.1. *The mean curvature of Q is H; in particular Q has constant mean curvature.*

Proof. We can suppose that γ is parameterized by arc length, let γ' be its vector velocity and denote by $\overrightarrow{n}$ the unit normal vector to γ. Let $\overline{\gamma}' = \mathit{lift}(\gamma')$ and $N = \mathit{lift}(\overrightarrow{n})$ be the horizontal lift of γ' and $\overrightarrow{n}$ respectively. Thus, $\overline{\gamma}'$ is tangent to Q. As E_3 is also tangent to Q, then $\{\overline{\gamma}', E_3\}$ is an orthonormal frame on Q. It is clear that N is a unit normal field along Q.

As the fibers are geodesic, we have, $\overline{\nabla}_{E_3}E_3 = 0$. Thus the *normal curvature* of E_3 is 0 (i.e. $k_N(E_3) = 0$).

Observe that

$$k_N(\overline{\gamma}') = g(-\overline{\nabla}_{\overline{\gamma}'}N, \overline{\gamma}') = ds(-\nabla_{\gamma'}\overrightarrow{n}, \gamma') = k_g = 2H.$$

Thus, the mean curvature of Q is H. □

We denote by K_{ext} the extrinsic curvature of $Q \equiv \pi^{-1}(\gamma)$.

Lemma 4.2. *The H-cylinder Q is flat and has constant extrinsic curvature $K_{ext} = -\tau^2$.*

Proof. We can suppose that the curve γ is parameterized by arc length, i.e., $\gamma(s) = (f(s), g(s)) \subset M^2(\kappa)$ with $ds^2(\gamma', \gamma') = 1$. Denote by n the normal to γ. Then

$$\begin{aligned} \gamma'(s) &= \lambda f'(s)e_1 + \lambda g'(s)e_2, \\ n(s) &= -\lambda g'(s)e_1 + \lambda f'(s)e_2. \end{aligned}$$

Let $\overline{\gamma}'$ and N be the horizontal lift of γ' and n, respectively. Thus $\overline{\gamma}'$ is tangent to Q and N is normal to Q. The other unit tangent vector field to Q, which is orthogonal to $\overline{\gamma}'$, is E_3. Thus we have an orthonormal frame:

$$\begin{aligned} X_1 &= \overline{\gamma}' = \lambda f'(s)E_1 + \lambda g'(s)E_2, \\ X_2 &= E_3, \\ N &= -\lambda g'(s)E_1 + \lambda f'(s)E_2. \end{aligned}$$

Using the Riemannian connection we obtain $[X_1, X_2] = 0$. Threfore, we have around $p \in Q$ a coordinate system $\Phi(s,t)$ such that

$$\begin{aligned} \Phi_s &= \lambda f'(s)E_1 + \lambda g'(s)E_2, \\ \Phi_t &= E_3, \\ N &= -\lambda g'(s)E_1 + \lambda f'(s)E_2. \end{aligned}$$

We denote by $\langle ., . \rangle$ the induced metric on Q:

$$\begin{aligned} \langle \Phi_s, \Phi_s \rangle &= \lambda^2 (f')^2 + \lambda^2 (g')^2 = 1, \\ \langle \Phi_s, \Phi_t \rangle &= 0, \\ \langle \Phi_t, \Phi_t \rangle &= 1. \end{aligned}$$

Then $K = 0$, i.e., the Gaussian curvature is 0, so the surface Q is flat. From the Gauss equation (see [1, Proposition 3.3]),

$$K = \det S + \tau^2 + (k - 4\tau^2)\nu^2.$$

Since $\nu = 0$, we obtain $K_{ext} = \det S = -\tau^2$. $\square$

We denote by $Ric(X)$ the Ricci curvature in the direction X.

Lemma 4.3. *The Ricci curvature in the direction N is given by*

$$Ric(N) = \kappa - 2\tau^2,$$

where N is the unit normal vector field along Q.

Proof. Let R denote the curvature tensor of $E = E^3(\kappa, \tau)$. We have an orthonormal frame on Q:

$$\begin{aligned} \Phi_s &= \lambda f'(s)E_1 + \lambda g'(s)E_2 c \\ \Phi_t &= E_3 c \\ N &= -\lambda g'(s)E_1 + \lambda f'(s)E_2. \end{aligned}$$

Let $Y \in TE$ be a vector field. By the linearity of R, we have

$$\begin{aligned} R(N,Y)N &= R(-\lambda g'(s)E_1 + \lambda f'(s)E_2, Y)(-\lambda g'(s)E_1 + \lambda f'(s)E_2) \\ &= [-\lambda g'(s)R(E_1,Y) + \lambda f'(s)R(E_2,Y)](-\lambda g'(s)E_1 + \lambda f'(s)E_2) \\ &= \lambda^2 (g')^2 R(E_1,Y)E_1 - \lambda^2 f'g'R(E_1,Y)E_2 - \lambda^2 f'g'R(E_2,Y)E_1 \\ &\quad + \lambda^2 (f')^2 R(E_2,Y)E_2. \end{aligned}$$

We use this expression to show the following two affirmations.

Affirmation 4.1. We have

$$g_E(R(N,\Phi_s)N, \Phi_s) = k - 3\tau^2,$$

where g_E is the metric of E.

Proof. Putting $Y = \Phi_s = \lambda f'(s)E_1 + \lambda, g'(s)E_2$ and using the linearity of the metric g_E of $E^3(\kappa,\tau)$ we obtain

$$\begin{aligned} g_E(R(N,\Phi_s)N,\Phi_s) &= \overline{K}(E_1,E_2)[\lambda^2(g')^2 + \lambda^2(f')^2]^2 \\ &= \overline{K}(E_1,E_2) \\ &= \kappa - 3\tau^2, \end{aligned}$$

where $\overline{K}(E_1,E_2)$ denotes the sectional curvature in the direction $\{E_1,E_2\}$. □

Affirmation 4.2. We have

$$g(R(N,\Phi_z)N,\Phi_z) = \tau^2.$$

Proof. Putting $Y = \Phi_t = E_3$ and using linearity of the metric g_E of $E^3(\kappa,\tau)$ we have

$$\begin{aligned} g_E(R(N,\Phi_t)N,\Phi_t) &= \overline{K}(E_1,E_3)[\lambda^2(g')^2 + \lambda^2(f')^2] \\ &= \overline{K}(E_1,E_3) \\ &= \tau^2, \end{aligned}$$

where $\overline{K}(E_1,E_3)$ denotes the sectional curvature in the direction $\{E_1,E_3\}$. □

This affirmation implies

$$\begin{aligned} Ric(N) &= g_E(R(N,\Phi_s)N,\Phi_s) + g_E(R(N,\Phi_z)N,\Phi_z) \\ &= (\kappa - 3\tau^2) + \tau^2 \\ &= \kappa - 2\tau^2, \end{aligned}$$

which proves the lemma. □

4.1. **Stability of H-cylinders in $E^3(\kappa,\tau)$.** Now we answer the question: When is Q an unstable surface?

Proposition 4.1. *The H-cylinder Q immersed in $E^3(\kappa,\tau)$ is unstable for $4H^2 + \kappa > 0$.*

Proof. The stability operator L of Q is given by

$$L = \Delta + |B|^2 + Ric(N),$$

where Δ is the Laplacian operator associated to the Riemannian metric induced on Q, $|B|^2$ is the square of the norm of the shape operator associated to Q and $Ric(N)$ is the Ricci curvature in the direction of N. Hence,

$$\begin{aligned} |B|^2 + Ric(N) &= 4H^2 - 2K_{ext} + \kappa - 2\tau^2 \\ &= 4H^2 + \kappa \\ &= 4H^2 + \kappa. \end{aligned}$$

Set L$= \Delta + a$, where $a = 4H^2 + \kappa > 0$.

We consider the operator L on $[0,T] \times [-r,r]$ for $r > 0$, where $[0,T]$ is an arc of length T on γ.

It is known that $\phi_1 = cos(\pi s/T)$ is a first eigenfunction of $\dfrac{\partial^2}{\partial^2 s}$ on $[0,T]$, with eigenvalue $\lambda_1 = \pi^2/T^2$.

Similarly, a first eigenfunction ϕ_2 of $\dfrac{\partial^2}{\partial^2 t}$ on $[-r,r]$ is $\phi_2 = cos(\pi t/2r)$, with eigenvalue $\lambda_2 = \pi^2/4r^2$.

Let $\phi = \phi_1 * \phi_2$ and observe that $\phi \in \mathfrak{F}$ (see [12] for notations) and that

$$\Delta\phi + (\lambda_1 + \lambda_2)\phi = 0 \quad \text{on} \quad [0, T] \times [-r, r].$$

Then,

$$L\phi + (\lambda_1 + \lambda_2 - a)\phi = 0 \quad \text{on} \quad [0, T] \times [-r, r].$$

Hence, if r and T satisfy $\lambda_1 + \lambda_2 - a < 0$, the domain is unstable; that is, there is a negative eigenvalue for L.

This condition is equivalent to

$$\frac{\pi^2}{T^2} + \frac{\pi^2}{4r^2} < 4H^2 + \kappa,$$

which is true for T or r large enough.

This completes the proof. $\square$

Recall $C(\mathbb{H}^2) \equiv 1$, $C(\mathbb{R}^2) \equiv 0$ and $C(\mathbb{S}^2) \equiv 0$. Hence, we have the following corollary.

Corollary 4.1. *The H-cylinder Q immersed in $E^3(\kappa, \tau)$ is unstable for $2H > C(M^2(\kappa))$.*

5. Multi-graphs in $E^3(\kappa, \tau)$

Throughout this section, we consider a surface Σ having constant mean curvature $H \neq 0$, immersed in $E^3(\kappa, \tau)$. Recall that we have denoted by N its unit normal vector field and by $\nu = g(N, E_3)$ its angle function.

Definition 5.1. Σ is called a *multi-graph* if the angle function ν does not change sign; i.e., either $\nu \geq 0$ or $\nu \leq 0$ on Σ.

Lemma 5.1. *Let Σ be a complete H-multi-graph immersed in $E^3(\kappa, \tau)$. Then one of the following conditions holds:*

(i) *Either $\nu > 0$ or $\nu < 0$ on Σ.*
(ii) *Σ is an H-cylinder.*

Proof. We can suppose $\nu \leq 0$. If there is a point $p \in \Sigma$ such that $\nu(p) = 0$, we will show that Σ is an H-cylinder.

Since E_3 is a Killing vector field, then ν is a Jacobi function on Σ; that is,

$$L\nu = \Delta\nu + (Ric(N) + |B|^2)\nu = 0.$$

Setting $c = Ric(N) + |B|^2$, we have the operator $L = \Delta + c$. Let U be a neighborhood of p, so $\nu \leq 0$ on U and $\nu(p) = 0$. Working on U, we have

$$\begin{aligned} \Delta\nu + \min_U(c, 0)\nu &= L\nu - c\nu + \min_U(c, 0)\nu \\ &= (\min_U(c, 0) - c)\nu + L\nu \\ &\geq 0, \end{aligned}$$

so $(\min_U(c, 0) - c)\nu \geq 0$. Thus, $P\nu = \Delta\nu + \min_U(c, 0)\nu \geq 0$ with $\min_U(c, 0) \leq 0$. Applying the maximum principle of Hopf, we conclude that $\nu \equiv 0$ on U.

This shows that the non-empty set $A = \{q \in \Sigma; \nu(q) = 0\}$ is open. On the other hand, A is closed. Since Σ is connected, $A = \Sigma$.

Thus, the angle function is nul on Σ. Using [9, Theorem 2.2], we conclude that Σ is an H-cylinder. $\square$

5.1. **The main theorem.** Recall that when $E^3(\kappa, \tau)$ is not compact, that is, $E^3(\kappa, \tau) \neq \mathbb{S}^3_\tau$, that the space $E^3(\kappa, \tau)$ is topologically $M^2(\kappa) \times \mathbb{R}$. In this case, there is a *trivial section*

$$s_0 : M^2(\kappa) \longrightarrow E^3(\kappa, \tau)$$

given by $s_0(p) = (p, 0) \in M^2(\kappa) \times \mathbb{R}$. We are identifying $M^2(\kappa)$ with its lift $s_0(M^2(\kappa))$.

Denote by $I : E^3(\kappa, \tau) \longrightarrow s_0(M^2(\kappa))$ the vertical translation along the fiber, taking the point (p_0, t_0) to the point $(p_0, 0)$. That is,

$$I(p_0, t_0) = (p_0, 0) \in s_0(M^2(\kappa)).$$

If Σ is a multi-graph, then ν is a non-negative Jacobi function. Thus Σ is stable, and this implies that Σ has bounded curvature. Thus, there is a real number $\delta > 0$ such that for each $p \in \Sigma$, a piece of Σ is a Euclidean graph (of bounded geometry) over the disk of radius $D_\delta(p) \subset T_p\Sigma$ (see [4]). We denote this graph by $G(p)$.

Let $I(p) = q$ and denote by $F(q) = I(G(p))$ the translated surface. This surface $F(q)$ is a Euclidean graph over the disk $D_\delta(q) \subset T_qF(q)$.

By Remark 3.2, there are entire graphs in $E^3(\kappa, \tau)$ having constant mean curvature H, with $2H = C(M^2(\kappa))$. However, there is no entire graph in $E^3(\kappa, \tau)$ having constant mean curvature H for $2H > C(M^2(\kappa))$ [see Proposition 3.1]. In a more general sense, we have the following theorem. (To prove this theorem, we follow ideas from [10, Theorem 4.1].)

Theorem 5.1. *Let $E^3(\kappa, \tau)$ be a complete simply connected homogeneous 3-manifold. Let Σ be a complete H-multi-graph immersed in $E^3(\kappa, \tau)$. If $2H > C(M^2(\kappa))$, then Σ is a vertical H-cylinder.*

Proof. If there is a point $p \in \Sigma$ such that $\nu(p) = 0$, then Σ is a vertical H-cylinder (see Lemma 5.1).

Thus, we can suppose that the angle function ν satisfies $\nu < 0$ on Σ and $g(N, \overrightarrow{H}) > 0$, where N is the unit normal vector field on Σ, $\overrightarrow{H}$ is the mean curvature vector field of Σ and g is the metric of $E^3(\kappa, \tau)$.

We divide the proof into two steps: First we will prove the theorem for the case $E^3(\kappa, \tau) \neq \mathbb{S}^3_\tau$. After that, we will prove the same result when $E^3(\kappa, \tau) = \mathbb{S}^3_\tau$.

The case $E^3(\kappa, \tau) \neq \mathbb{S}^3_\tau$. There is no complete H-multi-graph Σ immersed into $E^3(\kappa, \tau)$, having angle function $\nu < 0$, and $2H > C(M^2(\kappa))$.

Proof. The idea of the proof is to show that such a multi-graph is actually an entire graph. Hence, using Proposition 3.1, we conclude that such an entire graph does not exist.

Let $B_{\widetilde{R}}(0) = B_{\widetilde{R}}$ be the geodesic ball in $s_0(M^2(\kappa))$ centered at the origin of $s_0(M^2)$ with radius $\widetilde{R}$. Let $p \in \Sigma$ and assume there is a neighborhood U of p in Σ such that U is an H-section of $\pi : E^3(\kappa, \tau) \longrightarrow M^2(\kappa)$ (see Definition 3.2) over the ball $B_{\widetilde{R}}$. Let $f : B_{\widetilde{R}} \longrightarrow \mathbb{R}$ be such that U is equal to $graph(f)$.

If Σ is not an entire graph, let R be the largest such that f exists. Let $q \in \partial B_R$ be such that f cannot be extended to an H-section in any neighborhood of q.

Affirmation 5.1. For any sequence $q_n \subset B_R$ such that $q_n \longrightarrow q$, we have $f(q_n) \longrightarrow +\infty$ (or $f(q_n) \longrightarrow -\infty$).

Proof. Suppose there exists a sequence $q_n \subset B_R$ such that $q_n \longrightarrow q$ and $f(q_n)$ is bounded. Then, there exists a subsequence $f(q_{n_i})$ that converges to x; as Σ is complete we have $(q, x) \in \Sigma$. Since f cannot be extended to an H-section in any neighborhood of q, Σ must have a horizontal normal at (p, x), which contradicts $\nu < 0$.

Observe that there are no two subsequences (q_n) and $(\widetilde{q}_n)$ such that $f(q_n) \longrightarrow +\infty$ and $f(\widetilde{q}_n) \longrightarrow -\infty$. To see this, take n big and points q_n and $\widetilde{q}_n$ close to q. Now, consider the family of complete geodesics passing by q. This family gives rise to a family of vertical planes passing through q. Since Σ is connected and complete, there is a plane in this family such that the intersection of the plane with Σ is a connected curve. Supposing that $f(q_n) \longrightarrow +\infty$ and $f(\widetilde{q}_n) \longrightarrow -\infty$, this connected curve has a vertical tangent, which is impossible since $\nu < 0$. □

Now, consider the piece of curve $C_\delta(q)$ passing by q, tangent to ∂B_R at q, having geodesic curvature $2H$ ($k_g = 2H$) and such that B_R stays in the concave side of $C_\delta(q)$. Let $Q_\delta = \pi^{-1}(C_\delta(q))$ be the "δ-cylinder" which contains $C_\delta(q)$.

Using curvature estimates, we have for any $\widetilde{p} \in \Sigma$ that a neighborhood of $\widetilde{p}$ is a Euclidean graph over a disk $D_\delta(\widetilde{p}) \subset T_{\widetilde{p}}\Sigma$. We have denoted this graph by $G(\widetilde{p})$ and by $F(\widetilde{q}) = I(G(\widetilde{p}))$, where $\widetilde{q} = I(\widetilde{p})$. Using this notation we have

Affirmation 5.2 ([4, Theorem 3.3])**.** Let (p_n) be a sequence of points on Σ such that $I(p_n) = q_n$ converges to q. Then there exists a sub-sequence of $(F(q_n))$ that converges to a δ-piece of $Q_\delta = \pi^{-1}(C_\delta(q))$. This convergence is in C^2-topology.

Let γ_0 be the geodesic arc having length 2ϵ centered at q and orthogonal to $C_\delta(q)$. Denote by γ_0^+ the part of γ_0 lying in B_R.

Let (q_n) be a sequence converging to q with $q_n \in \gamma_0^+$.

Let C be the complete curve of $M^2(\kappa)$ having geodesic curvature $2H$ ($k_g = 2H$) containing $C_\delta(q)$. Parametrize C by arc length; denote by $q(s) \in C$ the point at distance s on C from $q(0) = q$, $s \in \mathbb{R}$. Note that C may be compact.

Denote by γ_s a geodesic arc, orthogonal to C at $q(s)$; that is, $q(s)$ is the midpoint of γ_s. Assume that the length of each γ_s is 2ϵ. Thus,

$$\bigcup_{s\in\mathbb{R}} \gamma_s = T_\epsilon(C)$$

is the ϵ-tubular neighborhood of C.

We denote by γ_s^+ the part of γ_s on the concave side of C.

Affirmation 5.3. For n large, each $F(q_n)$ is disjoint from $C \times \mathbb{R}$. Also, for $|s| < \delta$,

$$F(q_n) \cap (\gamma_s \times \mathbb{R})$$

is a vertical graph over an interval of γ_s.

Proof. Observe that the $F(q_n)$ are C^2 close to $C_\delta(q) \times J_\delta \subset Q_\delta$ (where J_δ is an interval of length 2δ). Choose n_0 so that for $n \geq n_0$, $\Gamma_n(s) = F(q_n) \cap (\gamma_s \times \mathbb{R})$ is one connected curve and the intersection of $F(q_n)$ with $\gamma_s \times \mathbb{R}$ is transverse for each $s \in [-\delta, \delta]$. Thus $\Gamma_n(s)$ has no horizontal or vertical tangents and is a graph over an interval in γ_s.

Now, we show that this interval is $\gamma_s^+ - q(s)$. If this is not the case, $\Gamma_n(s)$ goes beyond $C \times \mathbb{R}$ on the convex side. Recall that $p_n = (q_n, f(q_n))$. Lift each $\Gamma_n(s)$

to $G(p_n)$ by vertical translation I; we are still denoting by $\Gamma_n(s)$ its lift to Σ. The curve

$$\Gamma(s) = \bigcup_{n \geq n_0} \Gamma_n(s)$$

is a vertical graph over an interval in γ_s. This curve has points in the convex side of $C \times \mathbb{R}$ for some $s_0 \in [-\delta, \delta]$. For $s = 0$, $\Gamma(0)$ stays on the mean concave side of $C \times \mathbb{R}$. Thus, for some s_1, $0 < s_1 \leq s_0$, $\Gamma(s_1)$ has a point on $C \times \mathbb{R}$ and is inside the mean convex side of $C \times \mathbb{R}$. On the other hand, $(F(q_n))$ converges uniformly to $C_\delta(q) \times J_\delta$ as $n \longrightarrow +\infty$. Thus, the curve $\Gamma(s_1)$ converges to $q(s_1) \times \mathbb{R}$ as the height goes to $+\infty$. This ensures that $\Gamma(s_1)$ has a vertical tangent on the convex side of $C \times \mathbb{R}$. This is impossible since $\nu < 0$. $\square$

Now, we choose $\epsilon_1 < \epsilon$ (which we call ϵ as well) such that

$$\bigcup_{s \in [-\delta, \delta]} \Gamma(s)$$

is a vertical graph of a function g on

$$\bigcup_{s \in [-\delta, \delta]} (\gamma_s^+ - q(s)).$$

Here γ_s^+ has length ϵ_1. The graph of g converges to $C_\delta(q) \times \mathbb{R}$ as the height goes to infinity.

Observe that at the point $q(\delta)$ we have the same situation as for the point $q(0) = q$. Thus, replacing $\Gamma(0)$ by the curve $\Gamma(\delta)$, we begin this process again. This gives an analytic continuation for the graph g to a graph over

$$\bigcup_{s \in [-\delta, 2\delta]} (\gamma_s^+ - q(s)),$$

which converges uniformly to $C(q, [-\delta, 2\delta])$ as the height goes to infinity. Here $C(q, [-\delta, 2\delta])$ denotes the arc of C of length 3δ between the points $q(-\delta)$ and $q(2\delta)$. Now, we continue analytically by extending the graph about $\Gamma(2\delta)$. When we say analytic continuation, we mean the unique continuation of the local pieces of the surface. Continuing this argument, we obtain that a portion of Σ is a graphic of a function g definite over

$$\bigcup_{s \in \mathbb{R}} (\gamma_s^+ - q(s)).$$

Continue this process replacing $\Gamma(\delta)$ and $\Gamma(-\delta)$ by $\Gamma(2\delta)$ and $\Gamma(-2\delta)$, again, going up high enough on these curves so that the graph is within the ϵ-tubular neighborhood of $Q = \pi^{-1}(C)$, $T_\epsilon(C \times \mathbb{R})$.

Let M denote the surface obtained by this analytic continuation. Thus, the curve $\partial M := \beta$ is far away from Q and M is asymptotic to Q. By Proposition 4.1, $Q = C \times \mathbb{R}$ is an unstable H-surface.

Let K_0 be a compact stable domain of Q. Let K_0 expand until one reaches an unstable domain K of Q, with K compact. This means that there is a smooth function $f : K \longrightarrow \mathbb{R}$, $f = 0$ on ∂K, $f > 0$ on $int(K)$, and f satisfies

$$Lf + \lambda f = 0, \qquad \lambda < 0.$$

Let $K(t)$ be the variation of K given by $K(t) = exp_p(tf(p)Z(p))$, where $p \in K$ and $Z(p)$ is a unit normal to K, with Z pointing to the mean convex side of Q. $K(t)$ is a smooth surface with $\partial K(t) = \partial K \subset Q$, and for t small, $int(K(t)) \cap Q = \emptyset$.

Since the linearized operator L is the first variation of the mean curvature at $t = 0$, and $Lf(p) = -\lambda f(p) > 0$ for $p \in int(K)$, we conclude that $H(K(t)) > H$ for $t > 0$ and $H(K(t)) < H$ for $t < 0$.

Thus, for t small enough the surfaces $K(t)$ are disjoint from β, and they can be slid up and down Q to remain disjoint from M. However, M is asymptotic to Q. Hence for small $t < 0$, the surface $K(t)$ will touch M at a first point when $K(t)$ is slid up Q. However, this contradicts the maximum principle.

Thus Σ must be an entire graph. By Proposition 3.1 such an entire graph cannot exist. This completes the proof in this case. □

The case $E^3(\kappa,\tau) = \mathbb{S}^3_\tau$. There is no complete H-multi-graph Σ immersed into $E^3(\kappa,\tau)$, having angle function $\nu < 0$, and $2H > C(M^2(\kappa))$.

Proof. We will divide the proof into two subcases. In the first subcase, we will suppose that the angle function is far away from 0, that is, $\nu \ll 0$. In the second subcase, we will suppose that there is a sequence (p_n) such that $\nu(p_n) \to 0$.

In both cases, we will show that such an H-multi-graph cannot exist.

First subcase. If ν is far away from 0 ($\nu \ll 0$), then Σ cannot exist.

Let $\pi_1 = \pi_{|_\Sigma} : \Sigma \longrightarrow \mathbb{S}^2$ be the restriction of π to Σ, where π is the Killing submersion

$$\pi : E^3(\kappa,\tau) \longrightarrow M^2(\kappa).$$

As $\nu \ll 0$ for each $p \in \Sigma$, there is a neighborhood of p in Σ, which we denote by $V(p) \subset \Sigma$, such that $V(p)$ is the image of a section over a neighborhood of $\pi(p) \in \mathbb{S}^2$. Hence π_1 is a local homeomorphism. This implies that ($\mathbb{S}^2$ being simply connected) π_1 is a global homeomorphism. Thus, there is a global section of $\pi : \mathbb{S}^3_\tau \longrightarrow \mathbb{S}^2$, which is impossible (see Remark 3.1). Thus, in this subcase Σ cannot exist.

Second subcase. If there is a sequence $q_n \in \Sigma$ such that $\nu(q_n) \longrightarrow 0$, then such a Σ cannot exist.

Suppose that there exists a sequence $(q_n) \subset \Sigma$ such that $\nu(q_n) \longrightarrow 0$. Since $\mathbb{S}^3_\tau$ is compact, there is a subsequence (which we also denote by (q_n)) such that $q_n \longrightarrow p$, $p \in \mathbb{S}^3_\tau$.

Following the same arguments and notation from the case $E^3(\kappa,\tau) \neq \mathbb{S}^3_\tau$, and by using Affirmation 5.2 and Affirmation 5.3, we have:

Σ is asymptotic to an H-cylinder Q and the mean curvature vector of Σ has the same direction as that of Q at points of Σ converging to $Q = \pi^{-1}(C)$, where $C \subset \mathbb{S}^2$ is a complete curve having constant Gauss curvature $2H$. Furthermore, a portion of Σ is a graphic of a function g definite over

$$\bigcup_{s\in\mathbb{R}} (\gamma_s^+ - q(s)).$$

This graph is within a $T_\epsilon(Q) :=$ ϵ-tubular neighborhood of $Q = \pi^{-1}(C)$.

We denote by $\widetilde{C}$ the universal covering of C and by $\widetilde{Q}$ the universal covering of Q, setting a $\widetilde{Q}_\epsilon :=$ $\epsilon-$tubular neighborhood of $\widetilde{Q}$.

We give to $\widetilde{Q}_\epsilon$ the induced metric from $\mathbb{S}^3_\tau$. Again, following the ideas from the case $E^3(\kappa,\tau) \neq \mathbb{S}^3_\tau$, we denoted by M the portion of Σ, which is the graph of the function g, and by $\widetilde{M}$ the universal cover of M.

By construction $\widetilde{M} \subset \widetilde{Q}_\epsilon$; the boundary of $\widetilde{M}$, $\partial\widetilde{M}$ is far away from $\widetilde{Q}$; and $\widetilde{M}$ is asymptotic to $\widetilde{Q}$.

Now we apply the same argument as for the first case to conclude that such Σ cannot exist. $\square$

Considering that in the case $E^3(\kappa,\tau) \neq \mathbb{S}^3(\tau)$ as well as in the case $E^3(\kappa,\tau) = \mathbb{S}^3_\tau$ such an H-multi-graph Σ cannot exist, we conclude the proof of the theorem. $\square$

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