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RELATIVE RIEMANN MAPPING CRITERIA AND HYPERBOLIC CONVEXITY

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Abstract. Let R be a simply-connected Riemann surface with a simply-connected subdomain U. We give a criterion in terms of conformal reflections to determine whether R can be embedded in the complex plane so that U is mapped onto a disc. If it can, then U is convex with respect to the hyperbolic metric of R, by a theorem of Jørgensen. We discuss the close relationship of our criterion to two generalizations of Jørgensen's theorem by Minda and Solynin. We generalize our criterion to the quasiconformal setting and also give a criterion for the multiply-connected case, where an embedding is sought that maps a given subdomain onto a circle domain.

Let R be a simply-connected hyperbolic Riemann surface. It has a unique complete conformal metric of constant curvature -1, which we call the hyperbolic metric of R. (Some authors work with curvature -4, but this choice does not affect any of our arguments as we shall only be comparing hyperbolic metrics.) We say that a subset $D \subset \Omega$ is *hyperbolically convex* with respect to R if for any two points $z \neq w$ in D, the unique geodesic segment joining z to w in the hyperbolic metric of R is contained in D. More generally, if R is a possibly multiply-connected hyperbolic Riemann surface, we will say that a subset D is weakly hyperbolically convex with respect to R if each connected component of $\pi^{-1}(D)$ is hyperbolically convex with respect to the unit disc $\mathbb{D}$, where $\pi : \mathbb{D} \to R$ is an unbranched analytic universal covering map. If $\pi^{-1}(D)$ is connected and hyperbolically convex, we will say that D is strongly hyperbolically convex. Flinn [2] defined D to be h-convex if for any two points z, w in D, any shortest geodesic segment in R joining z to w is contained in D. This implies that D is weakly hyperbolically convex with respect to Ω, but not that it is strongly hyperbolically convex.

Let D be an open disc or half-plane contained in a hyperbolic plane domain Ω. Vilhelm Jørgensen [4] showed that D is h-convex in Ω.

A plane domain D has the *universal h-convexity property* (UHP) if D is h-convex with respect to each simply-connected hyperbolic plane domain Ω strictly containing D, and there is at least one such Ω. This definition is from Flinn [2], where it is shown that discs and half-planes are the only plane domains with the UHP. Alan Beardon observed to the author that the property of being a Euclidean

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disc or half-plane is not invariant with respect to conformal transformations of the larger domain. However the hyperbolic metric is conformally invariant and therefore we have a stronger criterion for hyperbolic convexity of a subdomain $D \subset \Omega$: if there is some conformal embedding $\varphi : \Omega \to \mathbb{C}$ such that $\varphi(D)$ is an open disc or half-plane, then D is h-convex in Ω. It is then natural to ask how we might recognize that there exists such an embedding. In section 1 we give a criterion for this relative Riemann mapping in terms of conformal reflections. We discuss how the criterion is related to two generalizations of Jørgensen's theorem due to Minda and to Solynin. In section 2 we generalize the criterion to quasiconformal mappings. In section 3 we generalize the criterion in a different direction to allow the subdomain to be multiply-connected.

Notation: $\overline{A}$ denotes the closure of the set A, and A^* denotes the image of A under complex conjugation. Although we mostly use z^* to denote the complex conjugate of z, we retain the standard bar notation for complex conjugation in partial derivatives $f_{\overline{z}}$. $\widehat{\mathbb{C}}$ denotes the Riemann sphere, $\mathbb{D}$ the open unit disc and $\mathbb{H}$ the upper half-plane.

1. Hyperbolic convexity and conformal reflections

We begin with the relative Riemann mapping criterion. This is known to complex analysts as folklore, being a simple consequence of the reflection principle.

Proposition 1. *Let R be a hyperbolic Riemann surface and let A and B be disjoint open subsets of R such that $\overline{A} \cup \overline{B} = R$, A is connected and simply-connected, $\partial A = \partial B$ and the common boundary γ of A and B is either a union of disjoint Jordan arcs or a single Jordan curve. Then the following are equivalent:*

(1) *There exists a conformal mapping $g : R \to \widehat{\mathbb{C}}$ such that $g(A) = \mathbb{D}$ and $g(\gamma) \subset \partial\mathbb{D}$.*
(2) *There exists an injective anti-analytic mapping $f : B \to A$ that extends continuously to the identity map on γ.*

Proof. To see that (1) implies (2), take $f(z) = g^{-1}(1/g(z)^*)$ for $z \in B$. For the converse, take any Riemann mapping $h : A \to \mathbb{D}$. We wish to extend h to all of R. Since f extends continuously to the identity on γ, we find that γ is locally an analytic arc and therefore h extends continuously to γ. Define $g(z) = h(z)$ for $z \in \overline{A}$ and $g(z) = 1/h(f(z))^*$ for $z \in B$. The Schwarz reflection principle shows that the extension of g to R is analytic. Moreover, g is injective because f is injective.

In the special case where R is a hyperbolic plane domain, we can arrange for the constructed map g to omit ∞ (so that $g(R)$ is also a hyperbolic plane domain) by altering h by an automorphism of $\mathbb{D}$ if necessary so that $h^{-1}(0) \notin f(B)$. This is possible since $f(B)$ is at least doubly-connected, so is not all of $\mathbb{D}$. $\square$

If we drop the injectivity in both conditions, the same proof yields the following:

Proposition 2. *Let R be a hyperbolic Riemann surface and let A and B be disjoint open subsets of R such that $\overline{A} \cup \overline{B} = R$, A is connected and simply-connected, $\partial A = \partial B$ and the common boundary γ of A and B is either a union of disjoint Jordan arcs or a single Jordan curve. Then the following are equivalent:*

(1) *There exists an analytic (but not necessarily injective)* $g : R \to \widehat{\mathbb{C}}$ *that maps* A *homeomorphically onto* $\mathbb{D}$*, maps* γ *into the unit circle, and maps* B *(not necessarily injectively) into* $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$*.*
(2) *There exists an anti-analytic mapping* $f : B \to A$ *that extends continuously to the identity map on* γ*.*

In the case where R is simply-connected, the first condition in Proposition 2 is equivalent to the sufficient condition for hyperbolic convexity that is given by Solynin:

Lemma A ([8, Lemma 2])**.** *Let* $\mathcal{R}$ *be a hyperbolic simply-connected Riemann surface over* $\widehat{\mathbb{C}}$*, which has exactly one sheet* $\mathcal{R}^0$ *over the lower half-plane* $\mathbb{H}^*$*. Then* $\mathcal{R}^0$ *is hyperbolically convex with respect to* $\mathcal{R}$*. It is hyperbolically strictly convex unless* $\mathcal{R}$ *coincides with the Riemann sphere slit along a closed segment of the extended real line* $\widehat{\mathbb{R}}$*.*

In the situation of Proposition 2, if R is simply-connected and there exists an anti-analytic mapping $f : B \to A$ that extends continuously to the identity map on γ, then applying Lemma A after a Möbius map, we deduce that A is hyperbolically convex in R.

Even this condition is far from being a *necessary* condition for hyperbolic convexity, as the example of a hyperbolic polygon shows: no conformal reflection exists across the boundary near to a vertex. However, the collection of hyperbolically convex sets in a given simply-connected domain is closed under intersection, so we can use conformal reflections to give a characterization of hyperbolic convexity, as follows:

Proposition 3. *Let* U *be a simply-connected hyperbolic Riemann surface and let* K *be a subset of* U*. Then* K *is hyperbolically convex in* U *if and only if for each point* z *in the boundary of* K *(relative to* U*), there exists an analytic Jordan curve* γ *through* z *with the following properties:* $U \setminus \gamma$ *is the union of two connected components* A *and* B *such that* $\overline{A}$ *is simply-connected and contains* K *and there exists an anti-analytic mapping of* B *into* A *that extends continuously to the identity map on* γ*.*

Proof. For necessity, we can just take the curve γ to be a hyperbolic geodesic through z that supports K. For sufficiency, we need to check that the given conditions imply that $\overline{A}$ is hyperbolically convex; since $\overline{A}$ contains K, this implies that $\overline{A}$ contains the hyperbolic convex hull of K, and hence z lies on the boundary of the hyperbolic convex hull of K. If this holds for each z on the boundary of K in U, then K must be hyperbolically convex, as required. To prove that $\overline{A}$ is hyperbolically convex, we use essentially the same proof as for Proposition 1. □

The utility of Proposition 3 lies in the fact that we may not have precise information about the hyperbolic geodesics in U, but can nevertheless check the hyperbolic convexity of a subdomain by finding enough conformal reflections in other curves. Here is a simple application. Let S be the interior of a rectangle, containing K, a smaller rectangle with sides parallel to S. Suppose that the closure of K contains the centre of S. Then K is hyperbolically convex in S. Indeed, the Euclidean reflections in the (extended) sides of K serve as the anti-conformal mappings required to apply Proposition 3.

2. Quasidiscs and quasiconformal reflections

A domain $U \subset \mathbb{C}$ is called a K-quasidisc when there exists a K-quasiconformal homeomorphism $f : \mathbb{C} \to \mathbb{C}$ such that $U = f(\mathbb{D})$. In this case, the map $g : z \mapsto f(1/f^{-1}(z)^*)$ is a K^2-quasiconformal reflection across the Jordan curve $\gamma = \partial U = f(\partial\mathbb{D})$. Conversely, any Jordan curve γ across which there is a K^2-quasiconformal reflection g of the whole of the Riemann sphere must in fact be a K-quasidisc (see [1, 6, 7]).

Here we will prove a result that includes this converse as a special case (when $U = \widehat{\mathbb{C}}$) and also contains Proposition 2 as a special case (when $K = 1$). We will use the upper half-plane rather than the disc, because it simplifies the appearance of the calculations with Beltrami coefficients and also avoids any confusion that might arise when we use the hyperbolic geometry of $\mathbb{D}$ as the natural space for the values of complex dilatations.

Proposition 4. *Let U be a domain in the Riemann sphere. Suppose γ is a Jordan curve in U, one of whose complementary components, A, is contained in U, and let $K \geq 1$. Then the following are equivalent:*

1. *there exists a K^2-quasiconformal reflection $g : U \setminus A \hookrightarrow \overline{A}$ that fixes each point of γ,*
2. *there exists a K-quasiconformal mapping $f : U \to \widehat{\mathbb{C}}$ that is a homeomorphism onto its image such that $f(A) = \mathbb{H}$.*

Proof. (2) implies (1): f is a homeomorphism onto its image, so $f(\gamma) = \partial\mathbb{D}$, and g fixes γ pointwise. For $z \in U \setminus A$, let $g(z) = f^{-1}(1/f(z)^*)$. Since f is injective, we have $f(z) \in \widehat{\mathbb{C}} \setminus \mathbb{D}$, so $1/f(z)^* \in \mathbb{D}$, and therefore g is well-defined and maps into $\overline{A}$, fixing each point of γ. Since g is the composition of two K-quasiconformal mappings and one orientation-reversing conformal mapping, it is an injective K^2-quasiconformal reflection.

(1) implies (2): Let $h : A \to \mathbb{H}$ be a Riemann mapping. Since A is bounded by a Jordan curve, h extends continuously to $h : \overline{A} \to \overline{\mathbb{H}}$ (where the closure is taken in the Riemann sphere). Now we extend h continuously to U by defining $h(z) = h(g(z))^*$ for each $z \in U \setminus A$. Now h is a K^2-quasiconformal homeomorphism onto its image, and it is conformal on A. We need to find a K-quasiconformal homeomorphism $k : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $k(\mathbb{H}) = \mathbb{H}$ and $k \circ h$ is K-quasiconformal on $U \setminus A$. Then $f = k \circ h$ will be the required mapping. We will obtain k by solving a measurable Riemann mapping problem for k fixing $0, 1$ and ∞, using a Beltrami coefficient μ such that $\mu(z^*) = \mu(z)^*$ for all $z \in \mathbb{C}$. The uniqueness of the solution guarantees that k maps $\mathbb{R}$ to $\mathbb{R}$, since $\tilde{k}(z) = k(z^*)^*$ is also a solution. Since the orientation is preserved, we find that $k(\mathbb{H}) = \mathbb{H}$, as required. To ensure that $k \circ h$ is K-quasiconformal on $U \setminus A$, the idea is to write h^{-1} as the composition $k_2 \circ k_1$ of two K-quasiconformal mappings, and then make the complex dilatation of k agree with that of k_1, so that $k \circ k_1^{-1}$ is conformal; then $k \circ h = (k \circ k_1^{-1}) \circ k_2^{-1}$ is K-quasiconformal.

To carry this out explicitly, define the Beltrami coefficient ν on $h(U)$ by

$$\nu = \frac{(h^{-1})_{\overline{z}}}{(h^{-1})_z},$$

so that $\nu = 0$ on $\mathbb{H}$. For each $z \in h(U \setminus \overline{A})$ define $\mu(z)$ to be the hyperbolic midpoint of 0 and $\nu(z)$ in $\mathbb{D}$.

For $z \in \widehat{\mathbb{C}} \setminus h(U)$, define $\mu(z) = 0$. Then for $z \in \mathbb{H}$ define $\mu(z) = \mu\left(z^*\right)^*$. We can leave μ undefined on the real line since it has measure zero. Then $\operatorname{ess\,sup} \frac{1+|\mu|}{1-|\mu|} \leq K$, and the complex dilatation $\frac{(k \circ h)_{\overline{z}}}{(k \circ h)_z}$ is the hyperbolic midpoint of 0 and the complex dilatation of h, so it also has dilatation at most K. $\square$

Remark. There is no uniqueness statement for the map f in condition 2, since at the step where we chose $\mu = 0$ on $\widehat{\mathbb{C}} \setminus h(U)$, we could have chosen instead any measurable Beltrami coefficient with dilatation at most K.

3. A criterion for multiply-connected domains

Minda [5] generalized Jorgensen's theorem to the multiply-connected case. Let $\overline{R}$ be a bordered Riemann surface with interior R and border ∂R, and let $\hat{R}$ be the double of R across ∂R, with associated anti-conformal involution j. For a subsurface $\Omega \subset R$ let Ω^* denote $j(\Omega)$, and let $\lambda_\Omega(z)|dz|$ be the hyperbolic metric of Ω.

Theorem B ([5, Theorem 3])**.** *Let Ω be a hyperbolic subsurface of $\hat{R}$ such that $\Omega \cap \partial R \neq \emptyset$ and $j(\Omega \setminus R) \subset \Omega$ or, equivalently, $\Omega \setminus R \subset \Omega^*$. Then $\lambda_{\omega^*}/\lambda_\Omega(a) \leq 1$ for $a \in \Omega \setminus \overline{R}$, with equality if and only if $\Omega = \Omega^*$.*

Minda's theorem gives a criterion for $\Omega \cap \overline{R}$ to be weakly hyperbolically convex in Ω. In Lemma 5 below we generalize Minda's Theorem B, Solynin's Lemma A, and part of our Proposition 2. No doubt further generalization is possible, but our formulation suffices for the application in Proposition 6.

Lemma 5. *Let R be a hyperbolic Riemann surface, and let A be a nonempty subdomain of R bounded by Γ, a union of disjoint Jordan curves and Jordan arcs with no accumulation point in R. Let B be the interior of $R \setminus A$ and suppose that $f : B \to R$ is an anti-analytic mapping extending continuously to the identity on the common boundary of A and B. Let S be the double of A across Γ. Then the natural inclusion $A \hookrightarrow S$ extends to an analytic map $g : R \to S$, and moreover A is weakly hyperbolically convex in R.*

Remark. The hypothesis that the components of Γ have no accumulation point in R means that there does not exist a sequence of points z_i, each in a different component of Γ, with a limit point in R. For example, it would suffice to assume that Γ has only finitely many components.

In Theorem B the restriction of j to $\Omega \setminus \overline{R}$ is anti-conformal, but in Lemma 5 the map f only has to be anti-analytic: it need not be injective and in particular can have critical points.

In the proof we will need to use a version of Ahlfors' Lemma, which is a generalization of the Schwarz-Pick Lemma. A conformal metric $\rho_a(z)|dz|$ is called a *supporting metric* for $\rho(z)|dz|$ at the point a if $\rho_a(z)$ is defined, positive and of class C^2 in a neighborhood U of a, has curvature at most -1 and $\rho/\rho_a \geq 1$ in U with equality at the point a.

Theorem C ([5, Theorem 1])**.** *Let Ω be a hyperbolic Riemann surface. Suppose that $\rho(z)|dz|$ is an upper semicontinuous nonnegative metric on Ω such that for any $a \in \Omega$ either $\rho/\lambda_\Omega(a) \leq 1$ or else $\rho(a) > 0$ and $\rho(z)|dz|$ has a supporting metric on a neighbourhood of a. Then $\rho/\lambda_\Omega \leq 1$.*

Proof of Lemma 5. We can define a continuous map $F : R \to R$ by

$$F(z) = \begin{cases} z & \text{if } z \in \overline{A}, \\ f(z) & \text{if } z \in \overline{B}. \end{cases}$$

We claim that for each $z \in R$, there exists n such that $F^{\circ n}(z) \in \overline{A}$. Let $N(z)$ be the least such n or 0 if $z \in \overline{A}$. Let j be the anti-analytic involution of S fixing Γ pointwise. Define

$$g(z) = \begin{cases} F^{N(z)}(z) & \text{if } N(z) \text{ is even}, \\ j\left(F^{N(z)}(z)\right) & \text{if } N(z) \text{ is odd}. \end{cases}$$

Then $g : R \to S$ is continuous and is locally the composition of an even number of anti-analytic mappings except at points where $g(z) \in \Gamma$; for such points the Schwarz reflection principle applies to show that g is analytic.

We follow a technique of Minda [5] to establish two weak contraction properties of F. Consider the conformal metric $\rho(z)|\mathrm{d}z| = F^*(\lambda_R(z)\,|\mathrm{d}z|)$, where $\lambda_R(z)|\mathrm{d}z|$ denotes the hyperbolic metric of R. This is well-defined on $\overline{A}$ and on B and is continuous across Γ, because the Jacobian of f tends to 1 as z approaches Γ. (To see this, note that f has an anti-analytic extension to some neighbourhood of Γ, and this fixes Γ pointwise.) So at each point of $R \setminus \Gamma$, either $\rho(z) = 0$ or $\rho(z)|dz|$ has curvature -4, so $\rho(z)|\mathrm{d}z|$ is its own supporting metric. On Γ we have $\rho(z) = \lambda_R(z)$. We can therefore apply Theorem C to deduce that $\rho(z) \leq \lambda_R(z)$ everywhere. It immediately follows that for all $z, w \in R$, we have

$$d_R(F(z), F(w)) \leq d_R(z, w)\,. \tag{1}$$

We will also need a strict contraction property:

$$d_R\left(f(z), \overline{A}\right) < d_R\left(z, \overline{A}\right), \quad \text{for } z \in B. \tag{2}$$

To prove this, let $\kappa : [0,1] \to R$ be a hyperbolic geodesic of minimal length, ℓ, say, such that $\kappa(0) = z$ and $\kappa(1) \in \overline{A}$. This exists since $d_R(z, \cdot)$ is a continuous function on the compact set $\overline{A} \cap B_R(z, 1 + d_R(z, \overline{A}))$, so its minimum is achieved. Now consider the curve $F \circ \kappa$. We have $F(\kappa(0)) = f(z)$, and the hyperbolic length of $F \circ \kappa$ with respect to λ_R is at most ℓ because $\rho \leq \lambda$. For t sufficiently close to 1, $F(\kappa(t)) \in A$, so $d(f(z), \overline{A})$ is strictly less than the length of $F \circ \kappa$, as required.

Pick a point $z_0 \in A$, and let z be any point of R. For each $n \in \mathbb{N}$ we have

$$d\left(F^{\circ n}(z), F\left(z_0\right)\right) = d\left(F^{\circ n}(z), z_0\right) \leq d\left(z, z_0\right),$$

so the orbit of z stays within a compact set and therefore has a subsequential limit point, $F^{\circ n_k}(z) \to z_\infty$, say. Suppose for a contradiction that $z_\infty \notin \overline{A}$. Then by (2), $d\left(F\left(z_\infty\right), \overline{A}\right) < d\left(z_\infty, \overline{A}\right)$, and so for n_k sufficiently large we also have

$$d\left(F\left(z_{n_k}\right), \overline{A}\right) < d\left(z_\infty, \overline{A}\right),$$

and therefore by (2) again

$$d\left(F\left(z_n\right), \overline{A}\right) < d\left(z_\infty, \overline{A}\right)$$

for all n sufficiently large, which is impossible as $d\left(\cdot, \overline{A}\right)$ is continuous at z_∞.

Since A is open, $F^{-1}(A)$ is open. Let $V = \overline{A} \cup F^{-1}(A)$. Note that V contains an open neighbourhood of each point of Γ, so V is an open neighbourhood of $\overline{A}$. Since $F^{\circ n_k}(z) \to z_\infty$ and $z_\infty \in \overline{A}$, we have $F^{\circ n_k} \in V$ for some k, and then $F^{\circ(1+\,n_k)} \in \overline{A}$, as required.

Finally, we show that $\overline{A}$ is weakly hyperbolically convex in R. Consider a hyperbolic geodesic κ in R joining two points $z, w \in \overline{A}$ with $d_R(z,w) = \ell$. Suppose that κ is homotopic relative to its endpoints to a curve κ' in A, via a homotopy $H : [0,1] \times [0,\ell] \to R$. That is, $\kappa(t) = H(0,t)$, parameterized by arc length, and $H(s,0) = z$, $H(s,1) = w$ and $H(1,t) \in \overline{A}$. Then $F \circ H$ is another such homotopy, between $F \circ \kappa$ and κ'. Moreover, $F \circ \kappa$ is no longer than κ, by (1). Since $F \circ \kappa$ is homotopic to κ and there is only one hyperbolic geodesic in each homotopy class, we must have $F \circ \kappa([0,\ell]) = \kappa([0,\ell])$. In fact, since $\rho \leq \lambda$, the curve $F \circ \kappa$ must also be parameterized by arc length, and therefore F fixes the image of κ pointwise. By (2), the only fixed points of F are the points of $\overline{A}$, and therefore κ is a geodesic in $\overline{A}$, as required. □

Koebe generalized the Riemann mapping theorem to the case of a finitely-connected plane domain, showing that it can be mapped conformally onto a finite circle domain (unique up to Möbius transformations). A finite circle domain is the complement in $\widehat{\mathbb{C}}$ of a union of finitely many disjoint closed discs and points. Koebe's theorem was generalized by He and Schramm [3] to the case of countably many complementary components, but we will restrict ourselves to the finite case here.

Suppose that R is a hyperbolic Riemann surface with a finitely-connected subdomain A of genus 0. Then A can be embedded conformally into the Riemann sphere. What are the necessary topological conditions for this embedding to extend to an embedding of R? R must also have genus 0, and the boundary components of A in R must belong to distinct connected components of $R \setminus A$. In this case we can ask whether the canonical conformal homeomorphism of A onto a circle domain in $\widehat{\mathbb{C}}$ can be extended to a conformal embedding of R into $\widehat{\mathbb{C}}$.

Proposition 6. *Let R be a hyperbolic Riemann surface with a subdomain A and a conformal homeomorphism $h : A \to U$, where U is a finitely-connected subdomain of $\widehat{\mathbb{C}}$. Let B be the interior of $R \setminus A$ and suppose that the common boundary Γ of A and B is a union of disjoint Jordan arcs or curves, lying in distinct connected components of $R \setminus A$ and with no accumulation point in R. Suppose that there exists an anti-analytic mapping $f : B \to R$ that extends continuously to the identity on Γ. Then there exists a finite circle domain V and a meromorphic $\psi : R \to \widehat{\mathbb{C}}$ such that the restriction of ψ to A is a homeomorphism onto V. The map ψ is essentially unique. Moreover A is weakly hyperbolically convex in R. If the restriction of f to each connected component of $R \setminus A$ is injective, then ψ is also injective.*

In contrast to Propositions 1 and 2, Proposition 6 gives only a sufficient condition for the canonical conformal mapping of a subdomain to extend to the whole of R, and not a necessary one. For example, if R is a round annulus and A is a ring domain such that $\pi_1(A) \hookrightarrow \pi_1(R)$ but A is disjoint from the closed hyperbolic geodesic of R, then no suitable anti-analytic mapping f will exist, so we cannot apply Theorem 6; nevertheless, A could already be a round annulus.

On the other hand, Proposition 6 is also more general than the sufficient condition in Proposition 2 because we do not require $f : B \to A$, merely $f : B \to R$.

Proof. Apply Koebe's circle domain theorem to obtain a conformal mapping k from $h(A)$ onto a circle domain V; this is determined up to a Möbius map. Let T be the double of V and let S be the double of A across Γ. Then $k \circ h : A \to V$ extends to a

conformal homeomorphism $\phi : S \to T$ by Schwarz reflection. Let Ω be the regular set of the Schottky group Λ^+ that is the orientation-preserving subgroup of the Möbius group Λ generated by conformal reflections in the boundary components of V. There is a natural unbranched covering map $\Omega \to T$. Lemma 5 gives us an analytic map $g : R \to S$. We claim that the composition $\phi \circ g : R \to T$ lifts to the required map $\psi : R \to \Omega$. Indeed, V is a fundamental domain for Λ and, for any point $z \in R$, we can determine which translate of V the lift should map to by looking at the sequence of boundary components in which one must reflect z under iteration of F in order to land in A. This also shows that if $\psi(z) = \psi(w)$, then z and w follow the same sequence of conformal reflections under iteration of F; and therefore if f is injective on each connected component of $R \setminus A$, it follows that $z = w$.

Finally, ψ is unique up to a Möbius map by the identity principle and uniqueness in Koebe's theorem. $\square$

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