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AN EXTREMAL DECOMPOSITION PROBLEM FOR HARMONIC MEASURE

VLADIMIR N. DUBININ AND MATTI VUORINEN

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Abstract. Let E be a continuum in the closed unit disk $|z| \leq 1$ of the complex z-plane which divides the open disk $|z| < 1$ into $n \geq 2$ pairwise nonintersecting simply connected domains D_k such that each of the domains D_k contains some point a_k on a prescribed circle $|z| = \rho$, $0 < \rho < 1$, $k = 1, \ldots, n$. It is shown that for some increasing function Ψ, independent of E and the choice of the points a_k, the mean value of the harmonic measures

$$\Psi^{-1}\left[\frac{1}{n}\sum_{k=1}^{k}\Psi(\omega(a_k, E, D_k))\right]$$

is greater than or equal to the harmonic measure $\omega(\rho, E^*, D^*)$, where $E^* = \{z : z^n \in [-1, 0]\}$ and $D^* = \{z : |z| < 1, |\arg z| < \pi/n\}$. This implies, for instance, a solution to a problem of R. W. Barnard, L. Cole, and A. Yu. Solynin concerning a lower estimate of the quantity $\inf_E \max_{k=1,\ldots,n} \omega(a_k, E, D_k)$ for arbitrary points of the circle $|z| = \rho$. These authors stated this hypothesis in the particular case when the points are equally distributed on the circle $|z| = \rho$.

1. Introduction and formulation of the result

Problems concerning extremal decomposition of domains on the complex sphere go back to the works of Lavrentiev, Goluzin, Nehari, Jenkins and have a rich history and are more or less directly associated with several areas of geometric function theory (see, e.g., [J2, Kuz, S]). A majority of extremal decompositions is related to estimates for the products whose factors are powers of inner radii of pairwise nonoverlapping domains. For the case of harmonic measure the problem concerning extremal decomposition was perhaps first formulated in the paper of R.W. Barnard, L. Cole, and A. Yu. Solynin [BCS]. Let E be a continuum in the closed unit disk $\overline{U}$, $U = \{z : |z| < 1\}$, and let it divide U into two subdomains $D_1 \ni \rho$ and $D_2 \ni -\rho$, $0 < \rho < 1$. Thus $U \setminus E = D_1 \cup D_2$. The authors of the paper [BCS] give the minimum of the sums

$$\frac{1}{2}(\omega(\rho, E, D_1) + \omega(-\rho, E, D_2)),$$

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taken over the set of all continua $E \subset \overline{U}$ that divide U into a pair of domains as described above. Here $\omega(z, E, D)$ denotes the harmonic measure of the set $E \cap \partial D$ with respect to the domain D, evaluated at the point z. For the properties of the harmonic measure we refer to [A, Ch 1.1]. In the paper [BCS] also a physical motivation of the problem is given. It is of interest to observe that the extremal configuration in this problem is not symmetric with respect to the imaginary axis. Among other things the authors of the paper [BCS] discuss the problem of finding the lower bound

$$\inf_E \max_{k=1,\dots,n} \omega(a_k^*, E, D_k)$$

over all continua E that divide U into n simply connected domains

$$D_k \ni a_k^* = \rho \exp(2\pi i(k-1)/n), \quad k = 1, \dots, n, n \geq 3\,.$$

They conjectured that the extremal configuration of the minmax problem consists of n circular sectors

$$D_k^* = \{z \in U : |\arg z - 2\pi(k-1)/n| < \pi/n\}, \;\; k = 1, \dots, n\,,$$

and observed that under the additional assumption that the closure of $\partial D_k \cap U$ is connected for all $k = 1, \dots, n$, this can be proved applying Jenkins's theory of extremal decomposition [J1, S](see [BCS, p. 246]). A. Yu. Solynin has informed the authors that this result has remained unpublished.

In this paper we prove the following theorem.

Theorem 1.1. *For a given ρ, $0 < \rho < 1$, let E be a continuum in the closed unit disk $\overline{U}$ dividing the open disk U into $n \geq 2$ pairwise nonintersecting simply connected domains D_k, such that each of the domains D_k intersects the circle $|z| = \rho, k = 1, \dots, n$. Then for arbitrary points $a_k \in D_k$ on the circle $|z| = \rho$, the following inequality holds:*

$$\frac{1}{n}\sum_{k=1}^{n} \log \frac{1+\sin(\pi\omega_k/2)}{1-\sin(\pi\omega_k/2)} \geq -n\log\rho \tag{1.2}$$

where $\omega_k = \omega(a_k, E, D_k), k = 1, \dots, n$. Equality holds in (1.2) if and only if

$$E = \{z : (e^{i\theta}z)^n \in [-1, 0]\}, D_k = \{z : ze^{i\theta} \in D_k^*\}$$

and $a_k = a_k^* e^{-i\theta}, k = 1, \dots, n$, for some fixed real θ.

Because the function

$$\Psi(x) = \log \frac{1+\sin(\pi\, x/2)}{1-\sin(\pi\, x/2)}$$

is strictly increasing on the interval $(0, 1)$, the inequality (1.2) implies a solution to the problem of R. W. Barnard, L. Cole, and A. Yu. Solynin [BCS], and furthermore, the points a_k need not coincide with $a_k^*, k = 1, \dots, n$. Our proof is based on the result of the first author concerning the product of inner radii of nonoverlapping domains with respect to "free" points on the given circle [D1]. We also make use of an idea of J. Krzyż about the transition from inner radius to conformal invariants [K]. If in the case of [K] the Green function was discussed, then in our case the harmonic measure is in focus. We note further that by the invariance of the harmonic measure under Möbius automorphisms of the disk U, the inequality (1.2) holds for all points a_k, located on an arbitrary hyperbolic circle in U of hyperbolic radius $\log((1+\rho)/(1-\rho))$.

2. Proof of Theorem 1.1

Fix an integer $k, 1 \le k \le n$. If all the boundary points of the domain D_k lie in the continuum E, then $\omega_k = 1$ and the left side of the inequality (1.2) is $+\infty$. In what follows we exclude this case for all $k = 1, \dots, n$. Now let z be some point of the boundary ∂D_k, not contained in E. Then $z \in T := \partial U$. In fact, in the opposite case, $z \in U \setminus E$, and therefore $z \in D_j$ for some $j, 1 \le j \le n$. Because of the openness of a domain, we see that $D_j \cap D_k \ne \emptyset$, which contradicts the hypothesis of the theorem. Further, because the set E is closed, on the circle T there is an open arc α containing the point z and not containing any points of E. We prove that all points of this arc are boundary points of the domain D_k. Suppose that there exists a point $z' \in \alpha \setminus \partial D_k$. Because $z' \notin E$, there exists a domain $D_{k'}$ such that $z' \in \partial D_{k'}, 1 \le k' \le n$. Choose sequences of points $z_m, z'_m, m = 1, 2, \dots,$ and circular arcs $\lambda_m, m = 1, 2, \dots,$ in U with z_m, z'_m as end points and satisfying the following condition. The points $z_m \in D_k, z_m \to z, m \to \infty$; $z'_m \in D_{k'}, z'_m \to z', m \to \infty$, and the arcs $\lambda_m \to T$ when $m \to \infty$. This last convergence means that the upper bound of the distances of the points of the arc λ_m to the circle T approaches 0 when $m \to \infty$. Because $D_k \cap D_{k'} = \emptyset$, we see that on every arc λ_m there exists a point $e_m \in E, m = 1, 2, 3, \dots$. The sequence of the points $\{e_m\}_{m=1}^{\infty}$ contains a subsequence converging to a point e of the arc α. Because E is closed, the point e must be in $\alpha \cap E$, a contradiction to the choice of α. Let us denote by α_k the maximal open subarc of the circle T, containing the arc α and not containing the points of the set E. Clearly, the end points of the arc α_k are contained in E. The domain D_k cannot contain on its boundary two different arcs of the aforementioned type (each corresponding to different points z) in view of the connectedness of the set E. We see that $\alpha_k = (\partial D_k) \setminus E, k = 1, 2, \dots, n$.

We introduce the notation $B_k = D_k \cup \alpha_k \cup \{z : 1/\overline{z} \in D_k\}, k = 1, \dots, n$. From what was said above it follows that the sets $B_k, k = 1, \dots, n$, are simply connected pairwise nonintersecting domains on the Riemann sphere $\overline{\mathbb{C}}$. By Riemann's theorem, there exists a function φ_k mapping conformally and univalently the corresponding domain B_k onto the domain G_k which is the complement of the arc $\beta_k := \{w : |w| = 1, |\arg w| \le \theta_k\}, \varphi_k(a_k) = 0,\ \varphi'_k(a_k) > 0,\ \ k = 1, \dots, n$. On the basis of symmetry, the image of the domain D_k is the disk $|w| < 1$, and further $\varphi_k(\alpha_k) = \{w \in G_k : |w| = 1\}$ and $\varphi_k(E \cap \partial D_k) = \beta_k$. Because the harmonic measure is conformally

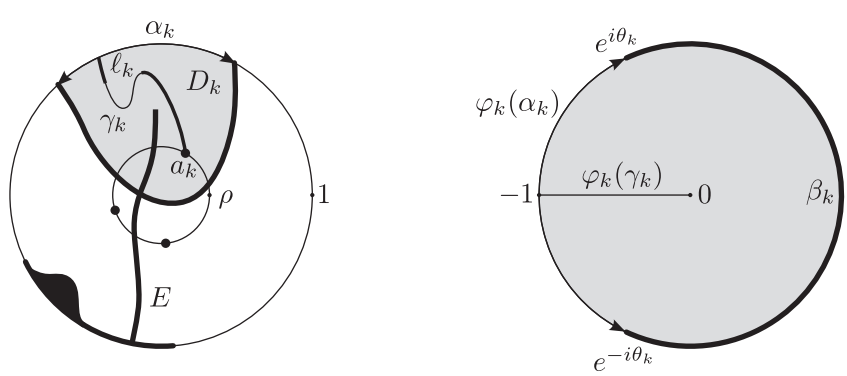

Figure 1. Left: The continuum E divides the unit disk U into three domains. Right: The image of the domain D_k under φ_k.

invariant, we have

$$\omega_k = \omega(0, \beta_k, U_w) = \theta_k/\pi, \qquad k = 1, \dots, n\,.$$

We denote by ℓ_k the union of a finite number of half-open arcs of the analytic curve $\gamma_k := \varphi_k^{-1}((-1,0])$, having the following property: for every $r, \rho \le r < 1$, the circle $|z| = r$ intersects with ℓ_k only at one point. We set $z_k(r) = \{z \in \ell_k : |z| = r\}$. The construction of the arcs ℓ_k can be made in several ways. We choose one of them. It follows from the paper [D1] (cf. also [D2, p. 53]) that for every $r, \rho \le r < 1$, the inequality

$$\sqrt[n]{\Pi_{k=1}^n r(B_k, z_k(r))} \le \frac{4r}{n} \tag{2.1}$$

holds and furthermore the equality in (2.1) holds if and only if the domains B_k and the points $z_k(r)$ coincide with the domains $\{z : |\arg z - 2\pi(k-1)/n| < \pi/n\}$ and the points $r\exp(2\pi(k-1)/n)$, respectively, with a possible rotation around the origin. Here $r(B, z)$ stands for the inner radius of the domain B with respect to a point $z \in B$; see [D2, Sec. 1]. The inequality (2.1) was proved by the piecewise separating symmetrization method of [D2].

By the arithmetic-geometric mean inequality this implies that

$$\frac{1}{n}\sum_{k=1}^n \frac{1}{r(B_k, z_k(r))} \ge \frac{n}{4r}\,.$$

Integration yields the conclusion that

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^n \int_{\gamma_k} \frac{|dz|}{r(B_k, z)} &\ge \frac{1}{n}\sum_{k=1}^n \int_{\ell_k} \frac{|dz|}{r(B_k, z)} \\ &\ge \frac{1}{n}\sum_{k=1}^n \int_\rho^1 \frac{dr}{r(B_k, z_k(r))} \ge \int_\rho^1 \frac{n}{4r}\,dr = -\frac{n}{4}\log\rho\,. \end{aligned} \tag{2.2}$$

On the other hand

$$\int_{\gamma_k} \frac{|dz|}{r(B_k, z)} = \int_{-1}^0 \frac{dw}{r(G_k, w)} \tag{2.3}$$

for all $k = 1, \dots, n$. For the computation of the inner radius $r(G_k, w)$ we map the domain G_k by a univalent conformal mapping onto the unit disk $|\zeta| < 1$. It is easy to see that for such a map we can take a superposition of the following functions:

$$w_1 = \frac{w-1}{w+1}\,, \qquad w_2 = -iw_1 \mathrm{ctg}\frac{\theta_k}{2}\,, \qquad \zeta = w_2 - \sqrt{w_2^2 - 1}\,,$$

where the last mapping is that branch of the function, inverse of the Joukowski transformation, which maps ∞ to zero. Because under a conformal univalent mapping the inner radius is multiplied by the modulus of the derivative at the corresponding point, we have

$$r(G_k, w)|w_1'(w)\mathrm{ctg}\frac{\theta_k}{2}\zeta'(w_2)| = r(U_\zeta, \zeta) = 1 - |\zeta|^2\,.$$

Simple calculations give from this

$$r(G_k, w) = \frac{1-w}{\sin(\theta_k/2)}\sqrt{w^2 - 2w\cos\theta_k + 1}\,.$$

Inserting this value of the inner radius in the inequality (2.3), we obtain

$$\int_{-1}^{0} \frac{dw}{r(G_k, w)} = \sin\frac{\theta_k}{2} \int_{-1}^{0} \frac{dw}{(1-w)\sqrt{w^2 - 2w\cos\theta_k + 1}} = \frac{1}{4}\log\frac{1+\sin(\theta_k/2)}{1-\sin(\theta_k/2)}\,.$$

The computation of the integral is carried out either with standard integration methods or using tables such as [PBM, p. 94]. Applying the relations (2.2) and (2.3) we arrive at the desired inequality (1.2).

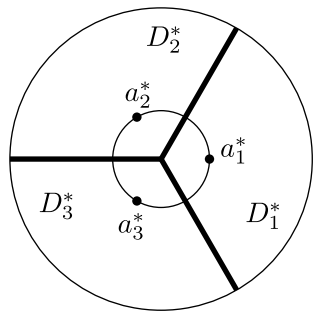

FIGURE 2. Extremal configuration for $n = 3$.

We now assume that in the inequality (1.2) we have equality. Then the equalities also hold in (2.2) and (2.1) for all $r, \rho \le r < 1$. From the equality in (2.2) it follows that the curves $\ell_k = \gamma_k$ are segments on radial rays, emanating from the origin, $k = 1, \dots, n$. Equality in (2.1) shows that the angles between neighboring rays are equal to $2\pi/n$ and the domains B_k are sectors with angle $2\pi/n$ for which these rays are bisectors. This observation implies that for some real θ, $a_k = a_k^* e^{-i\theta}$ and $D_k = \{z : ze^{i\theta} \in D_k^*\}$, $k = 1, \dots, n$. In this case the continuum E necessarily contains the set $E_\theta := \{z : (ze^{i\theta})^n \in [-1, 0]\}$. With a simple verification we see that for $E = E_\theta$ in (1.2) we have the sign of equality. It remains to observe that for other continua $E \supset E_\theta$, $E \ne E_\theta$ we have a strict inequality. The theorem is proved.

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Institute of Applied Mathematics, Far-Eastern Branch of the Russian Academy of Sciences, Vladivostok, Russia

E-mail address: dubinin@iam.dvo.ru

Department of Mathematics, University of Turku, Turku 20014, Finland

E-mail address: vuorinen@utu.fi