ASYMPTOTIC PERIODICITY OF GRADE ASSOCIATED TO MULTIGRADED MODULES

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Abstract. Let $R$ be a Noetherian $\mathbb{N}^r$-graded ring generated in degrees $d_1, \ldots, d_r$ which are linearly independent vectors over $\mathbb{R}$, and let $\mathfrak{a}$ be an ideal in $R_0$. In this paper, we investigate the asymptotic behavior of the grade of the ideal $\mathfrak{a}$ on the homogeneous components $M_n$ of a finitely generated $\mathbb{Z}^r$-graded $R$-module $M$ and show that the periodicity occurs in a cone.

1. Introduction

Let $A$ be a commutative Noetherian ring and $\mathfrak{a}$ an ideal in $A$. Let $R$ be a Noetherian $\mathbb{N}^r$-graded ring with $R_0 = A$, and let $M$ be a finitely generated $\mathbb{Z}^r$-graded $R$-module. In this paper, we study the asymptotic behavior of the numerical function $\text{grade}(\mathfrak{a}, M_n)$ associated to $M$.

The first result in this setting is due to McAdam and Eakin [7]. They proved that if $R$ is a standard $\mathbb{N}$-graded ring, then the set of primes $\text{Ass}_A M_n$ is stable for all large $n$, and hence, by using the technique due to Brodmann [3], we have that $\text{grade}(\mathfrak{a}, M_n)$ is constant for all large $n$. They also showed that Brodmann’s result [2] about the asymptotic prime divisors of an ideal followed from their results as a direct consequence. Afterwards, a number of authors have extended these results to multigraded cases, especially in connection with the study of the asymptotic prime divisors and the analytic spread of ideals. In particular, West [8] extended McAdam-Eakin’s results to multigraded cases. He proved that if $R$ is a standard $\mathbb{N}^r$-graded ring, then the set of primes $\text{Ass}_A M_n$ is eventually stable, and hence $\text{grade}(\mathfrak{a}, M_n)$ is constant for all large $n$. He also considered several interesting nonstandard multigraded cases. For more general results in the standard graded cases, see [1, 5].

On the other hand, when $R$ is a standard $\mathbb{N}$-graded ring with a local ring $A$, Herzog and Hibi [6] gave a direct proof of the stability of $\text{depth}_A M_n$ by using the Hilbert polynomial of Koszul homology modules of $M$ with respect to the maximal ideal of $A$ instead of the asymptotic stability of $\text{Ass}_A M_n$. More recently, Colomé-Nin and Elias [4] investigated in a similar way the asymptotic behavior of the depth associated to graded modules over certain nonstandard multigraded rings.
The purpose of this paper is to give a common generalization to all of the results concerning the asymptotic grade stated above with a more direct approach. The result is the following:

**Theorem 1.1.** Let $R$ be a Noetherian $\mathbb{N}^r$-graded ring generated in degrees $d_1, d_2, \ldots, d_r$, where $d_1, d_2, \ldots, d_r \in \mathbb{N}^r$ are linearly independent vectors over $\mathbb{R}$, with $R_0 = A$. Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a finitely generated $\mathbb{Z}^r$-graded $R$-module. Then, for any ideal $a$ in $A$, there exists a vector $k \in \mathbb{N}^r$ such that, in the cone $C_k$ with vertex $k$ generated by $d_1, d_2, \ldots, d_r$, grade($a, M_n$) is periodic with respect to $d_1, d_2, \ldots, d_r$. Namely, the equality

$$\text{grade}(a, M_n) = \text{grade}(a, M_{n+m})$$

holds true for all $n \in C_k$ and all $m \in \Gamma$, where $\Gamma$ is the semigroup generated by $d_1, d_2, \ldots, d_r$.

In the next section, we fix our notation and give some facts about cones and graded modules. In section 3, we will give a proof of Theorem 1.1.

Throughout this paper, $A$ is a commutative Noetherian ring with identity. $\mathbb{N}$ (resp. $\mathbb{R}$) denotes the set of nonnegative integers (resp. real numbers), and $r$ is any fixed positive integer. Vectors will always be written in bold-faced letters, e.g., $a$, and they will be represented by row vectors, e.g., $a = (a_1, a_2, \ldots, a_r)$.

2. Preliminaries

Let $d_1, d_2, \ldots, d_r \in \mathbb{N}^r$ be any fixed linearly independent vectors over $\mathbb{R}$. We denote by $\Gamma \subseteq \mathbb{N}^r$ the semigroup generated by $d_1, d_2, \ldots, d_r$, i.e.,

$$\Gamma = \left\{ \sum_{i=1}^{r} c_i d_i \mid c_i \in \mathbb{N} \right\}.$$

For any vector $k \in \mathbb{N}^r$, let

$$C_k := \left\{ k + \sum_{i=1}^{r} c_i d_i \mid c_i \in \mathbb{R}_{\geq 0} \right\} \cap \mathbb{N}^r$$

be the cone with vertex $k$ generated by $d_1, d_2, \ldots, d_r$ and

$$\Delta_k := \left\{ k + \sum_{i=1}^{r} c_i d_i \mid 0 \leq c_i < 1, \ c_i \in \mathbb{R} \right\} \cap \mathbb{N}^r$$

be the basic cell of the cone $C_k$. Then it is easy to see that

(i) $\Delta_k$ is a finite subset of $C_k$,

(ii) for any $n \in C_k$, there is a unique expression $n = \delta + m$ with $\delta \in \Delta_k$ and $m \in \Gamma$, and

(iii) $C_k = \bigcup_{\delta \in \Delta_k} (\delta + \Gamma)$.

Moreover, we have the following:

**Lemma 2.1.** For any vectors $k, k' \in \mathbb{N}^r$, there exists an integer $\ell_0 \geq 0$ such that

$$k' + \ell(d_1 + d_2 + \cdots + d_r) \in C_k$$

for all $\ell \geq \ell_0$.

In particular, $C_k \cap C_{k'} \neq \emptyset$, and hence there exists a cone $C_{k''} \subseteq C_k \cap C_{k'}$. 

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Proof. Consider the system
\[
xD = k - k',
\]
where $D$ is a square matrix of size $r$ whose $i$-th row is $d_i$. Since $d_1, d_2, \ldots, d_r$ are linearly independent, there is a unique solution $a = (a_1, a_2, \ldots, a_r) \in \mathbb{R}^r$ of the system (1). Let
\[
\ell_0 := \max \{ ||a_i|| \mid i = 1, 2, \ldots, r \},
\]
where $[\ast]$ denotes the least integer $\geq \ast$. Take $\ell \geq \ell_0$ and put $c_i := \ell - a_i \in \mathbb{R}_{\geq 0}$. Then $a = (\ell - c_1, \ell - c_2, \ldots, \ell - c_r)$ and we have
\[
k' + \ell(d_1 + d_2 + \cdots + d_r) = k + c_1d_1 + c_2d_2 + \cdots + c_r d_r \in C_k.
\]
The last assertions follow from the fact that the above vector is in $C_{k'}$.

Let $R = \bigoplus_{n \in \mathbb{N}^r} R_n$ be a Noetherian $\mathbb{N}^r$-graded ring with $R_0 = A$. Assume that $R$ is generated in degrees $d_1, d_2, \ldots, d_r$, i.e., $R = A[R_{d_1}, R_{d_2}, \ldots, R_{d_r}]$. Let $R_{+++} = (R_{d_1}R_{d_2} \cdots R_{d_r}) R$ be the irrelevant ideal of $R$, which is the ideal generated by homogeneous components of degree $d_1 + d_2 + \cdots + d_r$. For a finitely generated $\mathbb{Z}^r$-graded $R$-module $M = \bigoplus_{n \in \mathbb{N}^r} M_n$, we define the homogeneous support of $M$ as
\[
\text{Supp}_+ M := \{ P \in \text{Spec} R \mid P \text{ is a graded ideal, } M_P \neq (0), \text{ and } R_{+++} \not\subseteq P \}.
\]
For any vector $\delta \in \mathbb{N}^r$, we set $M^{(\delta + \Gamma)} = \bigoplus_{m \in \Gamma} M_{\delta + m}$, which is a graded submodule of $M$.

Lemma 2.2. Let $M = \bigoplus_{n \in \mathbb{N}^r} M_n$ be a finitely generated $\mathbb{Z}^r$-graded $R$-module. Then the following statements are equivalent:

1. $\text{Supp}_+ M = \emptyset$;
2. there exists a vector $k \in \mathbb{N}^r$ such that $M_n = (0)$ for all $n \in C_k$.

Proof. Suppose $\text{Supp}_+ M = \emptyset$. Let $k_0 \in \mathbb{N}^r$ be any fixed vector. Then we claim the following:

Claim. For any $\delta \in \Delta k_0$, there exists $k = k(\delta) \in \mathbb{N}^r$ such that $M_n = (0)$ for all $n \in C_k \cap (\delta + \Gamma)$.

Let $\delta \in \Delta k_0$. The assertion is clear if $M^{(\delta + \Gamma)} = (0)$. Assume $M^{(\delta + \Gamma)} \neq (0)$ and write $M^{(\delta + \Gamma)} = Rm_1 + Rm_2 + \cdots + Rm_t$, where $m_i \in M_{k_i}$ and $k_i \in \delta + \Gamma$. Since $\text{Supp}_+ M = \emptyset$, $\text{Supp}_+ M^{(\delta + \Gamma)} = \emptyset$ so that $R_{+++} \subseteq \sqrt{\text{Ann}_R(M^{(\delta + \Gamma)})}$. Therefore there exists an integer $\ell \geq 0$ such that $R_{+++}^\ell \cdot m_i = (0)$ for all $i = 1, 2, \ldots, t$. This implies that
\[
[Rm_i]_{k_0 + \ell(d_1 + \cdots + d_r)} = (0)
\]
for all $i = 1, 2, \ldots, t$. Let $l_i := k_i + \ell(d_1 + d_2 + \cdots + d_r) \in \delta + \Gamma$. Then $[Rm_i]_n = (0)$ for all $n \in C_{l_i} \cap (\delta + \Gamma)$. Hence, by taking a vector $k = k(\delta) \in \mathbb{N}^r$ such that
\[
C_k \subseteq C_{l_1} \cap C_{l_2} \cap \cdots \cap C_{l_t},
\]
we have that $M_n = [M^{(\delta + \Gamma)}]_n = (0)$ for all $n \in C_k \cap (\delta + \Gamma)$. This completes the proof of the claim.

Now, let $k \in \mathbb{N}^r$ be a vector such that
\[
C_k \subseteq C_{k_0} \cap \bigcap_{\delta \in \Delta k_0} C_{k(\delta)},
\]
where $k(\delta)$ is the vector in the claim. Then $M_n = (0)$ for all $n \in C_k$. Indeed, since $n \in C_{k_0}$, there is a unique expression $n = \delta + m$ for some $\delta \in \Delta_{k_0}$ and $m \in \Gamma$. Hence $n \in C_{k(\delta) \cap (\delta + \Gamma)}$ so that $M_n = (0)$ by the claim.

We prove the other implication. Suppose that there exists $k \in \mathbb{N}^r$ such that $M_n = (0)$ for all $n \in C_k$. We write $M = Rm_1 + Rm_2 + \cdots + Rm_t$, where $m_i \in M_{k_i}$. Then it is enough to show that for any $i = 1, 2, \ldots, t$, there exists an integer $\ell \geq 0$ such that $R^{\ell}_{k_i} \cdot m_i = (0)$. For the vectors $k, k_i \in \mathbb{N}^r$, there exists an integer $\ell_0 \geq 0$ such that

$$k_i + \ell(d_1 + d_2 + \cdots + d_r) \in C_k$$

for all $\ell \geq \ell_0$.

by Lemma 2.1 Thus

$$[Rm_i]_{k_i + \ell(d_1 + d_2 + \cdots + d_r)} \subseteq M_{k_i + \ell(d_1 + d_2 + \cdots + d_r)} = (0)$$

and hence $R^{\ell}_{k_i} \cdot m_i = (0)$. □

**Lemma 2.3.** Let $M = \bigoplus_{n \in \mathbb{Z}^r} M_n$ be a finitely generated $\mathbb{Z}^r$-graded $R$-module and let $k_0 \in \mathbb{N}^r$ and $\delta \in \Delta_{k_0}$. Then there exists a vector $k \in \mathbb{N}^r$ such that

$$M_n \neq (0) \text{ for all } n \in C_k \cap (\delta + \Gamma)$$

if $\text{Supp}_{++} M^{(\delta + \Gamma)} \neq \emptyset$.

**Proof.** Suppose $\text{Supp}_{++} M^{(\delta + \Gamma)} \neq \emptyset$. Assume the contrary, so that for any vector $k \in \mathbb{N}^r$, there exists $n \in C_k \cap (\delta + \Gamma)$ such that $M_n = (0)$. Write $M^{(\delta + \Gamma)} = Rm_1 + Rm_2 + \cdots + Rm_t$, where $m_i \in M_{k_i}$ and $k_i \in \delta + \Gamma$. Then, for each $k_i$, there exists $n_i \in C_{k_i} \cap (\delta + \Gamma)$ such that $M_{n_i} = (0)$ by the assumption. Therefore $[Rm_i]_{n_i} = (0)$ and hence $[Rm_i]_{n_i} = (0)$ for all $n \in C_{n_i}$. By taking a cone $C_k$ such that

$$C_k \subseteq C_{n_1} \cap C_{n_2} \cap \cdots \cap C_{n_t},$$

we have $[M^{(\delta + \Gamma)}]_n = (0)$ for all $n \in C_k$, which implies that $\text{Supp}_{++} M^{(\delta + \Gamma)} = \emptyset$ by Lemma 2.2. This is a contradiction. □

3. PROOF OF THEOREM 1.1

We are now ready to prove Theorem 1.1. Recall that $R$ is a Noetherian $\mathbb{N}^r$-graded ring generated in degrees $d_1, d_2, \ldots, d_r$, where $d_1, d_2, \ldots, d_r \in \mathbb{N}^r$ are linearly independent vectors over $\mathbb{R}$, with $R_0 = A$. Let $M = \bigoplus_{n \in \mathbb{Z}^r} M_n$ be a finitely generated $\mathbb{Z}^r$-graded $R$-module and let $a$ be an ideal in $A$.

**Proof of Theorem 1.1** Let $a = (a_1, a_2, \ldots, a_p)A$ and let $k_0 \in \mathbb{N}^r$ be any fixed vector. Then we claim the following:

**Claim.** For any $\delta \in \Delta_{k_0}$, there exist a vector $k = k(\delta) \in \mathbb{N}^r$ and a constant $c = c(\delta) \in \mathbb{N} \cup \{\infty\}$ such that $\text{grade}(a, M_n) = c$ for all $n \in C_k \cap (\delta + \Gamma)$.

Let $\delta \in \Delta_{k_0}$ and $L := M/aM$. If $\text{Supp}_{++} L^{(\delta + \Gamma)} = \emptyset$, then there exists $k \in \mathbb{N}^r$ such that $L_n = (0)$ for all $n \in C_k \cap (\delta + \Gamma)$ by Lemma 2.2. Therefore $M_n = aM_n$ so that $\text{grade}(a, M_n) = \infty$ for all $n \in C_k \cap (\delta + \Gamma)$. Suppose $\text{Supp}_{++} L^{(\delta + \Gamma)} \neq \emptyset$. By Lemma 2.3 there exists $k_1 \in \mathbb{N}^r$ such that for any $n \in C_{k_1} \cap (\delta + \Gamma)$, $L_{n} \neq (0)$ so that $M_n \neq aM_n$. Thus, by the grade sensitivity of the Koszul complex, we have that for any $n \in C_{k_1} \cap (\delta + \Gamma)$,

$$\text{grade}(a, M_n) = p - \max\{i \mid H_i(a; M_n) \neq (0)\},$$
where \( H_i(a; \ast) \) denotes the \( i \)-th Koszul homology module of \( \ast \) with respect to the sequence \( a = a_1, a_2, \ldots, a_p \). Let

\[
q := \max \left\{ i \mid \text{Supp}_+ \left( H_i \left( a; M^{(\delta + \Gamma)} \right) \right) \neq \emptyset \right\}.
\]

For any \( i > q \), since \( \text{Supp}_+ \left( H_i \left( a; M^{(\delta + \Gamma)} \right) \right) = \emptyset \), there exists \( I_i \in \mathbb{N}^r \) such that

\[
H_i(a; M_n) = (0) \quad \text{for all } n \in C_{I_i} \cap (\delta + \Gamma)
\]

by Lemma \[2.2\]. On the other hand, since \( \text{Supp}_+ \left( H_i(a; M^{(\delta + \Gamma)}) \right) \neq \emptyset \), there exists \( I_q \in \mathbb{N}^r \) such that

\[
H_q(a; M_n) \neq (0) \quad \text{for all } n \in C_{I_q} \cap (\delta + \Gamma)
\]

by Lemma \[2.3\]. Thus, by taking a cone \( C_{k_2} \) such that

\[
C_{k_2} \subseteq C_{I_q} \cap C_{I_{q+1}} \cap \cdots \cap C_{I_k},
\]

we have that for all \( n \in C_{k_2} \cap (\delta + \Gamma) \),

\[
H_q(a; M_n) \neq (0), \quad \text{and} \quad H_i(a; M_n) = (0) \quad \text{if } i > q.
\]

By taking a cone \( C_k \) such that \( C_k \subseteq C_{k_1} \cap C_{k_2} \), we get that

\[
\text{grade}(a, M_n) = p - q \quad \text{for all } n \in C_k \cap (\delta + \Gamma).
\]

This completes the proof of the claim.

Now let us take a cone \( C_k \) such that

\[
C_k \subseteq C_{k_0} \cap \bigcap_{\delta \in \Delta_{k_0}} C_{k(\delta)}
\]

where \( k(\delta) \) is the vector in the claim. Then we have the equality

\[
\text{grade}(a, M_n) = \text{grade}(a, M_{n+m})
\]

for all \( n \in C_k \) and all \( m \in \Gamma \). Indeed, since \( n \in C_{k_0} \), there is a unique \( \delta \in \Delta_{k_0} \) such that \( n \in \delta + \Gamma \). Thus \( n, n + m \in C_{k(\delta)} \cap (\delta + \Gamma) \) and hence \( \text{grade}(a, M_n) = \text{grade}(a, M_{n+m}) \) by the claim.

If we take each \( d_i = (0, \ldots, 0, 1, 0, \ldots, 0) \) to be the \( i \)-th standard basis element of \( \mathbb{N}^r \), then we can readily get the known results in the standard graded cases \[6, 7, 8\]. Moreover, as a direct consequence, we have the stability of the grade in the special nonstandard graded cases considered in \[8\].

**Corollary 3.1.** Let \( R, M, a \) be the same as in Theorem \[1.1\]. Assume that each vector \( d_i \) has the form \( d_i = (\ast, \ldots, \ast, 1, 0, \ldots, 0) \), where \( 1 \) is in the \( i \)-th column. Then there exists a vector \( k \in \mathbb{N}^r \) such that \( \text{grade}(a, M_n) \) is constant in the cone \( C_k \). Namely, the equality

\[
\text{grade}(a, M_n) = \text{grade}(a, M_k)
\]

holds true for all \( n \in C_k \).

**Proof.** Let \( k_0 \in \mathbb{N}^r \) be any fixed vector. Then \( \Delta_{k_0} = \{ k_0 \} \) because of the form of the vectors \( d_i \). Thus we have the assertion as a direct consequence of Theorem \[1.1\]. \( \square \)

Colomé-Nin and Elias studied in \[4\] the asymptotic behavior of depth \( A M_n \) in the case where each \( d_i \) has the form \( (\ast, \ldots, \ast, \lambda_i, 0, \ldots, 0) \) with \( \lambda_i \neq 0 \). This case is also a special case of Theorem \[1.1\].
REFERENCES


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