ISOMETRIES OF THE ZYGMUND $F$-ALGEBRA

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(Communicated by Richard Rochberg)

Abstract. In his monograph A. Zygmund introduced the space $\mathcal{N}\log^\alpha\mathcal{N}$ ($\alpha > 0$) of holomorphic functions on the unit ball that satisfy
$$\sup_{0 \leq r < 1} \int_S \varphi_\alpha(\log(1 + |f(r\zeta)|))d\sigma(\zeta) < \infty,$$
where $\varphi_\alpha(t) = t\{\log(\gamma_\alpha + t)^\alpha$ for $t \in [0, \infty)$ and $\gamma_\alpha = \max\{e, e^\alpha\}$. In 2002, O.M. Eminyan provided some basic properties of $\mathcal{N}\log^\alpha\mathcal{N}$. In this paper we will characterize injective and surjective linear isometries of $\mathcal{N}\log^\alpha\mathcal{N}$. As an application, we will consider isometrically equivalent composition operators or multiplication operators on $\mathcal{N}\log^\alpha\mathcal{N}$.

1. Introduction and main result

Throughout this paper, let $\mathbb{B}$ and $\mathbb{S}$ denote the open unit ball and the unit sphere in the $N$-dimensional complex Euclidean space $\mathbb{C}^N$, respectively, and $d\sigma$ the normalized Lebesgue measure on $\mathbb{S}$.

Let $X$ be a space of all holomorphic functions on some domain. The studies on linear isometries of $X$ have been studied since the 1960s. When $X$ is the Hardy space $\mathcal{H}^p$ ($0 < p \leq \infty, p \neq 2$) on the unit disc, D. deLeeuw, W. Rudin, and J. Wermer [16] ($p = 1$, $\infty$) and F. Forelli [6] ($1 \leq p < \infty$) characterized the linear isometries. J. Cima and W.R. Wogen [2] obtained the form of the isometries of the Bloch space. Also W. Hornor and J.E. Jamison [11] considered the isometries of the Dirichlet space and the $\mathcal{S}^p$-space. For the details on these studies, we can also refer to the monograph [5]. For the several variables case, Forelli [7] and Rudin [17] have determined the injective and/or surjective isometries of $\mathcal{H}^p$. For the case when $X$ is the weighted Bergman spaces $\mathcal{A}^p_\alpha$ ($0 < p < \infty, p \neq 2$), the isometries were completely characterized in a sequence of papers by C.J. Kolaski [13, 14, 15]. By these works we see that the isometries on these holomorphic function spaces are described as weighted composition operators defined by $\Psi C_\Phi(f) = \Psi \cdot (f \circ \Phi)$ for some holomorphic function $\Psi$ and holomorphic self-map $\Phi$ of the unit ball, which is one of the reasons why these operators have been investigated so much recently in the settings of the unit ball. The case when $X$ is not a Banach space has also been studied by many authors. The Smirnov class $\mathcal{N}^*$ and the Privalov space $\mathcal{N}^p$ ($1 < p < \infty$) which are contained in the Nevanlinna class $\mathcal{N}$ are examples of such spaces. These types of spaces are $F$-spaces in the sense of Banach with respect to a suitable metric on them. K. Stephenson [19], Y. Iida and N. Mochizuki [12], and A.V. Subbotin [21] have studied linear isometries of these spaces. Their works

Received by the editors October 1, 2010 and, in revised form, March 16, 2011.
2010 Mathematics Subject Classification. Primary 32A37; Secondary 47B33.
Key words and phrases. Isometries, Zygmund $F$-algebra, composition operators.

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showed that the injective isometries are weighted composition operators induced by some inner functions and inner maps of $B$ whose radial limit maps satisfy a measure-preserving property.

Motivated by these works, in this paper we will investigate injective and surjective linear isometries of the Zygmund $F$-algebra.

Take an arbitrary $\alpha > 0$ and consider the function $\varphi_\alpha(t) = t \{ \log(\gamma_\alpha + t) \}^\alpha$ for $t \in [0, \infty)$ where $\gamma_\alpha = e$ if $0 < \alpha \leq 1$, $\gamma_\alpha = e^\alpha$ if $\alpha > 1$. The Zygmund $F$-algebra $N \log^\alpha N$ consists of holomorphic functions $f$ on $B$ for which

$$\sup_{0 \leq r < 1} \int_{S} \varphi_\alpha(\log(1 + |f(r\zeta)|))d\sigma(\zeta) < \infty,$$

where $\log^+ x = \max\{0, \log x\}$ for $x \geq 0$. It is easily verified that the above condition is equivalent to the condition

$$\sup_{0 \leq r < 1} \int_{S} \varphi_\alpha(\log(1 + |f(r\zeta)|))d\sigma(\zeta) < \infty.$$

This class was considered by A. Zygmund [22] first. Recently O.M. Eminyan [4] studied linear space properties of this class. Since the function $\varphi_\alpha(\log(1 + x))$ satisfies

$$\varphi_\alpha(\log(1 + x)) \leq (\log \gamma_\alpha)^\alpha x \text{ for } x \geq 0,$$

we see that the inclusion $H^1 \subset N \log^\alpha N$ holds. More precisely it is known that it holds the following relation:

$$\bigcup_{p > 0} H^p \subset N \log^\alpha N \subset N^* \subset N.$$

This implies that the boundary function $f^*$ exists for any $f \in N \log^\alpha N$. By using this boundary value of $f$, we can define the quasi-norm $\|f\|_\alpha$ on $N \log^\alpha N$ by

$$\|f\|_\alpha = \int_{S} \varphi_\alpha(\log(1 + |f^*(\zeta)|))d\sigma(\zeta).$$

Since this quasi-norm satisfies the triangle inequality, $d_\alpha(f, g) := \|f - g\|_\alpha$ defines a translation invariant metric on $N \log^\alpha N$. So $N \log^\alpha N$ is an $F$-space in the sense of Banach with respect to this metric $d_\alpha$. Moreover Eminyan [4] proved that $N \log^\alpha N$ forms an $F$-algebra with respect to $d_\alpha$. We will consider a linear isometry of $N \log^\alpha N$ in this metric. The following is the main result in this paper.

**Theorem 1.** If $T$ is a linear isometry of $N \log^\alpha N$ into itself, then there exist an inner function $\Psi$ and an inner map $\Phi$ on $B$ which $\Phi^*$ satisfies the measure-preserving property on $S$ such that $T = \Psi C_\Phi$ on $N \log^\alpha N$.

Conversely, for given such $\Psi$ and $\Phi$, the weighted composition operator $\Psi C_\Phi$ is an injective linear isometry of $N \log^\alpha N$.

2. **Proofs of main result**

In this section we will prove Theorem 1. As a corollary, we also give the complete characterization of the surjective isometry of $N \log^\alpha N$. To prove Theorem 1 we need some lemmas.

**Lemma 1.** If $T$ is a linear isometry of $N \log^\alpha N$, then the restriction of $T$ to $H^1$ is also a linear isometry of $H^1$ into $H^1$. 

Proof. Take an $K$ constant $\alpha$ and $\omega > 0$. Since we see that

$$\lim_{\omega \to \infty} \frac{f^{*}(\omega)}{m}$$

again, we have

$$\int_{S} \varphi_{\alpha} \left( \log \left( 1 + \frac{|f^{*}(\zeta)|}{m} \right) \right) d\sigma(\zeta) = \int_{S} \varphi_{\alpha} \left( \log \left( 1 + \frac{|g^{*}(\zeta)|}{m} \right) \right) d\sigma(\zeta).$$

By inequality (1), we have

$$m \varphi_{\alpha} \left( \log \left( 1 + \frac{|f^{*}(\zeta)|}{m} \right) \right) \leq (\log \gamma_{\alpha})^{\alpha} |f^{*}(\zeta)|,$$

for each positive integer $m$ and almost all $\zeta \in S$. By using the definition of $\varphi_{\alpha}$, we see that

$$\lim_{m \to \infty} m \varphi_{\alpha} \left( \log \left( 1 + \frac{|f^{*}(\zeta)|}{m} \right) \right) = (\log \gamma_{\alpha})^{\alpha} |f^{*}(\zeta)|,$$

for almost all $\zeta \in S$. The Lebesgue dominated convergence theorem gives

$$\lim_{m \to \infty} \int_{S} m \varphi_{\alpha} \left( \log \left( 1 + \frac{|f^{*}(\zeta)|}{m} \right) \right) d\sigma(\zeta) = (\log \gamma_{\alpha})^{\alpha} \int_{S} |f^{*}(\zeta)| d\sigma(\zeta).$$

On the other hand, Fatou’s lemma, (2) and (3) give

$$(\log \gamma_{\alpha})^{\alpha} \int_{S} |g^{*}(\zeta)| d\sigma(\zeta) \leq \liminf_{m \to \infty} \int_{S} m \varphi_{\alpha} \left( \log \left( 1 + \frac{|g^{*}(\zeta)|}{m} \right) \right) d\sigma(\zeta)$$

$$= (\log \gamma_{\alpha})^{\alpha} \int_{S} |f^{*}(\zeta)| d\sigma(\zeta),$$

and so $g \in H^{1}$. By applying the Lebesgue dominated convergence theorem once again, we have

$$\lim_{m \to \infty} \int_{S} m \varphi_{\alpha} \left( \log \left( 1 + \frac{|g^{*}(\zeta)|}{m} \right) \right) d\sigma(\zeta) = (\log \gamma_{\alpha})^{\alpha} \int_{S} |g^{*}(\zeta)| d\sigma(\zeta).$$

By (2), (3) and (4), we see that $T$ is a linear isometry of $H^{1}$ into $H^{1}$. \hfill \Box

Lemma 2. There exist a bounded continuous function $\theta_{\alpha}$ on $[0, \infty)$ and a positive constant $K_{\alpha}$ such that

$$\varphi_{\alpha}(\log(1 + x)) = (\log \gamma_{\alpha})^{\alpha} x - K_{\alpha} x^{2} + x^{3} \theta_{\alpha}(x) \quad \text{for } x \in [0, \infty).$$

Proof. By the application of Taylor’s theorem of $\varphi_{\alpha}(\log(1 + x))$, we have

$$\varphi_{\alpha}(\log(1 + x)) = \omega'_{\alpha}(0)x + \frac{\omega''_{\alpha}(0)}{2!} x^{2} + \frac{\omega'''_{\alpha}(0)}{3!} x^{3} + R_{4}(x),$$

where $\omega_{\alpha}(x) = \varphi_{\alpha}(\log(1 + x))$ and $R_{4}(x)$ denotes the remainder term of order 4. Since we see that $\omega'_{\alpha}(0) = (\log \gamma_{\alpha})^{\alpha}$ and $\omega''_{\alpha}(0)/2!$ is equal to $\frac{2\alpha - \alpha}{2\alpha}$ ($< 0$) if $0 < \alpha \leq 1$, $\alpha^{2} \frac{2 - \alpha}{2\alpha} (< 0)$ if $\alpha > 1$, we put

$$\theta_{\alpha}(x) = \frac{\varphi_{\alpha}(\log(1 + x)) - (\log \gamma_{\alpha})^{\alpha} x + K_{\alpha} x^{2}}{x^{3}},$$

where $K_{\alpha} = -\omega''_{\alpha}(0)/2!$. Since $R_{4}(x)/x^{3} \to 0$ as $x \to 0^{+}$, $\theta_{\alpha}(x)$ has a finite limit $\omega'''_{\alpha}(0)/3!$ as $x \to 0^{+}$, and so this $\theta_{\alpha}$ is a desired function. \hfill \Box
Proof of Theorem 1. Assume that $T$ is a linear isometry of $N\log^\alpha N$. Since $T$ is also an isometry of $H^1$ by Lemma 1 and Rudin’s theorem 17 implies that $T = \Psi C_\Phi$ on $H^1$ where $\Psi = T(1)$ and $\Phi$ is an inner map of $\mathbb{B}$ which satisfies

$$\int_S h \, d\sigma = \int_S (h \circ \Phi^*) |\Psi^*| \, d\sigma,$$

for every bounded Borel function $h$ on $S$.

First we will prove that $T = \Psi C_\Phi$ on $N\log^\alpha N$. Fix an $f \in N\log^\alpha N$ and consider dilated functions $\{f_r\}_{0 < r < 1}$ of $f$ $(f_r(z) = f(rz))$. Since each $f_r$ are in the ball algebra, we have $T(f_r)(z) = \Psi(z) \cdot f(r\Phi(z))$ for all $r \in (0,1)$ and $z \in \mathbb{B}$. Since the convergence in $N\log^\alpha N$ implies the uniform convergence on compact subsets of $\mathbb{B}$ and $\|f_r - f\|_\alpha \to 0$ as $r \to 1^-$ (see 4),

$$\lim_{r \to 1} T(f_r)(z) = \lim_{r \to 1} \Psi(z) \cdot f(r\Phi(z)) = \Psi(z) \cdot f(\Phi(z)).$$

On the other hand, the assumption on $T$ which is an isometry of $N\log^\alpha N$ gives

$$\lim_{r \to 1} \|T(f_r) - T(f)\|_\alpha = \lim_{r \to 1} \|f_r - f\|_\alpha = 0.$$

Combining this with (5), we see that $T = \Psi C_\Phi$ on $N\log^\alpha N$.

Next we will prove that $\Psi$ is an inner function on $\mathbb{B}$. Since $\Psi \in H^1$, it is enough to prove that $|\Psi^*| = 1$ a.e. on $S$. Note that $1 = \|\Psi\|_{H^1} \leq \|\Psi\|_{H^2}$, $\|t\Psi\|_{H^1} = t$, and $\|t\Psi\|_\alpha = \varphi_\alpha(\log(1 + t))$ for any $t > 0$. By Lemma 2 we obtain

$$\int_S \{K_\alpha|\Psi^*|^2 - t|\Psi^*|^3\theta_\alpha(|t\Psi^*|)\} \, d\sigma = (\log \gamma_\alpha)^\alpha t - \varphi_\alpha(\log(1 + t))$$

$$= K_\alpha t^2 - t^3 \theta_\alpha(t),$$

and so we have

$$\int_S \{K_\alpha|\Psi^*|^2 - t|\Psi^*|^3\theta_\alpha(|t\Psi^*|)\} \, d\sigma = K_\alpha - \theta_\alpha(t).$$

Since $K_\alpha|\Psi^*|^2 - t|\Psi^*|^3\theta_\alpha(|t\Psi^*|) = \{(\log \gamma_\alpha)^\alpha |t\Psi^*| - \varphi_\alpha(\log(1 + |t\Psi^*|))\}/t^2 \geq 0$ a.e. on $S$, Fatou’s lemma gives

$$\int_S K_\alpha|\Psi^*|^2 \, d\sigma \leq \liminf_{t \to 0} \{K_\alpha - \theta_\alpha(t)\} = K_\alpha.$$

Thus we have $\|\Psi\|_{H^2} \leq 1$ and $\|\Psi\|_{H^2} = \|\Psi\|_{H^1} = 1$. This implies that $|\Psi^*| = 1$ a.e. on $S$. Furthermore by combining $|\Psi^*| = 1$ a.e. on $S$ and (5), we see that $\Phi^*$ is a measure-preserving map on $S$.

For the converse, we assume that $\Psi$ is an inner function and $\Phi$ is an inner map with $\Phi^*$ satisfy the measure-preserving property. If $f$ is in the ball algebra, then it holds that $\|\Psi C_\Phi(f)\|_\alpha = \|f\|_\alpha$ since $f \circ \Phi^* = (f \circ \Phi)^*$ a.e. on $S$. Fix $f \in N\log^\alpha N$ and take a sequence $\{r_j\}$ with $r_j \to 1$ as $j \to \infty$. Since $f_{r_j} \to f$ in $N\log^\alpha N$ as $j \to \infty$, $\{\Psi C_\Phi(f_{r_j})\}$ is a Cauchy sequence in $N\log^\alpha N$ and $g := \lim_{j \to \infty} \Psi C_\Phi(f_{r_j}) \in N\log^\alpha N$. The uniform convergence on compact subsets of $\mathbb{B}$ shows that $g(z) = \Psi C_\Phi(f(z))$ for each $z \in \mathbb{B}$ and $\|\Psi C_\Phi(f)\|_\alpha = \|f\|_\alpha$; namely $\Psi C_\Phi$ is an isometry of $N\log^\alpha N$. \qed

To obtain the form of the surjective isometry of $N\log^\alpha N$, we need some characterization for which a holomorphic function $f$ belongs to $N\log^\alpha N$. 

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Lemma 3. Let \( N \) denote the Nevanlinna class. For any function \( f \in N \), \( f \in N \log^a N \) if and only if \( \varphi_a(\log^+ |f^*|) \in L^1(\sigma) \) and
\[
\varphi_a(\log^+ |f(z)|) \leq \int_S P(z, \zeta) \varphi_a(\log^+ |f^*(\zeta)|) d\sigma(\zeta) \quad \text{for } z \in \mathbb{B},
\]
where \( P(z, \zeta) \) denotes the invariant Poisson kernel of \( \mathbb{B} \).

Proof. The result which replaced \( \varphi_a(\log^+ x) \) with \( \{\log^+ x\}^p (p > 1) \) can be found in [20]. For the reader’s benefit, however, we will give the proof.

Now we assume that \( f \in N \log^a N \). Fatou’s lemma shows that \( \varphi_a(\log^+ |f^*|) \in L^1(\sigma) \). The inclusion \( N \log^a N \subset N^* \) implies that \( \log^+ |f| \) has the least \( M \)-harmonic majorant. Since the least \( M \)-harmonic majorant of \( \log^+ |f| \) is the Poisson integral \( P[\log^+ |f^*|] \), we have the following inequality:
\[
\log^+ |f(z)| \leq \int_S P(z, \zeta) \log^+ |f^*(\zeta)| d\sigma(\zeta) \quad \text{for } z \in \mathbb{B}.
\]

Note that \( \varphi_a(t) \) is strictly increasing and convex on \([0, \infty)\) and the Poisson kernel has the normalization
\[
\int_S P(z, \zeta) d\sigma(\zeta) = 1.
\]

Applying Jensen’s inequality to (8), we obtain the inequality (7).

Conversely, we put \( z = r\eta \) \((0 \leq r < 1, \eta \in \mathbb{S})\) in (7). By integrating with respect to \( \eta \) and applying Fubini’s theorem, we have that
\[
\int_S \varphi_a(\log^+ |f(r\eta)|) d\sigma(\eta) \leq \int_S \varphi_a(\log^+ |f^*(\zeta)|) d\sigma(\zeta) \int_S P(r\eta, \zeta) d\sigma(\eta).
\]

By the symmetric property \( P(r\eta, \zeta) = P(r\zeta, \eta) \) and (6), we obtain that
\[
\sup_{0 \leq r < 1} \int_S \varphi_a(\log^+ |f(r\eta)|) d\sigma(\eta) \leq \int_S \varphi_a(\log^+ |f^*(\zeta)|) d\sigma(\zeta).
\]

Hence the condition \( \varphi_a(\log^+ |f^*|) \in L^1(\sigma) \) implies that \( f \in N \log^a N \). \( \square \)

Corollary 1. An isometry \( T \) of \( N \log^a N \) is surjective if and only if \( T = aC_U \), where \( a \in \mathbb{C} \) with \(|a| = 1\) and \( U \) is a unitary transformation.

Proof. Since the surface measure \( d\sigma \) is unitary invariant, it is clear that the operator \( aC_U \) is a surjective isometry of \( N \log^a N \).

Suppose that \( T \) is surjective. Then Theorem [11] gives that \( T = \Psi C_\Phi \). This assumption shows that \( \Phi \) is an automorphism of \( \mathbb{B} \). So it is enough to prove that \( \Phi \) fixes the origin because the automorphism which fixes the origin is a unitary transformation. Let \( \Phi_j \) \((1 \leq j \leq N)\) be the component of \( \Phi \). For each \( j \in \{1, \ldots, N\} \) and \( r \in (0, 1) \) we have
\[
\int_S \Phi_j(r\zeta) d\sigma(\zeta) = \int \frac{1}{2\pi} \Phi_j(re^{i\theta}) d\theta.
\]

Since the mean value theorem shows that the right-hand side term is equal to \( \Phi_j(0) \), the Lebesgue dominated convergence theorem gives \( \int_S \Phi_j^*(\zeta) d\sigma(\zeta) = \Phi_j(0) \).

On the other hand, by applying the measure-preserving property of \( \Phi^* \) to a bounded Borel function \( h(w) = \langle w, e_j \rangle \) where \( e_j \) is the standard orthonormal base.
vector in $\mathbb{C}^N$, we have
\[
\Phi_j(0) = \int_S \Phi_j^\ast(\zeta) d\sigma(\zeta) = \int_S \langle \Phi^\ast(\zeta), e_j \rangle d\sigma(\zeta) = \int_S \langle \zeta, e_j \rangle d\sigma(\zeta).
\]
From [13] p. 15, §1.4.5 (2) we have that
\[
\int_S \langle \zeta, e_j \rangle d\sigma(\zeta) = \frac{N-1}{\pi} \int_{-\pi}^{\pi} \int_0^{\pi} (1-t^2)^{N-2} t^2 e^{i\theta} dt d\theta = 0,
\]
and so $\Phi$ fixes the origin.

Next we will prove that $\Psi$ is a unimodular constant. If $f \in N_{\log^{\alpha}}N$ such that $1 = T(f) = \Psi C_\phi(f)$, then $1/\Psi = f \circ \Phi \in N_{\log^{\alpha}}N$. Inequality (7) in Lemma 3 gives that
\[
\varphi_\alpha \left( \log^+ \frac{1}{|\Psi(z)|} \right) \leq \int_S P(z, \zeta) \varphi_\alpha \left( \log^+ \frac{1}{|\Psi^\ast(\zeta)|} \right) d\sigma(\zeta) = 0,
\]
and so we have $1/|\Psi| \leq 1$ on $\mathbb{B}$. Since $\Psi$ is inner, $\Psi$ is a unimodular constant. \qed

3. ISOMETRICALLY EQUIVALENT COMPOSITION OPERATORS
AND MULTIPLICATION OPERATORS

For two continuous operators $S$, $T$ on a Banach space $X$, the isometric equivalence problem is defined as follows: What are the necessary and sufficient conditions such that $SU_1 = U_2 T$ on $X$ for some surjective isometries $U_1, U_2$ of $X$? This problem for composition operators on various holomorphic function spaces has been considered by several authors [11, 13, 10]. Recently N.J. Gal, J.E. Jamison, and A.G. Siskakis [9] considered this problem for an integral operator on the Hardy space and the Bergman space. By taking the form of the surjective isometry of $N_{\log^{\alpha}}N$ into consideration, we can also consider this problem for operators on $N_{\log^{\alpha}}N$. Hence we will define the $N_{\log^{\alpha}}N$-isometric equivalence for operators $S, T$ as follows.

**Definition.** For two continuous operators $S, T$ on $N_{\log^{\alpha}}N$, we say that $S$ and $T$ are $N_{\log^{\alpha}}N$-isometrically equivalent if and only if there are two surjective isometries $U_1, U_2$ of $N_{\log^{\alpha}}N$ such that $SU_1 = U_2 T$ on $N_{\log^{\alpha}}N$. In this section, we will consider the isometric equivalence problem for composition operators $C_\phi$ and multiplication operators $M_g$ on $N_{\log^{\alpha}}N$.

Composition operators are very popular studies in the field of analytic functions and operator theory. For the case $N_{\log^{\alpha}}N$ on the unit disc $\mathbb{D}$, since the function $\varphi_\alpha(\log(1 + |f|))$ is subharmonic on $\mathbb{D}$, Littlewood’s subordination principle shows that every analytic self-map of $\mathbb{D}$ induce a continuous composition operator on $N_{\log^{\alpha}}N$. For the unit ball case, however, every holomorphic self-map of $\mathbb{B}$ does not always induce a continuous composition operator on $N_{\log^{\alpha}}N$. This situation is the same as the Hardy space case (see [3]).

In contrast to the case of composition operators, multiplication operators $M_g$ defined by $M_g(f) = g \cdot f$ are always continuous on $N_{\log^{\alpha}}N$ for each $g \in N_{\log^{\alpha}}N$. Because $N_{\log^{\alpha}}N$ forms an algebra, the closed graph theorem implies that $M_g$ is continuous on $N_{\log^{\alpha}}N$.

Hence we will assume that a holomorphic self-map of $\mathbb{B}$ induces a continuous composition operator on $N_{\log^{\alpha}}N$ and a multiplier function $g$ belongs to $N_{\log^{\alpha}}N$ when we consider the isometric equivalence problem for these operators.
Theorem 2. Suppose that $Φ$ and $Ψ$ are holomorphic self-maps of $ℂ$ such that $C_Φ$ and $C_Ψ$ are continuous on $N\log^α N$. Then $C_Φ$ and $C_Ψ$ are $N\log^α N$-isometrically equivalent if and only if

$$Ψ(z) = (U_1 ∘ Φ ∘ U_2^*)(z),$$

for some unitary transformations $U_1, U_2$.

Proof. Now suppose that $C_Φ T_1 = T_2 C_Ψ$ for some surjective isometries $T_1, T_2$ of $N\log^α N$. By Corollary [1] $T_j (j = 1, 2)$ has the form $T_j = a_j C_{U_j}$, where $a_j$ are unimodular constants and $U_j$ are unitary transformations. Hence we have that $a_1 f(U_t ∘ Φ) = a_2 f(Ψ ∘ U_2)$ for any $f ∈ N\log^α N$. By taking $f ≡ 1$ in this relation, we see that $a_1 = a_2$. By taking $f(z) = z_j (1 ≤ j ≤ N)$ in this relation, we also obtain that $Ψ_j(U_2(z)) = ∑_{k=1}^N a_{jk}Φ_k(z)$, where $a_{jk}$ denotes components of the unitary matrix $U_j$. This implies that $Ψ = U_1 ∘ Φ ∘ U_2^*$ on $ℂ$.

To prove the other direction, take a unimodular constant $λ$ and unitary transformations $U_1$ and $U_2$ and consider the surjective isometries of $N\log^α N$ given by

$$T_1 = λC_{U_1} \text{ and } T_2 = λC_{U_2}.$$

Then we see that these isometries satisfy the relation $C_Φ T_1 = T_2 C_Ψ$ on $N\log^α N$. □

Theorem 3. Let $g, h ∈ N\log^α N$. Then $M_g$ and $M_h$ are $N\log^α N$-isometrically equivalent if and only if

$$g(z) = λ h(Uz),$$

for some unimodular constant $λ$ and unitary transformation $U$.

Proof. To prove the sufficiency, we may consider the surjective isometries

$$T_1 = λC_{U_1} \text{ and } T_2 = C_{U_2}.$$

To prove the necessity, we may apply the relation $M_g T_1 = T_2 M_h$ to the constant function $f ≡ 1$. □

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