PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 140, Number 8, August 2012, Pages 2873–2882
S 0002-9939(2011)11342-X
Article electronically published on December 19, 2011

A SPLITTING THEOREM FOR HIGHER ORDER PARALLEL IMMERSIONS

INES KATH AND PAUL-ANDI NAGY

(Communicated by Ken Ono)

Abstract. We consider isometric immersions into space forms having the second fundamental form parallel at order k. We show that this class of immersions consists of local products, in a suitably defined sense, of parallel immersions and normally flat immersions of flat spaces.

1. Introduction

A basic question in submanifold geometry is the study of isometric immersions of Riemannian manifolds having their second fundamental form subject to certain geometric or analytic properties. One of the best known examples is the class of immersions having parallel second fundamental form. Such immersions are called parallel. Their study is part of well-established theories; for instance, the Euclidean case has been fully described by Ferus [5, 6, 7]. Parallel immersions in space forms are the extrinsic counterpart of locally symmetric spaces [21].

In this spirit, a natural class to consider is that of *k-parallel* immersions, by requiring the second fundamental form to be parallel at order k for $k \geqslant 1$. At a pure analogy level, while each Riemannian manifold whose Riemannian curvature tensor is parallel at higher order is locally symmetric [20] the class of k-parallel immersions is sensibly larger than that of parallel ones. Curves such as the Cornu spiral provide simple 2-parallel, nonparallel examples.

Early work by Mirzojan [15] on k-parallel immersions has been taken up by Dillen and Lumiste among others; see [14] for an overview. In the case of immersions into space forms, classification results have been obtained for low dimension or codimension or for small k in [9, 2, 8, 12, 10, 11, 13]. A case-by-case inspection of instances therein reveals that the immersion is a product and each factor is either a k-parallel curve, a k-parallel flat and normally flat surface, an affine subspace or a sphere.

That this reflects the general structure of k-parallel immersions into space forms is the main result of this paper. We prove

Theorem. *Any k-parallel isometric immersion of a Riemannian manifold (M, g) into a simply connected space form is the local product of a parallel immersion and a k-parallel immersion of a flat space with flat normal bundle.*

Received by the editors March 14, 2011.
2010 *Mathematics Subject Classification.* Primary 53B21, 53C42.
Key words and phrases. Submanifold, higher order parallel fundamental form, splitting theorems.

This is Theorem 2.3 in the paper. Products of immersions are understood in the sense of Nölker [19]; see Definition 2.4. Open manifolds are naturally considered, since in the compact case integration shows that k-parallel immersions are parallel.

To prove the theorem we first note that having the ambient space locally symmetric forces the Riemann curvature to be parallel at higher order, hence parallel by [20]. We refine the natural local product decomposition of (M, g) into a flat space and a locally symmetric space with nondegenerate curvature. The key ingredient is to take into account the algebraic structure of the normal bundle reflecting the structure of higher order derivatives of the second fundamental form. It is used to modify the above-mentioned splitting into one that satisfies the requirements in the generalised Moore splitting criterion [16, 19].

Our theorem reduces the study of k-parallel immersions into space forms to that of parallel immersions and k-parallel normally flat immersions of flat space. This is an essential step towards the classification of k-parallel immersions of space forms. Indeed, parallel immersions of space forms are known. The Euclidean case was solved by Ferus as already mentioned above. If the target space is a sphere one can easily get a classification from the Euclidean case. For the hyperboloid as target a classification was achieved independently by Takeuchi [22] and Backes and Reckziegel [1]. Furthermore, in [13] Lumiste describes a general method (the so-called polynomial map method) for the investigation of k-parallel immersions in Euclidean spaces that are flat and normally flat; see also [18], where the latter are interpreted as certain integrable systems of hydrodynamic type. In conclusion, to obtain a full classification there remains to understand the structure of flat, normally flat k-parallel immersions in hyperbolic space.

2. Structure results

Let (N^n, g) be a Riemannian manifold and let us consider an isometric immersion $(M^m, g) \hookrightarrow (N, g), m \leqslant n$. We will denote by ∇ and ∇^N, respectively, the Levi-Civita connections and by R, R^N the Riemann curvature tensors, with the convention that $R^N(X, Y, Z, U) = g((\nabla^N)^2_{Y,X} Z - (\nabla^N)^2_{X,Y} Z, U)$ for all X, Y, Z, U in TN. The orthogonal splitting

$$TN = TM \oplus NM$$

along M enables us to consider the second fundamental form $\alpha : TM \times TM \to NM$, where NM is the normal bundle of M. We denote by $\nabla^\perp$ the induced covariant derivative in NM and by $R^\perp$ the corresponding curvature tensor. Furthermore, we write $\nabla\alpha$ for the covariant derivative of α, which is defined by ∇ and $\nabla^\perp$.

Definition 2.1. Let $(M, g) \hookrightarrow (N, h)$ be an isometric immersion. It is called k-parallel for some $k \geqslant 1$ if and only if

$$\nabla^k \alpha = 0. \tag{2.1}$$

A particular case thereof consists in the so-called *parallel* immersions, when one requires $\nabla\alpha = 0$.

In most of what follows we will look at immersions into standard spaces $M^n(c)$, that is, simply connected, complete spaces of constant sectional curvature c and dimension n, realised by their standard models in $\mathbb{R}^{n+1}$: the sphere, the hyperboloid and the hyperplane.

The main goal of this paper is to prove that the study of k-parallel immersions into standard spaces reduces to that of parallel immersions and k-parallel immersions of flat space. The latter class will be shown later on to be normally flat as well.

One of our main tools is Moore's decomposition criterion [17] and its generalisations to the case of standard spaces due to [16] (see also [19]). Its application requires setting up the notion of product immersion, which we recall below in the Euclidean case to begin with.

Definition 2.2. An isometric immersion $f : M_1 \times M_2 \to \mathbb{R}^n$ of a Riemannian product is called a product immersion if and only if $f = F \circ (f_1 \times f_2)$ for isometric immersions $f_i : M_i \to N_i, i = 1, 2$, where N_1, N_2 are affine subspaces of $\mathbb{R}^n$ such that $F : N_1 \times N_2 \to \mathbb{R}^n$ is isometric.

The next definition captures necessary conditions for an immersion to be a product. In the rest of this paper we will systematically use the natural identification $T(M_1 \times M_2) = TM_1 \oplus TM_2$ whenever M_1, M_2 are smooth manifolds.

Definition 2.3. The second fundamental form α of an isometric immersion $f : M_1 \times M_2 \to M$ is called decomposable if $\alpha(\mathscr{D}_1, \mathscr{D}_2) = 0$, where $\mathscr{D}_i = df(TM_i), i = 1, 2$.

Having decomposable second fundamental form has been shown, for Euclidean target spaces, to yield a product structure in [17]; explicitly:

Theorem 2.1. *An isometric immersion $f : M_1 \times M_2 \to \mathbb{R}^n$, where M_1, M_2 are connected and f has decomposable second fundamental form, is a product immersion.*

To present the generalisation of the above result to a standard space target, we first recall that a submanifold N in $M^n(c)$ is called spherical if its second fundamental form $\alpha = g \otimes \zeta$ for some parallel normal vector field, where g is the induced metric on N. For $c \neq 0$ this is equivalent to N being the intersection of the standard space and some affine subspace of $\mathbb{R}^{n+1}$.

Definition 2.4. An isometric immersion $f : M_1 \times M_2 \to M^n(c)$ of a Riemannian product is called a product immersion if and only if $f = F \circ (f_1 \times f_2)$ for isometric immersions $f_i : M_i \to N_i, i = 1, 2$, where:

(i) N_1, N_2 are isometric to standard spaces and admit isometric embeddings $\varphi_i : N_i \to M^n(c), i = 1, 2$, with spherical image such that
$$\varphi_1(N_1) \ \cap \ \varphi_2(N_2) = \{\overline{p}\};$$
(ii) the map $F : N_1 \times N_2 \to \mathbb{R}^{n+1}$, $(p_1, p_2) \mapsto \overline{p} + (\varphi_1(p_1) - \overline{p}) + (\varphi_2(p_2) - \overline{p})$ is an isometric embedding with image contained in $M^n(c)$.

The curved counterpart of Moore's decomposition criterion in Theorem 2.1 is

Theorem 2.2 ([16]; see also [19])**.** *An isometric immersion $f : M_1 \times M_2 \to M^n(c)$, where M_1, M_2 are connected and f has decomposable second fundamental form, is a product immersion.*

Whereas Definition 2.2 is a particular case of Definition 2.4, the latter allows a broader treatment of the case when $c = 0$, e.g. when N_1, N_2 are taken to be spheres.

An isometric immersion $f : M \to N$ will be called a local product if for each x in M there exists a product open set $M_1 \times M_2$ around x such that $f_{|M_1 \times M_2}$ is a product immersion.

The main result in this paper can now be formulated as follows.

Theorem 2.3. *Let $(M, g) \to M^n(c)$ be a k-parallel isometric immersion. It is the local product of a parallel immersion and a k-parallel immersion of a flat space with flat normal bundle. When (M, g) is simply connected and complete, the splitting is global.*

The proof requires a few technical steps, which we will outline below. When not specified otherwise we will work under the assumptions of Theorem 2.3.

We define at each point of M,

$$E_0 = \{X \in TM : R(TM, TM)X = 0\},\ E_1 = E_0^{\perp},$$

and note that $E_1 = span\{R(X, Y)Z : X, Y, Z \in TM\}$.

Proposition 2.1. *Let $(M, g) \hookrightarrow (N, h)$ be a k-parallel isometric immersion, where (N, h) is locally symmetric. The following hold:*

(i) *E_0 and E_1 are parallel in TM;*
(ii) *the distribution E_0 is flat;*
(iii) *$E_1 \subset \{X \in TM : \nabla_X \alpha = 0\}$.*

Proof. (i) We differentiate the Gauss equation

$$\begin{aligned} R^N(X_1, X_2, X_3, X_4) =& R(X_1, X_2, X_3, X_4) + \langle \alpha_{X_2} X_3, \alpha_{X_1} X_4\rangle \\ &- \langle \alpha_{X_1} X_3, \alpha_{X_2} X_4\rangle, \end{aligned} \tag{2.2}$$

where $X_i, 1 \le i \le 4$, belong to TM, to find that

$$(\nabla^{2k-1} R)(X_1, X_2, X_3, X_4) = (\nabla^{2k-1} R^N)(X_1, X_2, X_3, X_4)$$

in directions tangent to M. Now, taking into account that $\nabla^N R^N = 0$ yields

$$-(\nabla_{X_0} R^N)(X_1, X_2, X_3, X_4) = \sum_{i=1}^{4} R^N(X_1, \dots, \alpha_{X_0} X_i, \dots, X_4)$$

for all X_0 in TM. Now the Codazzi-Mainardi formula

$$-R^N(X_1, X_2, X_3, \xi) = \langle (\nabla_{X_1} \alpha)_{X_2} X_3 - (\nabla_{X_2} \alpha)_{X_1} X_3, \xi\rangle \tag{2.3}$$

for $\xi \in NM$ gives, for instance, that

$$-R^N(X_1, X_2, X_3, \alpha_{X_0} X_4) = \langle (\nabla_{X_1} \alpha)_{X_2} X_3 - (\nabla_{X_2} \alpha)_{X_1} X_3, \alpha_{X_0} X_4\rangle;$$

hence clearly $\nabla^{2k-1} R^N = 0$ and then $\nabla^{2k-1} R = 0$. By [20] we have that $\nabla R = 0$, whence the claim.

(ii) follows from the definition of E_0.

(iii) We exploit an argument used in [20, 23] under slightly different assumptions. The function $t = |\alpha|^2 : M \to \mathbb{R}$ satisfies

$$\langle \nabla_X grad\, t, Y\rangle = 2\langle \nabla_X \alpha, \nabla_Y \alpha\rangle \tag{2.4}$$

for all X, Y in TM; it follows that $\nabla^{2k-2} grad\, t = 0$ along M, after differentiating at higher order. In particular, $\nabla^{2k-2} X_1 = 0$, where X_1 denotes the orthogonal projection of $grad\, t$ onto the parallel distribution E_1. Now we can use the following result due to Tanno. Suppose that at some point x of a Riemannian manifold and for some tangent vectors X, Y at x, $R(X, Y)$ is not singular. Then having

$\nabla^k T = 0$ for an arbitrary tensor T and for some $k \geq 1$ implies that $\nabla T = 0$; see [23], Theorem 1. Applied to the integral manifolds of E_1 this yields $\nabla_U X_1 = 0$ for all U in E_1; in particular, $R(E_1, E_1)X_1 = 0$. Since, moreover, $R(E_0, TM) = 0$ we get $R(TM, TM)X_1 = 0$; thus $X_1 \in E_0$ showing that $X_1 = 0$. In other words, *grad* t belongs to E_0, which is parallel in TM. When taking X, Y in E_1 it follows that the right-hand side of (2.4) vanishes and the claim follows by a positivity argument. □

Under additional assumptions we have the following alternative by using essentially that a Ricci flat, locally symmetric space is flat.

Corollary 2.1. *If we have a k-parallel isometric immersion of an Einstein or locally irreducible manifold M into a locally symmetric space, then either the immersion is parallel or M is flat.*

In what follows the action of the curvature tensor on a tensor field Q in $(T^\star M)^l \otimes NM$ is given by

$$(R(X,Y)\cdot Q)(Z_1,\dots,Z_l) = R^\perp(X,Y)Q(Z_1,\dots,Z_l) - \sum_{i=1}^{l} Q(Z_1,\dots,R(X,Y)Z_i,\dots,Z_l)$$

whenever $X, Y, Z_i, 1 \leq i \leq l$, belong to TM. For $s \geqslant 0$ we will frequently use the shorthand notation $\operatorname{Im}\nabla^s\alpha$ to refer to $span\{\nabla^s_{X_1,\dots,X_s}\alpha\} \subseteq NM$, where $X_i, 1 \leq i \leqslant s$ belong to TM.

Lemma 2.1. *Let (M, g) be a k-parallel isometric immersion into a space form. We have*

(i) $R^\perp(E_0, TM)(\operatorname{Im}\nabla^{k-s}\alpha) = 0, s \geqslant 0$;
(ii) $R^\perp(E_0, TM) = 0$.

Proof. Because $\nabla^2(\nabla^{k-2}\alpha) = 0$ it follows after anti-symmetrisation that the curvature operator $R(X, Y)$ acts trivially on $\nabla^{k-2}\alpha$. Since, moreover, $R(E_0, TM)TM = 0$ we obtain that

$$R^\perp(E_0, TM)(\operatorname{Im}\nabla^{k-2}\alpha) = 0.$$

Further anti-symmetrisation in the first two arguments in $\nabla^{k-2}\alpha = \nabla^2(\nabla^{k-4}\alpha)$ yields

$$R^\perp(E_0, TM)(R^\perp(E_0, TM)(\operatorname{Im}\nabla^{k-4}\alpha)) = 0.$$

After taking scalar products with vectors in $\operatorname{Im}\nabla^{k-4}\alpha$ a positivity argument shows that

$$R^\perp(E_0, TM)(\operatorname{Im}\nabla^{k-4}\alpha) = 0.$$

Continuing this procedure leads to $R^\perp(E_0, TM)(\operatorname{Im}\nabla^{k-2s}\alpha) = 0, s \geqslant 0$. Since k-parallel manifolds are also $(k+1)$-parallel this equation holds also for $k+1$, and (i) follows.

In particular, (i) gives $R^\perp(E_0, TM)(\operatorname{Im}\alpha) = 0$. We will now use the Ricci formula

$$\langle R^\perp(X,Y)\xi_1, \xi_2\rangle = \langle [A_{\xi_1}, A_{\xi_2}]X, Y\rangle \tag{2.5}$$

for X, Y in TM and ξ_1, ξ_2 in NM, where $\langle A_\xi X, Y\rangle = -\langle \alpha_X Y, \xi\rangle$ is the shape operator. It shows that also $(\operatorname{Im}\alpha)^\perp = \{\xi \in NM : A_\xi = 0\}$ is annihilated by $R^\perp(E_0, TM)$, and (ii) follows. □

Proposition 2.2. *Let (M, g) be a k-parallel isometric immersion into a space form. Then we have that*

(i) *(M, g) is semi-parallel; that is, $R(X, Y) \cdot \alpha = 0$ whenever X, Y belong to TM;*
(ii) $\nabla R^{\perp} = 0$.

Proof. (i) From the lemma and the definition of E_0 it follows that the tensor

$$(R(X,Y)\cdot\alpha)(Z_1,Z_2) = R^{\perp}(X,Y)(\alpha_{Z_1}Z_2) - \alpha(R(X,Y)Z_1,Z_2) - \alpha(Z_1,R(X,Y)Z_2)$$

vanishes when X is in E_0. For X, Y in E_1 we have $\nabla^2_{X,Y}\alpha = 0$ by (iii) in Proposition 2.1 and $R(X, Y) \cdot \alpha = 0$ follows.

(ii) By differentiation in the lemma above,

$$(\nabla_{TM} R^{\perp})(E_0, TM) = 0$$

since E_0 is parallel inside TM. There remains to show that $(\nabla_{TM} R^{\perp})(E_1, E_1) = 0$. For X, Y in E_1, U in TM as well as ξ_1, ξ_2 in NM, we have

$$\langle(\nabla_U R^{\perp})(X,Y)\xi_1,\xi_2\rangle = \langle[(\nabla_U A)_{\xi_1}, A_{\xi_2}]X,Y\rangle + \langle[A_{\xi_1},(\nabla_U A)_{\xi_2}]X,Y\rangle$$

by differentiation in the Ricci equation (2.5). Moreover

$$\begin{aligned}\langle[(\nabla_U A)_{\xi_1}, A_{\xi_2}]X,Y\rangle &= \langle(\nabla_U A)_{\xi_1}A_{\xi_2}X,Y\rangle - \langle A_{\xi_2}(\nabla_U A)_{\xi_1}X,Y\rangle \\ &= \langle(\nabla_U A)_{\xi_1}Y,A_{\xi_2}X\rangle - \langle A_{\xi_2}(\nabla_U A)_{\xi_1}X,Y\rangle.\end{aligned}$$

Now for space forms, the right-hand side in the Codazzi-Mainardi equation (2.3) vanishes. Since, moreover, $\nabla_{E_1} A = 0$ by (iii) in Proposition 2.1, each summand above is zero. A similar argument shows that the same holds for the second commutator in the expression of $\nabla R^{\perp}$ and the claim is proved. □

We consider

$$N_0M = \{\xi \in NM : R^{\perp}(TM, TM)\xi = 0\}$$

with orthogonal complement N_1M in NM.

Proposition 2.3. *Let (M, g) be a k-parallel isometric immersion into a space form. We have:*

(i) *N_0M, N_1M are parallel;*
(ii) *N_0M is flat.*

Proof. (i) is a direct consequence of (ii) in Proposition 2.2, while (ii) follows from (i) and the definition of N_0M. □

Let us consider the following subbundle of the normal bundle of M:

$$F_0 := span\{(\nabla^l_{X_1,\dots,X_l}\alpha)(Y,Z) : 1 \le l \le k-1, X_1,\dots,X_l,Y,Z \in TM\} \subseteq NM.$$

By applying Proposition 2.1(iii) and Lemma 2.1(i) and using the Codazzi-Mainardi equation we can also write this as

(2.6)
$$F_0 = span\{(\nabla^l_{X_1,\dots,X_l}\alpha)(Y,Z) : 1 \le l \le k-1, X_1,\dots,X_l,Y,Z \in E_0\} \subseteq NM.$$

Proposition 2.4. *Let (M, g) be a k-parallel isometric immersion into a space form. The following hold:*

(i) *the subbundle $F_0 \subseteq NM$ is parallel w.r.t. the normal connection;*
(ii) *the bundle F_0 is flat, that is, $R^{\perp}(TM, TM)F_0 = 0$.*

Proof. (i) is a direct consequence of $\nabla^k\alpha = 0$.

(ii) We use (2.6) for the description of F_0. Let $X_1, \dots, X_l, Y, Z$ be in E_0 and $1 \le l \le k-1$. We have $R(U,V)\cdot\nabla^l\alpha = 0$ for all $U, V \in TM$ by differentiating the equation $R(U,V)\cdot\alpha = 0$, which we know from Proposition 2.2(i). This gives

$$R^\perp(U,V)((\nabla^l_{X_1,\dots,X_l}\alpha)(Y,Z)) = \sum_{j=1}^{l}(\nabla^l_{X_1,\dots,R(U,V)X_j,\dots,X_l}\alpha)(Y,Z)$$
$$+(\nabla^l_{X_1,\dots,X_l}\alpha)(R(U,V)Y,Z) + (\nabla^l_{X_1,\dots,X_l}\alpha)(Y,R(U,V)Z)$$

Since E_0 is flat, the right-hand side of the above equation vanishes. □

Lemma 2.2. *If (M,g) is a k-parallel isometric immersion into a space form, then*

(i) $\alpha(E_0,E_0) \subseteq N_0M$;
(ii) $\alpha(E_0,E_1) \subseteq N_1M$;
(iii) $A_{N_0M}E_i \subseteq E_i$, $i = 1,2$;
(iv) $A_{F_0}E_1 = 0$.

Proof. (i) Since (M,g) is semiparallel by Proposition 2.2 and since E_0 is flat, we have

$$R(U,V)^\perp(\alpha(X,Y)) = \alpha(R(U,V)X,Y) + \alpha(X,R(U,V)Y) = 0$$

for all $X, Y \in E_0$ and all $U, V \in TM$.

(ii) Using again that (M,g) is semiparallel and that E_0 is flat we obtain

$$R^\perp(U,V)(\alpha(X,Y)) = \alpha(R(U,V)X,Y)$$

for all $U, V, X \in TM$ and $Y \in E_0$. The claim now follows from $E_1 = span\{R(X,Y)Z : X, Y, Z \in TM\}$.

(iii) follows from (ii). Indeed,

$$\langle A_{N_0M}E_1, E_0\rangle = \langle E_1, A_{N_0M}E_0\rangle = \langle\alpha(E_1,E_0), N_0M\rangle = 0.$$

(iv) We have to show that $\alpha(E_1,TM) \perp F_0$. Obviously, $\alpha(E_1,E_0) \perp F_0$ by (ii) since Proposition 2.4(ii) gives $F_0 \subset N_0M$. It remains to show that $\alpha(E_1,E_1) \perp F_0$. Take $Y_0, Z_0 \in E_0$ and $Y_1, Z_1 \in E_1$. Because $R(E_0,E_1,E_0,E_1) = 0$ the Gauss equation (2.2) yields

$$\langle\alpha(Y_1,Y_0),\alpha(Z_1,Z_0)\rangle - \langle\alpha(Y_1,Z_1),\alpha(Y_0,Z_0)\rangle = c\langle Y_0,Z_0\rangle\langle Y_1,Z_1\rangle.$$

After differentiation, using that $\nabla_{E_1}\alpha = 0$, the Codazzi-Mainardi equation leads to

$$\langle\alpha(Y_1,Z_1),(\nabla^l_{X_1,\dots,X_l}\alpha)(Y_0,Z_0)\rangle = 0$$

for all $1 \le l \le k-1$ and $X_1, \dots, X_l \in E_0$. Now it follows from (2.6) that $\alpha(E_1,E_1)$ is orthogonal to F_0. □

Now we will define a further decomposition of E_0 into subbundles $E_0' \oplus E_0''$ such that Moore's criterion applies to $D_0 := E_1 \oplus E_0'$ and $D_1 := E_0''$. Let E_0'' be spanned by elements of the form

$$A_\xi X,\ (\nabla^l_{U_1,\dots U_l}A)_\xi X, \quad l = 1,\dots,k-1,$$

where ξ belongs to F_0, X is in TM and $U_1,\dots,U_l$ are in TM. Equivalently, it suffices to take $X \in E_0$. Indeed, this follows from Lemma 2.2(iv).

Obviously, $E_0'' \subset E_0$ by Lemma 2.2(iii). We will denote by E_0' the orthogonal complement of E_0'' in E_0.

Proposition 2.5. *If (M, g) is a k-parallel isometric immersion into a space form, then*

(i) *E_0' and E_0'' are parallel w.r.t. ∇;*
(ii) *$E_0' \subseteq \{X \in E_0 : A_{F_0} X = 0\}$;*
(iii) *$E_0' \subseteq \{X \in E_0 : \nabla_X \alpha = 0\}$.*

Proof. (i) follows directly from having E_0 and F_0 parallel and $\nabla^k \alpha = 0$.

(ii) is a direct consequence of the definition of E_0'.

(iii) Take $X \in E_0'$. We first note that $(\nabla_U A)_\xi X = 0$; hence also $\langle (\nabla_U \alpha)(X, \cdot), \xi \rangle = 0$ holds for all $U \in TM$ and $\xi \in F_0$ by the definition of E_0'. By the Codazzi-Mainardi equation we obtain $\langle (\nabla_X \alpha)(U, \cdot), \xi \rangle = 0$ for all $U \in TM$ and $\xi \in F_0$. Since $(\nabla_X \alpha)(TM, TM) \subseteq F_0$ the claim follows. □

Proposition 2.6. $\alpha(E_1 \oplus E_0', E_0'') = 0$.

Proof. The orthogonal complement F_1 of F_0 clearly contains $N_1 M$. Take $\xi_1 \in F_1$. Then

$$\langle (\nabla^l_{U_1,\dots,U_l} A)_{\xi_1} X, Y \rangle = \langle (\nabla^l_{U_1,\dots,U_l} \alpha)(X, Y), \xi_1 \rangle = 0$$

holds for all $X, Y, U_1, \dots, U_l \in TM$ since $(\nabla^l_{U_1,\dots,U_l} \alpha)(X, Y) \in F_0$. Thus

$$(\nabla^l_{U_1,\dots,U_l} A)_{\xi_1} = 0. \tag{2.7}$$

The Ricci equation (2.5) gives

$$[A_\xi, A_{\xi_1}] = 0 \tag{2.8}$$

for all $\xi \in N_0 M$ and $\xi_1 \in F_1$, which implies that

$$A_{\xi_1}(\operatorname{Im} A_\xi) \subseteq \operatorname{Im} A_\xi. \tag{2.9}$$

Differentiating (2.8) and taking into account (2.7) we obtain

$$[(\nabla^l_{U_1,\dots,U_l} A)_\xi, A_{\xi_1}] = 0.$$

This gives

$$A_{\xi_1}(\operatorname{Im}(\nabla^l_{U_1,\dots,U_l} A)_\xi) \subseteq \operatorname{Im}(\nabla^l_{U_1,\dots,U_l} A)_\xi \tag{2.10}$$

for all $U_1, \dots, U_l \in TM$. Applying (2.9) and (2.10) to all $\xi \in F_0$ we obtain

$$A_{F_1}(E_0'') \subseteq E_0''. \tag{2.11}$$

Since $N_1 M \subseteq F_1$ this implies that

$$\langle \alpha(E_0'', E_1), N_1 M \rangle = \langle A_{N_1 M} E_0'', E_1 \rangle = 0,$$

which combined with $\alpha(E_0, E_1) \subseteq N_1 M$ shows that $\alpha(E_0'', E_1) = 0$.

At the same time from (2.11),

$$\langle \alpha(E_0', E_0''), F_1 \rangle = \langle A_{F_1} E_0'', E_0' \rangle = 0,$$

which shows that $\alpha(E_0', E_0'')$ is contained in F_0. However, $\langle \alpha(E_0', E_0''), F_0 \rangle = \langle A_{F_0} E_0', E_0'' \rangle = 0$ by (ii) in Proposition 2.5 leads to $\alpha(E_0', E_0'') = 0$. □

Proof of Theorem 2.3. The orthogonal splitting $TM = (E_1 \oplus E_0') \oplus E_0''$ is parallel w.r.t. to the Levi-Civita connection and hence induces a local splitting of M with corresponding factors M_1 and M_2. By Proposition 2.6, Molzan's generalisation of Moore's Theorem applies to the immersion of $M_1 \times M_2$ in $M^n(c)$; it yields a product decomposition of f according to Definition 2.2 with factors $f_i : M_i \to N_i, i = 1, 2$.

From (iii) in Propositions 2.1 and 2.5 it follows that $(\varphi_1 \circ f_1)(M_1)$ is parallel in $M^n(c)$. Since the image of N_1 in $M^n(c)$ is spherical, it follows by a standard argument that $(\varphi_1 \circ f_1)(M_1)$ is parallel in $\varphi_1(N_1)$ as well. Hence $f_1 : M_1 \to N_1$ is parallel since $\varphi_1 : N_1 \to \varphi_1(N_1)$ is an isometry.

Since the distributions E_0 and hence E_0'' are flat, so is M_2 and the same argument as above shows that it is immersed in N_2 as a k-parallel immersion.

In the absence of a factor of type E_1 it follows from (ii) in Lemma 2.1 that the normal bundle of M_2 is flat. □

Acknowledgement

It is a pleasure to thank A. J. di Scala for useful comments on an earlier version of this paper.

References

[1] E. Backes and H. Reckziegel, *On symmetric submanifolds of spaces of constant curvature*, Math. Ann. **263** (1983), 419-433. MR707240 (84i:53050)

[2] F. Dillen, *The classification of hypersurfaces of a Euclidean space with parallel higher order fundamental form*, Math. Z. **203** (1990), 635-643. MR1044069 (91d:53076)

[3] F. Dillen and S. Nölker, *Semi-parallelity, multi-rotation surfaces and the helix-property*, J. Reine Angew. Math. **435** (1993), 33-63. MR1203910 (94d:53083)

[4] N. Ejiri, *Minimal immersions of Riemannian products into real space forms*, Tokyo J. Math. vol. 2, no. 1, 1979. MR541897 (80h:53059)

[5] D. Ferus, *Produkt–Zerlegung von Immersionen mit paralleler zweiter Fundamentalform*, Math. Ann. **211** (1974), 1-5. MR0367876 (51:4118)

[6] D. Ferus, *Immersions with parallel second fundamental form*, Math. Z. **140** (1974), 87-92. MR0370437 (51:6664)

[7] D. Ferus, *Symmetric Submanifolds of Euclidean Space*, Math. Ann. **247** (1980), 81-93. MR565140 (81i:53040)

[8] Ü. Lumiste, *Small-dimensional irreducible submanifolds with parallel third fundamental form* (Russian, summary in English), Tartu Ülik. Toim. Acta Comm. Univ. Tartuensis **734** (1986), 50-62. MR851351 (87k:53040)

[9] Ü. Lumiste, *Submanifolds with a Van der Waerden-Bortolotti plane connection and parallelism of the third fundamental form*, Izv. Vyssh. Uchebn. Zaved. Mat. **31** (1987), 18-27. MR891995 (88e:53033)

[10] Ü. Lumiste, *Normally flat submanifolds with parallel third fundamental form*, Proc. Estonian Acad. Sci. Phys. Math. **38**, no. 2 (1989), 129-138. MR1046139 (91b:53022)

[11] Ü. Lumiste, *Three dimensional submanifolds with parallel third fundamental form in Euclidean spaces*, Tartu Ülikooli Toimetised. Acta et Comm. Univ. Tartuensis **889** (1990), 45-56. MR1082922 (92c:53013)

[12] Ü. Lumiste, *Submanifolds with parallel fundamental form*, in F. Dillen and L. Verstraelen, eds., Handbook of Differential Geometry, Vol. I, Elsevier Science, Amsterdam, 2000, 779-864. MR1736858 (2000j:53071)

[13] Ü. Lumiste, *On submanifolds with parallel higher order fundamental form in Euclidean spaces*, in B. Wegner, D. Ferus, U. Pinkall, and U. Simon, eds., Global Differential Geometry and Global Analysis, Lecture Notes in Mathematics, Vol. 1481, Springer-Verlag, Berlin, New York, 1991, 126–137. MR1178526 (93i:53054)

[14] Ü. Lumiste, *Semiparallel Submanifolds in Space Forms*, Springer, 2008. MR2454627 (2010h:53080)

[15] V. A. Mirzojan, *Submanifolds with a higher order parallel fundamental form* (Russian), Akad. Nauk Armjan. SSR Dokl. **66** (1978), no. 2, 71–75. MR0487907 (58:7499)

[16] R. Molzan, *Extrinsische Produkte und symmetrische Untermannigfaltigkeiten in Standardräumen konstanter und konstanter holomorpher Krümmung*, Dissertation, Köln, 1983.

[17] J. D. Moore, *Isometric immersions of riemannian products*, J. Differential Geometry **5** (1971), 159-168. MR0307128 (46:6249)

[18] O. I. Mokhov, *Nonlocal Hamiltonian operators of hydrodynamic type with flat metrics, integrable hierarchies and the equations of associativity*, Funct. Anal. Appl. **40** (2006), no. 1, 1-23. MR2223246 (2007b:37147)

[19] S. Nölker, *Isometric immersions of warped products*, Diff. Geom. Appl. **6** (1996), 1-30. MR1384876 (97d:53064)

[20] K. Nomizu and H. Ozeki, *A theorem on curvature tensor fields*, Proc. Nat. Acad. Sci. **48** (1962), 206-207. MR0132507 (24:A2347)

[21] W. Strübing, *Symmetric submanifolds of Riemannian manifolds*, Math. Ann. **245** (1979), 37-44. MR552577 (81a:53027)

[22] M. Takeuchi, *Parallel submanifolds of space forms*, in Manifolds and Lie Groups, Papers in Honor of Y. Matsushima, Progr. Math. **14** Birkhäuser, Boston, MA, 1981, 429-447. MR642871 (83h:53078)

[23] S. Tanno, *Curvature tensors and covariant derivatives*. Ann. Mat. Pura Appl. (4) **96** (1972), 233-241. MR0326619 (48:4962)

Institut für Mathematik und Informatik, Ernst-Moritz-Arndt Universität Greifswald, Walther-Rathenau str. 47, 17487 Greifswald, Germany

E-mail address: ines.kath@uni-greifswald.de

Institut für Mathematik und Informatik, Ernst-Moritz-Arndt Universität Greifswald, Walther-Rathenau str. 47, 17487 Greifswald, Germany

E-mail address: nagyp@uni-greifswald.de