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# EQUATIONS AND SYZYGIES OF THE FIRST SECANT VARIETY TO A SMOOTH CURVE

PETER VERMEIRE

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Abstract. We show that if $C$ is a linearly normal smooth curve embedded by a line bundle of degree at least $2g+3+p$, then the secant variety to the curve satisfies $N_{3,p}$.

## 1. Introduction

We work throughout over an algebraically closed field $k$ of characteristic zero. If $X \subset \mathbb{P}^n$ is a smooth variety, then we let $\Sigma_i(X)$ (or just $\Sigma_i$ if the context is clear) denote the (complete) variety of $(i{+}1)$-secant $i$-planes. Though secant varieties are a classical subject, the majority of the work done involves determining the dimensions of secant varieties to well-known varieties. Perhaps the two most well-known results in this direction are the solution by Alexander and Hirschowitz (completed in [1]) of the Waring problem for homogeneous polynomials and the classification of the Severi varieties by Zak [16].

More recently there has been great interest, e.g. related to algebraic statistics and algebraic complexity, in determining the equations defining secant varieties (e.g. [2], [4], [5], [11], [14], [19], [21]). In this work, we use the detailed geometric information concerning secant varieties developed by Bertram [3], Thaddeus [22], and the author [23] to study not just the equations defining secant varieties, but the syzygies among those equations as well.

It was conjectured in [8] and it was shown in [18] that if $C$ is a smooth curve embedded by a line bundle of degree at least $4g+2k+3$, then $\Sigma_k$ is set theoretically defined by the $(k{+}2)\times(k{+}2)$ minors of a matrix of linear forms. This was extended in [12], where the degree bound was improved to $4g+2k+2$ and it was shown that the secant varieties are *scheme* theoretically cut out by the minors. It was further shown in [24] that if $X \subset \mathbb{P}^n$ satisfies condition $N_2$, then $\Sigma_1(v_d(X))$ is set theoretically defined by cubics for $d \geq 2$.

In [25] it was shown that if $C$ is a smooth curve embedded by a line bundle of degree at least $2g+3$, then $\mathcal{I}_{\Sigma_1}$ is 5-regular, and under the same hypothesis it was shown in [20] that $\Sigma_1$ is arithmetically Cohen-Macaulay. Together with the analogous well-known facts for the curve $C$ itself ([10], [13], [17]), this led to the following conjecture, extending the one found in [24].

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**Conjecture 1.1** ([20]). *Suppose that $C \subset \mathbb{P}^n$ is a smooth linearly normal curve of degree $d \geq 2g + 2k + 1 + p$, where $p, k \geq 0$. Then:*

1. *$\Sigma_k$ is ACM and $\mathcal{I}_{\Sigma_k}$ has regularity $2k+3$ unless $g = 0$, in which case the regularity is $k+2$.*
2. *$\beta_{n-2k-1,n+1}(\Sigma_k) = \binom{g+k}{k+1}$.*
3. *$\Sigma_k$ satisfies $N_{k+2,p}$.* □

*Remark* 1.2. Recall [7] that a variety $Z \subset \mathbb{P}^n$ satisfies $\mathrm{N}_{r,p}$ if the ideal of $Z$ is generated in degree $r$ and the syzygies among the equations are linear for $p-1$ steps. Note that the better-known condition $N_p$ [13] implies $N_{2,p}$.

By the work of Green and Lazarsfeld [13], [15], the conjecture holds for $k = 0$. Further, by [9] and by [27] it holds for $g \leq 1$, and by [20] parts (1) and (2) hold for $k = 1$. In this work, we show that part (3) holds for $k = 1$ (Theorem 3.5). Some analogous results for higher dimensional varieties can be found in [26].

Our approach combines the geometric knowledge of secant varieties mentioned above with the well-known Koszul approach of Green and Lazarsfeld. To fix notation, if $L$ is a vector bundle on a smooth curve $C$, then we let $\mathcal{E}_L = d_*(L \boxtimes \mathcal{O})$, where $d : C \times C \to S^2C$ is the natural double cover, and if $\mathcal{F}$ is a globally generated coherent sheaf on a variety $X$, then we have the coherent sheaf $M_{\mathcal{F}}$ defined via the exact sequence $0 \to M_{\mathcal{F}} \to \Gamma(X, \mathcal{F}) \to \mathcal{F} \to 0$. As we will be interested only in the first secant variety for the remainder of the paper, we write $\Sigma$ for $\Sigma_1$.

## 2. Preliminaries

Our starting point is the familiar:

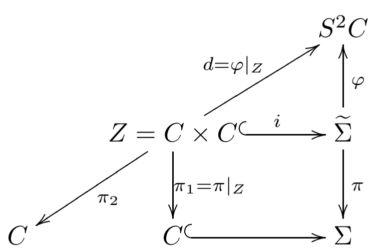
**Proposition 2.1.** *Let $C \subset \mathbb{P}^n$ be a smooth curve embedded by a line bundle $L$ of degree at least $2g+3$. Then $\Sigma$ satisfies $\mathrm{N}_{3,p}$ if and only if $H^1(\Sigma, \wedge^a M_L(b)) = 0$, $2 \leq a \leq p+1$, $b \geq 2$.*

*Proof.* Because $L$ also induces an embedding $\Sigma \subset \mathbb{P}^n$, we abuse notation and denote the associated vector bundle on $\Sigma$ by $M_L$. Letting $F = \bigoplus \Gamma(\Sigma, \mathcal{O}_\Sigma(n))$ and applying [6, 5.8] to $\mathcal{O}_\Sigma$ give the exact sequence:

$$0 \to \mathrm{Tor}_{a-1}(F,k)_{a+b} \to H^1(\Sigma, \bigwedge^a M_L(b)) \to H^1(\Sigma, \bigwedge^a \Gamma(\mathcal{O}(1)) \otimes \mathcal{O}_\Sigma(b)).$$

As $\Sigma$ is ACM [20], the term on the right vanishes. □

*Notation and Terminology* 2.2. Under the hypothesis that $\deg(L) \geq 2g+3$, the reader should keep in mind the following morphisms [23]:




where

- $\pi$ is the blow up of $\Sigma$ along $C$.
- $i$ is the inclusion of the exceptional divisor of the blow-up.

- $d$ is the double cover; $\pi_i$ are the projections.
- $\varphi$ is the morphism induced by the linear system $|2H-E|$ which gives $\widetilde{\Sigma}$ the structure of a $\mathbb{P}^1$-bundle over $S^2C$; note in particular that $\widetilde{\Sigma}$ is smooth.

We make frequent use of the rank 2 vector bundle $\mathcal{E}_L = \varphi_*\mathcal{O}(H) = d_*\,(L \boxtimes \mathcal{O})$ and the fact [25, Proposition 9] that $R^i\pi_*\mathcal{O}_{\widetilde{\Sigma}} = H^i(C, \mathcal{O}_C) \otimes \mathcal{O}_C$ for $i \geq 1$.

**Proposition 2.3.** *If $C$ is a smooth curve embedded by a line bundle $L$ with* $\deg(L) \geq 2g+3$*, then $\Sigma$ satisfies $\mathrm{N}_{3,p}$ if and only if*

$$H^1(\widetilde{\Sigma}, \pi^* \bigwedge^a M_L(b)) \to H^0(\Sigma, \bigwedge^a M_L(b) \otimes R^1\pi_*\mathcal{O}_{\widetilde{\Sigma}})$$

*is injective for* $2 \leq a \leq p+1$, $b \geq 2$.

*Proof.* This follows immediately from the 5-term sequence associated to the Leray-Serre spectral sequence,

$$0 \to H^1(\Sigma, \bigwedge^a M_L(b)) \to H^1(\widetilde{\Sigma}, \pi^* \bigwedge^a M_L(b)) \to H^0(\Sigma, \bigwedge^a M_L(b) \otimes R^1\pi_*\mathcal{O}_{\widetilde{\Sigma}}),$$

and Proposition 2.1. □

We will need a cohomological result:

**Lemma 2.4.** *Let $C \subset \mathbb{P}^n$ be a smooth curve embedded by a line bundle $L$ with* $\deg(L) \geq 2g+3$*. Then $H^i(\widetilde{\Sigma}, \mathcal{O}_{\widetilde{\Sigma}}(bH-E)) = 0$ for $i, b \geq 1$.*

*Proof.* Because $C$ is projectively normal we have $H^i(\widetilde{\mathbb{P}}^n, \mathcal{O}_{\widetilde{\mathbb{P}}^n}(bH-E)) = 0$ for $i, b \geq 1$. Thus $H^i(\widetilde{\Sigma}, \mathcal{O}_{\widetilde{\Sigma}}(bH-E)) = H^{i+1}(\widetilde{\mathbb{P}}^n, \mathcal{O}_{\widetilde{\mathbb{P}}^n}(bH-E) \otimes \mathcal{I}_{\widetilde{\Sigma}})$, but by [20, 2.4(6)], we know that $H^{i+1}(\widetilde{\mathbb{P}}^n, \mathcal{O}_{\widetilde{\mathbb{P}}^n}(bH-E) \otimes \mathcal{I}_{\widetilde{\Sigma}}) = H^{i+1}(\mathbb{P}^n, \mathcal{I}_\Sigma(b)) = 0$ for $i \geq 0$, $b \in \mathbb{Z}$. □

## 3. Main result

We first reinterpret the injection in Proposition 2.3 as a cohomological vanishing statement on $\widetilde{\Sigma}$ (Proposition 3.1), then on $S^2C$ (Corollary 3.3), and finally on $C \times C$ (Theorem 3.5).

**Proposition 3.1.** *Let $C \subset \mathbb{P}^n$ be a smooth curve satisfying $N_p$ embedded by a line bundle $L$ with* $\deg(L) \geq 2g+3$*. Then $\Sigma$ satisfies $\mathrm{N}_{3,p}$ if*

$$H^i(\widetilde{\Sigma}, \pi^* \bigwedge^{a-1+i} M_L \otimes \mathcal{O}(2H-E)) = 0$$

*for* $2 \leq a \leq p+1$, $i \geq 1$.

*Proof.* We use Proposition 2.3. Consider the sequence on $\widetilde{\Sigma}$,

$$0 \to \pi^* \bigwedge^a M_L(bH-E) \to \pi^* \bigwedge^a M_L(bH) \to \pi^* \bigwedge^a M_L(bH) \otimes \mathcal{O}_Z \to 0.$$

We know that

$$\begin{aligned} H^1(Z, \pi^* \bigwedge^a M_L(bH) \otimes \mathcal{O}_Z) &= H^1\left(Z, \left(\bigwedge^a M_L \otimes L^b\right) \boxtimes \mathcal{O}_C\right) \\ &= H^1(C, \mathcal{O}_C) \otimes H^0(C, \bigwedge^a M_L \otimes L^b) \\ &= H^0(\Sigma, \bigwedge^a M_L(b) \otimes R^1\pi_*\mathcal{O}_{\widetilde{\Sigma}}). \end{aligned}$$

The first equality follows as the restriction of $\pi^* \bigwedge^a M_L(bH)$ to $Z$ is $\bigwedge^a M_L(bH) \boxtimes \mathcal{O}_C$. For the second we use the Künneth formula together with the fact that $h^1(C, \bigwedge^a M_L \otimes L^b) = 0$ as $C$ satisfies $N_p$ [13]. The third is the last part of subsection 2.2.

Thus

$$h^1(\Sigma, \bigwedge^a M_L(b)) = \operatorname{Rank}\Bigg( H^1(\widetilde{\Sigma}, \pi^* \bigwedge^a M_L(bH - E)) \\ \to H^1(\widetilde{\Sigma}, \pi^* \bigwedge^a M_L(bH)) \Bigg),$$

and so by Proposition 2.3 it is enough to show that $H^1(\widetilde{\Sigma}, \pi^* \bigwedge^a M_L \otimes \mathcal{O}(bH - E)) = 0$ for $2 \leq a \leq p+1$, $b \geq 2$.

From the sequence

$$0 \to \pi^* \bigwedge^{a+1} M_L \otimes \mathcal{O}(bH - E) \to \bigwedge^{a+1} \Gamma \otimes \mathcal{O}(bH - E) \\ \to \pi^* \bigwedge^a M_L \otimes \mathcal{O}((b+1)H - E) \to 0$$

and the fact (Lemma 2.4) that $H^i(\widetilde{\Sigma}, \mathcal{O}(bH - E)) = 0$, we see that

$$H^1(\widetilde{\Sigma}, \pi^* \bigwedge^a M_L \otimes \mathcal{O}(bH - E)) = H^{b-2}(\widetilde{\Sigma}, \pi^* \bigwedge^{a+b-2} M_L \otimes \mathcal{O}(2H - E))$$

for $b \geq 2$. □

**Lemma 3.2.** *Let $C \subset \mathbb{P}^n$ be a smooth curve embedded by a line bundle $L$ with $\deg(L) \geq 2g+3$ and consider the morphism $\varphi : \widetilde{\Sigma} \to S^2C \subset \mathbb{P}^s$ induced by the linear system $|2H - E|$. Then $\varphi_* \bigwedge^a M_L = \bigwedge^a M_{\mathcal{E}_\mathcal{L}}$, and hence*

$$H^i(\widetilde{\Sigma}, \pi^* \bigwedge^a M_L \otimes \mathcal{O}(2H - E)) = H^i(S^2C, \bigwedge^a M_{\mathcal{E}_L} \otimes \mathcal{O}_{S^2C}(1)).$$

*Proof.* Consider the diagram on $\widetilde{\Sigma}$:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & & K & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \varphi^* M_{\mathcal{E}_L} & \longrightarrow & \Gamma(S^2C, \mathcal{E}_L) \otimes \mathcal{O}_{\widetilde{\Sigma}} & \longrightarrow & \varphi^* \mathcal{E}_L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi^* M_L & \longrightarrow & \Gamma(C, L) \otimes \mathcal{O}_{\widetilde{\Sigma}} & \longrightarrow & \pi^* L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & K & & 0 & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

The vertical map in the middle is surjective, as we have $\Gamma(S^2C, \mathcal{E}_L) = \Gamma(\widetilde{\Sigma}, \mathcal{O}(H)) = \Gamma(C \times C, L \boxtimes \mathcal{O}) = \Gamma(C, L)$. Therefore, surjectivity of the lower right horizontal map and commutativity of the diagram show that the right-hand vertical map is surjective.

Note that $R^i\varphi_*\varphi^*\mathcal{E}_L = \mathcal{E}_L \otimes R^i\varphi_*\mathcal{O}_{\widetilde{\Sigma}}$ by the projection formula and that the higher direct image sheaves $R^i\varphi_*\mathcal{O}_{\widetilde{\Sigma}}$ vanish as $\widetilde{\Sigma}$ is a $\mathbb{P}^1$-bundle over $S^2C$. For the higher direct images, we have $R^i\varphi_*\pi^*L = 0$ as the restriction of $L$ to a fiber of $\varphi$ is $\mathcal{O}(1)$, and hence the cohomology along the fibers vanishes. From the rightmost column, we see that $R^i\varphi_*K = 0$. From the leftmost column, we have the sequence

$$0 \to \varphi^* \bigwedge^{a} M_{\mathcal{E}_L} \to \pi^* \bigwedge^{a} M_L \to \varphi^* \bigwedge^{a-1} M_{\mathcal{E}_L} \otimes K \to 0,$$

but as $R^i\varphi_*\left(K \otimes \varphi^* \bigwedge^{a-1} M_{\mathcal{E}_L}\right) = R^i\varphi_*K \otimes \bigwedge^{a-1} M_{\mathcal{E}_L} = 0$, we have $\varphi_* \bigwedge^a M_L = \bigwedge^a M_{\mathcal{E}_L}$. □

Combining Proposition 3.1 with Lemma 3.2 yields:

**Corollary 3.3.** *Let $C \subset \mathbb{P}^n$ be a smooth curve satisfying $N_p$ embedded by a line bundle $L$ with $\deg(L) \geq 2g+3$. Then $\Sigma$ satisfies $\mathrm{N}_{3,p}$ if*

$$H^i(S^2C, \bigwedge^{a-1+i} M_{\mathcal{E}_L} \otimes \mathcal{O}(1)) = 0$$

*for $2 \leq a \leq p+1$, $i \geq 1$.* □

We need a technical lemma, completely analogous to [15, 1.4.1].

**Lemma 3.4.** *Let $X \subset \mathbb{P}^n$ be a smooth curve embedded by a non-special line bundle $L$ satisfying $N_{2,2}$, let $x_1, \cdots, x_{n-2}$ be a general collection of distinct points, and let $D = x_1 + \cdots + x_{n-2}$. Then there is an exact sequence of vector bundles on $X \times X$,*

$$0 \to L^{-1}(D) \boxtimes L^{-1}(D)(\Delta) \to d^*M_{\mathcal{E}_L} \to \bigoplus_i (\mathcal{O}(-x_i) \boxtimes \mathcal{O}(-x_i)) \to 0.$$

*Proof.* Choose a general point $x_1 \in X$ and consider the following diagram on $X \times X$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & d^*M_{\mathcal{E}_{L(-x_1)}} & \longrightarrow & M_{L(-x_1)} \boxtimes \mathcal{O} & \longrightarrow & (\mathcal{O} \boxtimes L(-x_1))(-\Delta) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & d^*M_{\mathcal{E}_L} & \longrightarrow & M_L \boxtimes \mathcal{O} & \longrightarrow & \mathcal{O} \boxtimes L(-\Delta) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(-x_1) \boxtimes \mathcal{O}(-x_1) & \longrightarrow & \mathcal{O}(-x_1) \boxtimes \mathcal{O} & \longrightarrow & \mathcal{O} \boxtimes (L \otimes \mathcal{O}_{x_1})(-\Delta) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the center column comes from [15, 1.4.1]. Following just as in that proof, we obtain

$$0 \to d^* M_{\mathcal{E}_{L(-D)}} \to d^* M_{\mathcal{E}_L} \to \bigoplus_i (\mathcal{O}(-x_i) \boxtimes \mathcal{O}(-x_i)) \to 0$$

from the left column. Note however that $d^* M_{\mathcal{E}_{L(-D)}}$ is a line bundle, and one checks that $d^* M_{\mathcal{E}_{L(-D)}} = \bigwedge^2 \mathcal{E}^*_{L(-D)} = L^{-1}(D) \boxtimes L^{-1}(D)(\Delta)$. □

**Theorem 3.5.** *Let $C \subset \mathbb{P}^n$ be a smooth curve embedded by a line bundle $L$ with $\deg(L) \geq 2g + p + 3$, $p \geq 0$. Then $\Sigma$ satisfies $\mathrm{N}_{3,p}$.*

*Proof.* First note [13] that such a curve satisfies $N_{p+2}$. We verify the condition in Corollary 3.3. Pulling the sequence on $S^2C$,

$$0 \to M_{\mathcal{E}_L} \to \Gamma(S^2C, \mathcal{E}_L) \to \mathcal{E}_L \to 0,$$

back to $Z = C \times C$ yields the diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & & K & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & d^* M_{\mathcal{E}_L} & \longrightarrow & d^*\Gamma(S^2C, \mathcal{E}_L) & \longrightarrow & d^*\mathcal{E}_L & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_{L \boxtimes \mathcal{O}} & \longrightarrow & \Gamma(C, L) & \longrightarrow & L \boxtimes \mathcal{O} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & K & & 0 & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

As in [20] we have $d_*\mathcal{O}_Z = \mathcal{O}_{S^2C} \oplus M$ where $d^*M = \mathcal{O}(-\Delta)$, $d_*K = \mathcal{E}_L \otimes M$, and $K = d^* \bigwedge^2 \mathcal{E}_L \otimes (L^* \boxtimes \mathcal{O}) = \mathcal{O} \boxtimes L(-\Delta)$. From the left vertical sequence we have

$$0 \to \bigwedge^a d^* M_{\mathcal{E}_L} \to \bigwedge^a M_{L\boxtimes\mathcal{O}} \to \bigwedge^{a-1} d^* M_{\mathcal{E}_L} \otimes K \to 0,$$

and pushing down to $S^2C$ yields

$$0 \to \bigwedge^a M_{\mathcal{E}_L} \oplus \left(\bigwedge^a M_{\mathcal{E}_L} \otimes M\right) \to d_* \bigwedge^a M_{L\boxtimes\mathcal{O}}$$
$$\to \bigwedge^{a-1} M_{\mathcal{E}_L} \otimes \mathcal{E}_L \otimes M \to 0.$$

Twisting this sequence by $\mathcal{O}_{S^2C}(1) \otimes M^*$ gives

$$0 \to \bigwedge^a M_{\mathcal{E}_L}(1) \otimes M^* \oplus \bigwedge^a M_{\mathcal{E}_L}(1)\mathcal{O}_{S^2C}(1) \otimes M^* \otimes d_* \bigwedge^a M_{L\boxtimes\mathcal{O}}$$
$$\to \bigwedge^{a-1} M_{\mathcal{E}_L}(1) \otimes \mathcal{E}_L \to 0.$$

Since $d^*\mathcal{O}_{\mathrm{Hilb}^2X}(1) \otimes M^* = L \boxtimes L \otimes \mathcal{O}(-\Delta)$, it suffices to show that

$$H^i(Z, \bigwedge^{a-1+i} d^* M_{\mathcal{E}_L} \otimes L \boxtimes L \otimes \mathcal{O}(-\Delta)) = 0$$

for $2 \leq a \leq p+1$, $i = 1, 2$.

Now, by Lemma 3.4 we have exact sequences

$$\begin{aligned} 0 \to \bigwedge^{r-1} Q \otimes \mathcal{O}(D) \boxtimes \mathcal{O}(D) &\to \bigwedge^{r} d^* M_{\mathcal{E}_L} \otimes L \boxtimes L \otimes \mathcal{O}(-\Delta) \\ &\to \bigwedge^{r} Q \otimes L \boxtimes L \otimes \mathcal{O}(-\Delta) \to 0, \end{aligned}$$

where $Q = \bigoplus_i (\mathcal{O}(-x_i) \boxtimes \mathcal{O}(-x_i))$.

On the right, we have a direct sum of vector bundles of the form $F \boxtimes F(-\Delta)$, where $F$ is a line bundle of degree $\deg(L) - r$. Thus $H^1$ and $H^2$ of the right side will vanish when $\deg(L) - r \geq 2g + 1$.

On the left, we have a direct sum of vector bundles of the form $F \boxtimes F$, where $F$ is a line bundle of degree $n - 2 - (r-1) = \deg(L) - g - r - 1$. Because $x_1, \cdots, x_{n-2}$ are general, $H^1$ and $H^2$ of the left side will vanish when $\deg(L) - g - r - 1 \geq g$. Combining these, we see that $H^i(Z, \bigwedge^{a-1+i} d^* M_{\mathcal{E}_L} \otimes L \boxtimes L \otimes \mathcal{O}(-E_\Delta)) = 0$ for $2 \leq a \leq p+1$, $i = 1, 2$, as long as $\deg(L) \geq 2g + p + 3$. $\square$

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Department of Mathematics, 214 Pearce Hall, Central Michigan University, Mount Pleasant, Michigan 48859

*E-mail address*: p.vermeire@cmich.edu