COEFFECTIVE COHOMOLOGY
OF SYMPLECTIC ASPHERICAL MANIFOLDS

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(Communicated by Lei Ni)

Abstract. We prove a generalization of the theorem which has been proved by Fernandez, Ibanez, and de Leon. By this result, we give examples of non-Kähler manifolds which satisfy the property of compact Kähler manifolds concerning the coeffective cohomology.

1. Introduction

Let \((M, \omega)\) be a compact \(2n\)-dimensional symplectic manifold. Denote by \(A^*(M)\) the de Rham complex of \(M\). We call a differential form \(\alpha \in A^*(M)\) coeffective if \(\omega \wedge \alpha = 0\), and denote the sub-DGA by

\[ A^*_{coE}(M) = \{ \alpha \in A^*(M) | \omega \wedge \alpha = 0 \}. \]

We call the cohomology \(H^*(A^*_{coE}(M))\) the coeffective cohomology of \(M\). We also denote

\[ \tilde{H}^*(A^*(M)) = \{ [\alpha] \in H^*(A^*(M)) | [\omega] \wedge [\alpha] = 0 \}. \]

Theorem 1.1 (\cite{4}). Let \((M, \omega)\) be a compact Kähler manifold. For \(p \geq n + 1\), we have an isomorphism

\[ H^p(A^*_{coE}(M)) \cong \tilde{H}^p(A^*(M)). \]

However for general symplectic manifolds, the isomorphism \(H^p(A^*_{coE}(M)) \cong \tilde{H}^p(A^*(M))\) does not hold. In fact, counterexamples are given in \cite{5}. So far we have hardly found examples of non-Kähler manifolds such that isomorphisms \(H^p(A^*_{coE}(M)) \cong \tilde{H}^p(A^*(M))\) hold. The purpose of this paper is to compute the coeffective cohomology of some class of symplectic manifolds by use of a finite-dimensional cochain complex and to give non-Kähler examples such that the isomorphisms \(H^p(A^*_{coE}(M)) \cong \tilde{H}^p(A^*(M))\) hold.

2. Preliminary: Coeffective cohomology of subcomplex

Let \((M, \omega)\) be a compact \(2n\)-dimensional symplectic manifold.

Proposition 2.1 (\cite{5}, \cite{11}). Then the map \(\omega \wedge : A^p(M) \to A^{p+2}(M)\) is injective for \(p \leq n - 1\) and surjective for \(p \geq n - 1\).
By this proposition we have $H^p(A^*_{coE}(M)) = \{0\}$ for $p \leq n - 1$, and so it is sufficient to consider $H^p(A^*_{coE}(M))$ for $p \geq n$. Since $\omega$ is closed, we have the short exact sequence of cochain complexes

$$0 \rightarrow A^*_{coE}(M) \rightarrow A^*(M) \overset{\omega^*}{\rightarrow} \omega \wedge A^*(M) \rightarrow 0,$$

where we consider $\omega \wedge A^*(M)$, the cochain complex which is graded as $(\omega \wedge A^*)^p = \omega \wedge A^p(M)$. By this sequence we have the long exact sequence of cohomology

$$\cdots \rightarrow H^{p-1}(A^*(M)) \overset{(\omega^\wedge)^*}{\rightarrow} H^p(\omega \wedge A^*(M)) \rightarrow H^p(A^*_{coE}(M)) \rightarrow H^p(A^*(M)) \overset{(\omega^\wedge)^*}{\rightarrow} \cdots .$$

By Proposition 2.1 we have $\omega \wedge A^{p-1}(M) = A^{p+1}(M)$ for $p \geq n$ and so the exact sequence is given by

$$\cdots \rightarrow H^{p-1}(A^*(M)) \overset{(\omega^\wedge)^*}{\rightarrow} H^p(A^*(M)) \rightarrow H^p(A^*_{coE}(M)) \rightarrow H^p(A^*(M)) \overset{(\omega^\wedge)^*}{\rightarrow} \cdots .$$

**Proposition 2.2.** Let $A^* \subset A^*(M)$ be a subcomplex such that the inclusion $\Phi : A^* \rightarrow A^*(M)$ induces a cohomology isomorphism. Assume $\omega \in A^*$ and the map $\omega^\wedge : A^p \rightarrow A^{p+2}$ is surjective for $p \geq n - 1$. Denote $A^*_{coE} = \ker((\omega^\wedge)|_{A^*})$. Then the inclusion $\Phi : A^*_{coE} \rightarrow A^*_{coE}(M)$ induces an isomorphism

$$H^p(A^*_{coE}) \cong H^p(A^*_{coE}(M))$$

for $p \geq n$.

**Proof.** As above, we have the exact sequence of cochain complexes

$$0 \rightarrow A^*_{coE} \rightarrow A^* \overset{\omega^\wedge}{\rightarrow} \omega \wedge A^* \rightarrow 0.$$

By the assumption, for $p \geq n$ we have the long exact sequence of cohomology

$$\cdots \rightarrow H^{p-1}(A^*) \overset{(\omega^\wedge)^*}{\rightarrow} H^p(A^*) \rightarrow H^p(A^*_{coE}) \rightarrow H^p(A^*) \overset{(\omega^\wedge)^*}{\rightarrow} \cdots .$$

By the inclusion $\Phi : (\wedge_{coE} A^*)^T \rightarrow A^*_{coE}(M)$, we have the commutative diagram

$$\begin{array}{cccccc}
H^{p-1}(A^*(M)) & \overset{(\omega^\wedge)^*}{\rightarrow} & H^p(A^*(M)) & \overset{\Phi^*}{\rightarrow} & H^p(A^*_{coE}(M)) & \overset{(\omega^\wedge)^*}{\rightarrow} & H^p(A^*(M)) \\
\uparrow{\Phi^*} & & \uparrow{\Phi^*} & & \uparrow{\Phi^*} & & \uparrow{\Phi^*} \\
H^{p-1}(A^*) & \overset{(\omega^\wedge)^*}{\rightarrow} & H^p(A^*) & \overset{\Phi^*}{\rightarrow} & H^p(A^*_{coE}) & \overset{(\omega^\wedge)^*}{\rightarrow} & H^p(A^*) \\
\end{array}$$

By the assumption, $\Phi^* : H^*(A^*) \rightarrow H^*(A^*(M))$ is an isomorphism, and so by this diagram $\Phi^* : H^p(A^*_{coE}) \rightarrow H^p(A^*_{coE}(M))$ is an isomorphism. \qed
3. Background: Fernandez-Ibanez-de Leon’s Theorem

Let $G$ be a simply connected Lie group with a lattice $\Gamma$ (i.e. a cocompact discrete subgroup of $G$). We call $G/\Gamma$ a nilmanifold (resp. solvmanifold) if $G$ is nilpotent (resp. solvable). Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\bigwedge \mathfrak{g}^*$ be the cochain complex of $\mathfrak{g}$ with the differential which is induced by the dual of the Lie bracket. As we regard $\bigwedge \mathfrak{g}^*$ as the left-invariant forms on $G/\Gamma$, we consider the inclusion $\bigwedge \mathfrak{g}^* \subset \mathcal{A}^*(G/\Gamma)$. Let $\omega \in \bigwedge^2 \mathfrak{g}^*$ be a left-invariant symplectic form. Then the map $\omega \wedge : \bigwedge^n \mathfrak{g}^* \rightarrow \bigwedge^{n+2} \mathfrak{g}^*$ is surjective for $p \geq n - 1$ (see [3]). In [14] Nomizu showed that if $G$ is nilpotent, then the inclusion $\bigwedge \mathfrak{g}^* \subset \mathcal{A}^*(G/\Gamma)$ induces an isomorphism of cohomology. Hence by Proposition 2.2 we have the following theorem, which was noted in [5] and [6].

**Theorem 3.1.** Let $G$ be a simply connected nilpotent Lie group with a lattice $\Gamma$ and a left-invariant symplectic form $\omega$. Then the inclusion $\bigwedge \mathfrak{g}^* \subset \mathcal{A}^*(G/\Gamma)$ induces an isomorphism

$$H^p(\bigwedge^*_{coE} \mathfrak{g}^*) \cong H^p(\mathcal{A}^*(G/\Gamma))$$

for $p \geq n$, where $\bigwedge^*_{coE} \mathfrak{g}^* = \{ \alpha \in \bigwedge \mathfrak{g}^* | \omega \wedge \alpha = 0 \}$.

In [8] Hattori showed that the isomorphism $H^*(\bigwedge \mathfrak{g}^*) \cong H^*(\mathcal{A}^*(G/\Gamma))$ also holds if $G$ is completely solvable (i.e. $G$ is solvable and for any $g \in G$ all the eigenvalues of the adjoint operator $\text{Ad}_g$ are real). Thus we can extend this theorem for completely solvmanifolds. However for a general solvmanifold $G/\Gamma$, the isomorphism $H^*(\bigwedge^*_{coE} \mathfrak{g}^*) \cong H^*(\mathcal{A}^*(G/\Gamma))$ does not hold and we can’t compute the coeffective cohomology by using $\bigwedge \mathfrak{g}^*$.

In [2] Baues constructed compact aspherical manifolds $M_\Gamma$ such that the class of these aspherical manifolds contains the class of solvmanifolds and showed that the de Rham cohomology of these aspherical manifolds can be computed by certain finite-dimensional cochain complexes. In the next section, by using Baues’s results, we will show a generalization of Theorem 3.1.

4. Main Results: Coeffective Cohomology of Aspherical Manifolds with Torsion-free Virtually Polycyclic Fundamental Groups

4.1. Notation and Conventions. Let $k$ be a subfield of $\mathbb{C}$. A group $G$ is called a $k$-algebraic group if $G$ is a Zariski-closed subgroup of $GL_n(\mathbb{C})$ which is defined by polynomials with coefficients in $k$. Let $G(k)$ denote the set of $k$-points of $G$ and $U(G)$ the maximal Zariski-closed unipotent normal $k$-subgroup of $G$ called the unipotent radical of $G$. If $G$ consists of semi-simple elements, we call $G$ a d-group. Let $U_n(k)$ denote the $n \times n$ $k$-valued upper triangular unipotent matrix group.

4.2. Baues’s results. A group $\Gamma$ is called polycyclic if it admits a sequence

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_k = \{e\}$$

of subgroups such that each $\Gamma_i$ is normal in $\Gamma_{i-1}$ and $\Gamma_{i-1}/\Gamma_i$ is cyclic. We denote $\text{rank} \Gamma = \sum_{i=1}^k \text{rank} \Gamma_{i-1}/\Gamma_i$. We define an infra-solvmanifold as a manifold of the form $G/\Delta$, where $G$ is a simply connected solvable Lie group and $\Delta$ is a torsion-free subgroup of $\text{Aut}(G) \ltimes G$ such that for the projection $p : \text{Aut}(G) \ltimes G \rightarrow \text{Aut}(G)$ $p(\Delta)$ is contained in a compact subgroup of $\text{Aut}(G)$. By a result of Mostow in [12], the fundamental group of an infra-solvmanifold is virtually polycyclic (i.e. it contains...
a finite index polycyclic subgroup). In particular, a lattice $\Gamma$ of a simply connected solvable Lie group $G$ is a polycyclic group with rank $\Gamma = \dim G$ (see [15]).

Let $k$ be a subfield of $\mathbb{C}$. Let $\Gamma$ be a torsion-free virtually polycyclic group. For a finite index polycyclic subgroup $\Delta \subset \Gamma$, we denote rank $\Gamma = \text{rank} \Delta$.

**Definition 4.1.** We call a $k$-algebraic group $H_\Gamma$ a $k$-algebraic hull of $\Gamma$ if there exists an injective group homomorphism $\psi : \Gamma \to H_\Gamma(k)$ and $H_\Gamma$ satisfies the following conditions:

1. $\psi(\Gamma)$ is Zariski-dense in $H_\Gamma$.
2. $Z_{H_\Gamma}(U(H_\Gamma)) \subset U(H_\Gamma)$, where $Z_{H_\Gamma}(U(H_\Gamma))$ is the centralizer of $U(H_\Gamma)$.
3. $\dim U(H_\Gamma) = \text{rank} \Gamma$.

**Theorem 4.2 ([2] Theorem A.1).** There exists a $k$-algebraic hull of $\Gamma$, and a $k$-algebraic hull of $\Gamma$ is unique up to a $k$-algebraic group isomorphism.

Let $\Gamma$ be a torsion-free virtually polycyclic group and $H_\Gamma$ the $\mathbb{Q}$-algebraic hull of $\Gamma$. Denote $H_\Gamma = H_\Gamma(\mathbb{R})$. Let $U_\Gamma$ be the unipotent radical of $H_\Gamma$ and $T$ a maximal $\mathbb{R}$-subgroup. Then $H_\Gamma$ decomposes as a semi-direct product $H_\Gamma = T \ltimes U_\Gamma$; see [2] Proposition 2.1. Let $u$ be the Lie algebra of $U_\Gamma$. Since the exponential map $\exp : u \to U_\Gamma$ is a diffeomorphism, $U_\Gamma$ is diffeomorphic to $\mathbb{R}^n$ such that $n = \text{rank} \Gamma$. For the semi-direct product $H_\Gamma = T \ltimes U_\Gamma$, we denote $\phi : T \to \text{Aut}(U_\Gamma)$, the action of $T$ on $U_\Gamma$. Then we have the homomorphism $\alpha : H_\Gamma \to \text{Aut}(U_\Gamma) \ltimes U_\Gamma$ such that $\alpha(t,u) = (\phi(t),u)$ for $(t,u) \in T \ltimes U_\Gamma$. By the property (2) in Definition 4.1 $\phi$ is injective and hence $\alpha$ is injective.

In [2] Baues constructed a compact aspherical manifold $M_\Gamma = \alpha(\Gamma) \backslash U_\Gamma$ with $\pi_1(M_\Gamma) = \Gamma$. We call $M_\Gamma$ a standard $\Gamma$-manifold.

**Theorem 4.3 ([2] Theorems 1.2, 1.4).** A standard $\Gamma$-manifold is unique up to diffeomorphism. A compact infra-solvmanifold with the fundamental group $\Gamma$ is diffeomorphic to the standard $\Gamma$-manifold $M_\Gamma$. In particular, a solvmanifold $G/\Gamma$ is diffeomorphic to the standard $\Gamma$-manifold $M_\Gamma$.

Let $A^*(M_\Gamma)$ be the de Rham complex of $M_\Gamma$. Then $A^*(M_\Gamma)$ is the set of the $\Gamma$-invariant differential forms $A^*(U_\Gamma)^T$ on $U_\Gamma$. Let $(\bigwedge u^*)^T$ be the left-invariant forms on $U_\Gamma$ which are fixed by $T$. Since $\Gamma \subset H_\Gamma = U_\Gamma \cdot T$, we have the inclusion

$$(\bigwedge u^*)^T = A^*(U_\Gamma)^H \subset A^*(U_\Gamma)^\Gamma = A^*(M_\Gamma).$$

**Theorem 4.4 ([2] Theorem 1.8).** This inclusion induces a cohomology isomorphism.

4.3. **Main results.** Let $\omega \in (\bigwedge u^*)^T$ be a symplectic form. Denote

$$\bigwedge_{coE} u^* = \{ \alpha \in \bigwedge u^* | \omega \wedge \alpha = 0 \}$$

and

$$\tilde{H}^*((\bigwedge u^*)^T) = \{ [\alpha] \in H^*((\bigwedge u^*)^T) | [\omega] \wedge [\alpha] = 0 \}.$$ 

By Theorem 4.4 we have $\tilde{H}^*((\bigwedge u^*)^T) \cong \tilde{H}^*(A^*(M_\Gamma)).$

**Lemma 4.5.** For $p \geq n - 1$, the linear map $\omega \wedge : (\bigwedge^p u^*)^T \to (\bigwedge^{p+2} u^*)^T$ is surjective.
Proof. First we notice that the map \( \omega \wedge : \bigwedge^p u^* \to \bigwedge^{p+2} u^* \) is surjective (see [5 Lemma 2.1]). Since \( T \) is a d-group, for \( t \in T \) the \( t \)-action on \( \bigwedge u^* \) is diagonalizable (see [2]). Hence we have a decomposition

\[
\bigwedge u^* = A^p \oplus B^p
\]
such that \( A^p \) is the subspace of \( t \)-invariant elements and \( B^p \) is its complement. Since the \( t \)-action is diagonalizable, we have a basis \( \{x_1, \ldots, x_{2n}\} \) of \( u^* \otimes C \) such that the \( t \)-action is represented by a diagonal matrix. Then we have

\[
A^p \otimes C = \langle x_{i_1} \wedge \cdots \wedge x_{i_p} \mid 1 \leq i_1 < \cdots < i_p \leq 2n, t \cdot (x_{i_1} \wedge \cdots \wedge x_{i_p}) = x_{i_1} \wedge \cdots \wedge x_{i_p} \rangle
\]
and

\[
B^p \otimes C = \langle x_{i_1} \wedge \cdots \wedge x_{i_p} \mid 1 \leq i_1 < \cdots < i_p \leq 2n, t \cdot (x_{i_1} \wedge \cdots \wedge x_{i_p}) = \alpha_{i_1 \ldots i_p}(t)x_{i_1} \wedge \cdots \wedge x_{i_p}, \alpha_{i_1 \ldots i_p}(t) \neq 1 \rangle.
\]

By \( \omega \in (\bigwedge u^*)^T \), we have \( \omega = \sum a_{kl}x_k \wedge x_l \) such that if \( a_{kl} \neq 0 \), then \( x_k \wedge x_l \in A^p \otimes C \). Then for \( x_{i_1} \wedge \cdots \wedge x_{i_p} \in B^p \otimes C \) we have

\[
\omega \wedge x_{i_1} \wedge \cdots \wedge x_{i_p} = \sum a_{kl}x_k \wedge x_l \wedge x_{i_1} \wedge \cdots \wedge x_{i_p}.
\]
If \( a_{kl} \neq 0 \), we have

\[
t \cdot (x_k \wedge x_l \wedge x_{i_1} \wedge \cdots \wedge x_{i_p}) = \alpha_{i_1 \ldots i_p}(t)x_k \wedge x_l \wedge x_{i_1} \wedge \cdots \wedge x_{i_p}.
\]
Thus \( \omega \wedge x_{i_1} \wedge \cdots \wedge x_{i_p} \in B^{p+2} \otimes C \). By this we have \( (\omega \wedge B^p) \subset B^{p+2} \). Since \( T \) acts semi-simply on \( \bigwedge^p u^* \), we consider the decomposition

\[
\bigwedge u^* = (\bigwedge^p u^*)^T \oplus C^p
\]
such that \( C^p \) is a complement of \( (\bigwedge^p u^*)^T \) for \( T \)-action. By the above argument we have \( (\omega \wedge C^p) \subset C^{p+2} \). Clearly we have \( (\omega \wedge (\bigwedge^p u^*)^T) \subset (\bigwedge^{p+2} u^*)^T \). Since for \( p \geq n - 1 \) the map \( \omega \wedge : \bigwedge^p u^* \to \bigwedge^{p+2} u^* \) is surjective, we have

\[
\bigwedge^p u^* \wedge C^{p+2} = \omega \wedge \bigwedge^p u^* = (\omega \wedge (\bigwedge^p u^*)^T) \oplus (\omega \wedge C^p).
\]
Thus we have \( \omega \wedge (\bigwedge^p u^*)^T = (\bigwedge^{p+2} u^*)^T \). Hence the lemma follows.

By this lemma and Proposition 2.2 we have:

**Theorem 4.6.** Let \( \Gamma \) be a torsion-free virtually polycyclic group and \( M_\Gamma \) the standard \( \Gamma \)-manifold with a symplectic form \( \omega \) such that \( \omega \in (\bigwedge u^*)^T \). Then for \( p \geq n \), the inclusion \( \Phi : (\bigwedge_{coE} u^*)^T \to A^*_{coE}(M_\Gamma) \) induces an isomorphism \( \Phi^* : H^*((\bigwedge_{coE} u^*)^T) \cong H^*(A^*_{coE}(M_\Gamma)) \).

**Remark 1.** In [10], the author showed that if there exists \([\omega] \in H^2(M_\Gamma, \mathbb{R})\) such that \([\omega]^{1/2} \dim M_\Gamma \neq 0\), then an invariant form \( \omega \in (\bigwedge u^*)^T \) which represents the cohomology class \([\omega]\) is a symplectic form on \( M_\Gamma \). Hence if \( M_\Gamma \) is cohomologically symplectic (i.e. there exists \([\omega] \in H^2(M_\Gamma, \mathbb{R})\) such that \([\omega]^{1/2} \dim M_\Gamma \neq 0\), then \( M_\Gamma \) admits a symplectic form \( \omega \) such that \( \omega \in (\bigwedge u^*)^T \).

**Corollary 4.7.** Under the same assumption of Theorem 4.6, if \( U_\Gamma \) is abelian, then for \( p \geq n \) we have an isomorphism

\[
H^p(A^*_{coE}(M_\Gamma)) \cong \tilde{H}^p(A^*(M_\Gamma)).
\]
Proof. If $U_{\Gamma}$ is abelian, then the differential of $\bigwedge u^*$ is 0. Hence we have

$$H^*(A^*(M_\Gamma)) \cong H^*(\bigwedge u^*) = (\bigwedge u^*)^T$$

and

$$H^*(\bigwedge_{coE} u^*)^T = (\bigwedge_{coE} u^*)^T.$$  

This gives

$$\tilde{H}^*(A^*(M_\Gamma)) \cong \tilde{H}^*(\bigwedge u^*)^T = \{ \alpha \in (\bigwedge u^*)^T | \alpha \wedge \omega = 0 \}$$

$$= (\bigwedge_{coE} u^*)^T = H^*(\bigwedge_{coE} u^*)^T).$$

Hence by the above theorem the corollary follows. \qed

In [9] the author showed the following theorem.

**Theorem 4.8 ([9]).** Let $\Gamma$ be a torsion-free virtually polycyclic group. Then the following two conditions are equivalent:

1. $U_{\Gamma}$ is abelian.
2. $\Gamma$ is a finite extension group of a lattice of a Lie group $G = \mathbb{R}^n \rtimes_\phi \mathbb{R}^m$ such that the action $\phi : \mathbb{R}^n \to \text{Aut}(\mathbb{R}^m)$ is semi-simple.

Hence we have:

**Corollary 4.9.** Under the same assumptions of Theorem 4.6 if $\Gamma$ satisfies the condition (2) in Theorem 4.8 then for $p \geq n$ we have an isomorphism

$$H^p(A_{coE}^*(M_\Gamma)) \cong \tilde{H}^p(A^*(M_\Gamma)).$$

**Remark 2.** In fact by Arapura and Nori’s theorem ([1]) a virtually polycyclic group $\Gamma$ must be virtually abelian if the standard $\Gamma$-manifold is Kähler. Therefore $G/\Gamma$ is finitely covered by a torus and the assumptions of Theorem 4.8 are satisfied. By Arapura and Nori’s theorem, if a solvmanifold $G/\Gamma$ admits a Kähler structure, then $G$ is (I)-type (i.e. for any $g \in G$ all eigenvalues of the adjoint operator $\text{Ad}_g$ have absolute value 1). Thus in the above corollary if $G$ is not (I)-type, then $M_\Gamma$ does not admit a Kähler structure. The author gave such non-Kähler examples in [9].

5. Examples

**Example 1.** First we give examples of solvmanifolds such that $H^p(A_{coE}^*(M_\Gamma)) \cong \tilde{H}^p(A^*(M_\Gamma))$ by using Corollary 4.9. We notice that if a solvmanifold $G/\Gamma$ has a symplectic form $\omega$, then we have a closed two-form $\omega_0 \in (\bigwedge u^*)^T$ which is homologous to $\omega$ and $\omega_0$ is also a symplectic form as we note in Remark 1. Let $G = \mathbb{C} \times_\phi \mathbb{C}^2$ with $\phi(x) = \begin{pmatrix} x^2 & 0 \\ 0 & e^{-x} \end{pmatrix}$. Then it is known that $G$ has a left-invariant symplectic form and a lattice $\Gamma$ (see [13]). Thus we have a symplectic form $\omega \in (\bigwedge u^*)^T$, and by Corollary 4.9 we have an isomorphism $H^p(A_{coE}^*(G/\Gamma)) \cong \tilde{H}^p(A^*(G/\Gamma))$.

**Remark 3.** $G$ is not completely solvable. In fact the de Rham cohomology of $G/\Gamma$ varies according to a choice of a lattice $\Gamma$. Thus it is not easy to compute the coeffective cohomology of $G/\Gamma$ by using $\bigwedge g^*$.  

**Remark 4.** $G$ is not (I)-type, and hence $G/\Gamma$ does not admit a Kähler structure.
Example 2. We give an example of a symplectic manifold $M_\Gamma$ such that the isomorphism $H^p(A^{*}_{\text{coE}}(M_\Gamma)) \cong \hat{H}^p(A^{*}(M_\Gamma))$ holds but $U_\Gamma$ is not abelian. Let $\Gamma = \mathbb{Z} \times \mathbb{Z}$ such that for $t \in \mathbb{Z}$,

$$\phi(t) = \begin{pmatrix} (-1)^t & (-1)^t t \\ 0 & (-1)^t \end{pmatrix}.$$

Then we have $H_\Gamma = \{ \pm 1 \} \times U_3(\mathbb{R})$ such that

$$( -1 ) \cdot \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & (-1)z \\ 0 & 1 & (-1)y \\ 0 & 0 & 1 \end{pmatrix}$$

(see [9, Section 7]). The dual space of the Lie algebra $u$ of $U_3(\mathbb{R})$ is given by $u^* = \langle x_1, x_2, x_3 \rangle$ such that the differential is given by

$$dx_1 = dx_2 = 0, \ dx_3 = -x_1 \wedge x_2.$$

The action of $\{ \pm 1 \}$ on $U_\Gamma$ is given by

$$( -1 ) \cdot x_1 = x_1, \quad ( -1 ) \cdot x_2 = -x_2, \quad ( -1 ) \cdot x_3 = -x_3.$$

Then we have $(\bigwedge u^*)^{\{ \pm 1 \}} = \bigwedge \langle x_1, x_2 \wedge x_3 \rangle$. By this the differential on $(\bigwedge u^*)^{\{ \pm 1 \}}$ is 0. We consider the product $M_\Gamma \times M_\Gamma$ for this $\Gamma$. Then by the cochain complex $(\bigwedge u^*)^{\{ \pm 1 \}} \otimes (\bigwedge u^*)^{\{ \pm 1 \}} = \bigwedge \langle x_1, x_2 \wedge x_3 \rangle \otimes \bigwedge \langle y_1, y_2 \wedge y_3 \rangle$ we can compute the de Rham cohomology and coeffective cohomology of $M_\Gamma \times M_\Gamma$, where we denote by $y_1, y_2, y_3$ the copy of $x_1, x_2, x_3$. We have a symplectic form

$$\omega = x_1 \wedge y_1 + x_2 \wedge x_3 + y_2 \wedge y_3$$

on $M_\Gamma \times M_\Gamma$. Then we have:

**Proposition 5.1.** For $p \geq n$ we have an isomorphism

$$H^p(A^{*}_{\text{coE}}(M_\Gamma \times M_\Gamma)) \cong \hat{H}^p(A^{*}(M_\Gamma \times M_\Gamma)).$$

**Proof.** Since the differential on $(\bigwedge u^*)^{\{ \pm 1 \}} \otimes (\bigwedge u^*)^{\{ \pm 1 \}}$ is 0 as above, the proposition follows as the proof of Corollary 4.7. \qed

**Remark 5.** $M_\Gamma$ is finitely covered by a quotient of $U_3(\mathbb{R})$ by a lattice. Thus $M_\Gamma \times M_\Gamma$ is finitely covered by the product of such nilmanifolds. The de Rham cohomology and coeffective cohomology of this covering space are computed by $\bigwedge u^* \otimes \bigwedge u^*$. This space does not satisfy the isomorphism in this proposition. Indeed $x_1 \wedge x_2 \wedge y_2 \wedge y_3$ is coeffective and its coeffective cohomology class is not 0. But we have $d(x_3 \wedge y_2 \wedge y_3) = x_1 \wedge x_2 \wedge y_2 \wedge y_3$, and hence its de Rham cohomology class is 0. Thus we have

$$H^4(A^{*}_{\text{coE}}((U_3(\mathbb{R})/\Gamma') \times (U_3(\mathbb{R})/\Gamma'))) \not\cong \hat{H}^4(A^{*}((\langle U_3(\mathbb{R})/\Gamma' \rangle \times (U_3(\mathbb{R})/\Gamma')))$$.
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