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COEFFECTIVE COHOMOLOGY OF SYMPLECTIC ASPHERICAL MANIFOLDS

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Abstract. We prove a generalization of the theorem which has been proved by Fernandez, Ibanez, and de Leon. By this result, we give examples of non-Kähler manifolds which satisfy the property of compact Kähler manifolds concerning the coeffective cohomology.

1. Introduction

Let (M,ω) be a compact $2n$-dimensional symplectic manifold. Denote by $A^*(M)$ the de Rham complex of M. We call a differential form $\alpha \in A^*(M)$ coeffective if $\omega\wedge\alpha = 0$, and denote the sub-DGA by

$$A^*_{coE}(M) = \{\alpha \in A^*(M) | \omega\wedge\alpha = 0\}.$$

We call the cohomology $H^*(A^*_{coE}(M))$ the coeffective cohomology of M. We also denote

$$\tilde{H}^*(A^*(M)) = \{[\alpha] \in H^*(A^*(M)) | [\omega]\wedge[\alpha] = 0\}.$$

Theorem 1.1 ([4]). *Let (M,ω) be a compact Kähler manifold. For $p \geq n+1$, we have an isomorphism*

$$H^p(A^*_{coE}(M)) \cong \tilde{H}^p(A^*(M)).$$

However for general symplectic manifolds, the isomorphism $H^p(A^*_{coE}(M)) \cong \tilde{H}^p(A^*(M))$ does not hold. In fact, counterexamples are given in [5]. So far we have hardly found examples of non-Kähler manifolds such that isomorphisms $H^p(A^*_{coE}(M)) \cong \tilde{H}^p(A^*(M))$ hold. The purpose of this paper is to compute the coeffective cohomology of some class of symplectic manifolds by use of a finite-dimensional cochain complex and to give non-Kähler examples such that the isomorphisms $H^p(A^*_{coE}(M)) \cong \tilde{H}^p(A^*(M))$ hold.

2. Preliminary: Coeffective cohomology of subcomplex

Let (M,ω) be a compact $2n$-dimensional symplectic manifold.

Proposition 2.1 ([5], [11]). *Then the map $\omega\wedge : A^p(M) \to A^{p+2}(M)$ is injective for $p \leq n-1$ and surjective for $p \geq n-1$.*

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By this proposition we have $H^p(A^*_{coE}(M)) = \{0\}$ for $p \leq n-1$, and so it is sufficient to consider $H^p(A^*_{coE}(M))$ for $p \geq n$. Since ω is closed, we have the short exact sequence of cochain complexes

$$0 \longrightarrow A^*_{coE}(M) \longrightarrow A^*(M) \xrightarrow{\omega\wedge} \omega \wedge A^*(M) \longrightarrow 0,$$

where we consider $\omega \wedge A^*(M)$, the cochain complex which is graded as $(\omega \wedge A^*(M))^p = \omega \wedge A^p(M)$. By this sequence we have the long exact sequence of cohomology

$$\longrightarrow H^{p-1}(A^*(M)) \xrightarrow{(\omega\wedge)^*} H^{p+1}(\omega \wedge A^*(M)) \longrightarrow H^p(A^*_{coE}(M)) \longrightarrow H^p(A^*(M)) \xrightarrow{(\omega\wedge)^*} .$$

By Proposition 2.1, we have $\omega \wedge A^{p-1}(M) = A^{p+1}(M)$ for $p \geq n$ and so the exact sequence is given by

$$\longrightarrow H^{p-1}(A^*(M)) \xrightarrow{(\omega\wedge)^*} H^{p+1}(A^*(M)) \longrightarrow H^p(A^*_{coE}(M)) \longrightarrow H^p(A^*(M)) \xrightarrow{(\omega\wedge)^*} .$$

Proposition 2.2. *Let $A^* \subset A^*(M)$ be a subcomplex such that the inclusion $\Phi : A^* \to A^*(M)$ induces a cohomology isomorphism. Assume $\omega \in A^*$ and the map $\omega\wedge : A^p \to A^{p+2}$ is surjective for $p \geq n-1$. Denote $A^*_{coE} = \ker(\omega\wedge)_{|A^*}$. Then the inclusion $\Phi : A^*_{coE} \to A^*_{coE}(M)$ induces an isomorphism*

$$H^p(A^*_{coE}) \cong H^p(A^*_{coE}(M))$$

for $p \geq n$.

Proof. As above, we have the exact sequence of cochain complexes

$$0 \longrightarrow A^*_{coE} \longrightarrow A^* \xrightarrow{\omega\wedge} \omega \wedge A^* \longrightarrow 0.$$

By the assumption, for $p \geq n$ we have the long exact sequence of cohomology

$$\longrightarrow H^{p-1}(A^*) \xrightarrow{(\omega\wedge)^*} H^{p+1}(A^*) \longrightarrow H^p(A^*_{coE}) \longrightarrow H^p(A^*) \xrightarrow{(\omega\wedge)^*} .$$

By the inclusion $\Phi : (\bigwedge_{coE} A^*)^T \to A^*_{coE}(M)$, we have the commutative diagram

$$\begin{tikzcd}
H^{p-1}(A^*(M)) \arrow[r, "(\omega\wedge)^*"] & H^{p+1}(A^*(M)) \arrow[r] & H^p(A^*_{coE}(M)) \arrow[r] & H^p(A^*(M)) \arrow[r, "(\omega\wedge)^*"] & H^{p+2}(A^*(M)) \\
H^{p-1}(A^*) \arrow[u, "\Phi^*"] \arrow[r, "(\omega\wedge)^*"] & H^{p+1}(A^*) \arrow[u, "\Phi^*"] \arrow[r] & H^p(A^*_{coE}) \arrow[u, "\Phi^*"] \arrow[r] & H^p(A^*) \arrow[u, "\Phi^*"] \arrow[r, "(\omega\wedge)^*"] & H^{p+2}(A^*). \arrow[u, "\Phi^*"]
\end{tikzcd}$$

By the assumption, $\Phi^* : H^*(A^*) \to H^*(A^*(M))$ is an isomorphism, and so by this diagram $\Phi^* : H^p(A^*_{coE}) \to H^p(A^*_{coE}(M))$ is an isomorphism. $\square$

3. Background: Fernandez-Ibanez-de Leon's theorem

Let G be a simply connected Lie group with a lattice Γ (i.e. a cocompact discrete subgroup of G). We call G/Γ a nilmanifold (resp. solvmanifold) if G is nilpotent (resp. solvable). Let $\mathfrak{g}$ be the Lie algebra of G and $\bigwedge \mathfrak{g}^*$ be the cochain complex of $\mathfrak{g}$ with the differential which is induced by the dual of the Lie bracket. As we regard $\bigwedge \mathfrak{g}^*$ as the left-invariant forms on G/Γ, we consider the inclusion $\bigwedge \mathfrak{g}^* \subset A^*(G/\Gamma)$. Let $\omega \in \bigwedge^2 \mathfrak{g}^*$ be a left-invariant symplectic form. Then the map $\omega\wedge : \bigwedge^p \mathfrak{g}^* \to \bigwedge^{p+2} \mathfrak{g}^*$ is surjective for $p \geq n-1$ (see [5]). In [14] Nomizu showed that if G is nilpotent, then the inclusion $\bigwedge \mathfrak{g}^* \subset A^*(G/\Gamma)$ induces an isomorphism of cohomology. Hence by Proposition 2.2, we have the following theorem, which was noted in [5] and [6].

Theorem 3.1. *Let G be a simply connected nilpotent Lie group with a lattice Γ and a left-invariant symplectic form ω. Then the inclusion $\bigwedge \mathfrak{g}^* \subset A^*(G/\Gamma)$ induces an isomorphism*

$$H^p(\bigwedge_{coE} \mathfrak{g}^*) \cong H^p(A^*_{coE}(G/\Gamma))$$

for $p \geq n$, where $\bigwedge_{coE} \mathfrak{g}^* = \{\alpha \in \bigwedge \mathfrak{g}^* | \omega \wedge \alpha = 0\}$.

In [8] Hattori showed that the isomorphism $H^*(\bigwedge \mathfrak{g}^*) \cong H^*(A^*(G/\Gamma))$ also holds if G is completely solvable (i.e. G is solvable and for any $g \in G$ all the eigenvalues of the adjoint operator Ad_g are real). Thus we can extend this theorem for completely solvmanifolds. However for a general solvmanifold G/Γ, the isomorphism $H^*(\bigwedge_{coE} \mathfrak{g}^*) \cong H^*(A^*_{coE}(G/\Gamma))$ does not hold and we can't compute the coeffective cohomology by using $\bigwedge \mathfrak{g}^*$.

In [2] Baues constructed compact aspherical manifolds M_Γ such that the class of these aspherical manifolds contains the class of solvmanifolds and showed that the de Rham cohomology of these aspherical manifolds can be computed by certain finite-dimensional cochain complexes. In the next section, by using Baues's results, we will show a generalization of Theorem 3.1.

4. Main results: Coeffective cohomology of aspherical manifolds with torsion-free virtually polycyclic fundamental groups

4.1. **Notation and conventions.** Let k be a subfield of $\mathbb{C}$. A group $\mathbf{G}$ is called a k-algebraic group if $\mathbf{G}$ is a Zariski-closed subgroup of $GL_n(\mathbb{C})$ which is defined by polynomials with coefficients in k. Let $\mathbf{G}(k)$ denote the set of k-points of $\mathbf{G}$ and $\mathbf{U}(\mathbf{G})$ the maximal Zariski-closed unipotent normal k-subgroup of $\mathbf{G}$ called the unipotent radical of $\mathbf{G}$. If $\mathbf{G}$ consists of semi-simple elements, we call $\mathbf{G}$ a d-group. Let $U_n(k)$ denote the $n \times n$ k-valued upper triangular unipotent matrix group.

4.2. **Baues's results.** A group Γ is called polycyclic if it admits a sequence

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_k = \{e\}$$

of subgroups such that each Γ_i is normal in Γ_{i-1} and Γ_{i-1}/Γ_i is cyclic. We denote $\operatorname{rank}\Gamma = \sum_{i=1}^{i=k} \operatorname{rank}\Gamma_{i-1}/\Gamma_i$. We define an infra-solvmanifold as a manifold of the form G/Δ, where G is a simply connected solvable Lie group and Δ is a torsion free subgroup of $\mathrm{Aut}(G) \ltimes G$ such that for the projection $p : \mathrm{Aut}(G) \ltimes G \to \mathrm{Aut}(G)$ $p(\Delta)$ is contained in a compact subgroup of $\mathrm{Aut}(G)$. By a result of Mostow in [12], the fundamental group of an infra-solvmanifold is virtually polycyclic (i.e. it contains a finite index polycyclic subgroup). In particular, a lattice Γ of a simply connected solvable Lie group G is a polycyclic group with $\operatorname{rank}\Gamma = \dim G$ (see [15]).

Let k be a subfield of $\mathbb{C}$. Let Γ be a torsion-free virtually polycyclic group. For a finite index polycyclic subgroup $\Delta \subset \Gamma$, we denote $\operatorname{rank}\Gamma = \operatorname{rank}\Delta$.

Definition 4.1. We call a k-algebraic group $\mathbf{H}_\Gamma$ a k-algebraic hull of Γ if there exists an injective group homomorphism $\psi : \Gamma \to \mathbf{H}_\Gamma(k)$ and $\mathbf{H}_\Gamma$ satisfies the following conditions:

(1) $\psi(\Gamma)$ is Zariski-dense in $\mathbf{H}_{\mathbf{\Gamma}}$.
(2) $Z_{\mathbf{H}_\Gamma}(\mathbf{U}(\mathbf{H}_\Gamma)) \subset \mathbf{U}(\mathbf{H}_\Gamma)$, where $Z_{\mathbf{H}_\Gamma}(\mathbf{U}(\mathbf{H}_\Gamma))$ is the centralizer of $\mathbf{U}(\mathbf{H}_\Gamma)$.
(3) $\dim \mathbf{U}(\mathbf{H}_\Gamma) = \operatorname{rank}\Gamma$.

Theorem 4.2 ([2, Theorem A.1])**.** *There exists a k-algebraic hull of Γ, and a k-algebraic hull of Γ is unique up to a k-algebraic group isomorphism.*

Let Γ be a torsion-free virtually polycyclic group and $\mathbf{H}_{\mathbf{\Gamma}}$ the $\mathbb{Q}$-algebraic hull of Γ. Denote $H_\Gamma = \mathbf{H}_{\mathbf{\Gamma}}(\mathbb{R})$. Let U_Γ be the unipotent radical of H_Γ and T a maximal d-subgroup. Then H_Γ decomposes as a semi-direct product $H_\Gamma = T \ltimes U_\Gamma$; see [2, Proposition 2.1]. Let $\mathfrak{u}$ be the Lie algebra of U_Γ. Since the exponential map $\exp : \mathfrak{u} \longrightarrow U_\Gamma$ is a diffeomorphism, U_Γ is diffeomorphic to $\mathbb{R}^n$ such that $n = \operatorname{rank}\Gamma$. For the semi-direct product $H_\Gamma = T \ltimes U_\Gamma$, we denote $\phi : T \to \operatorname{Aut}(U_\Gamma)$, the action of T on U_Γ. Then we have the homomorphism $\alpha : H_\Gamma \longrightarrow \operatorname{Aut}(U_\Gamma) \ltimes U_\Gamma$ such that $\alpha(t, u) = (\phi(t), u)$ for $(t, u) \in T \ltimes U_\Gamma$. By the property (2) in Definition 4.1, ϕ is injective and hence α is injective.

In [2] Baues constructed a compact aspherical manifold $M_\Gamma = \alpha(\Gamma)\backslash U_\Gamma$ with $\pi_1(M_\Gamma) = \Gamma$. We call M_Γ a standard Γ-manifold.

Theorem 4.3 ([2, Theorems 1.2, 1.4])**.** *A standard Γ-manifold is unique up to diffeomorphism. A compact infra-solvmanifold with the fundamental group Γ is diffeomorphic to the standard Γ-manifold M_Γ. In particular, a solvmanifold G/Γ is diffeomorphic to the standard Γ-manifold M_Γ.*

Let $A^*(M_\Gamma)$ be the de Rham complex of M_Γ. Then $A^*(M_\Gamma)$ is the set of the Γ-invariant differential forms $A^*(U_\Gamma)^\Gamma$ on U_Γ. Let $(\bigwedge \mathfrak{u}^*)^T$ be the left-invariant forms on U_Γ which are fixed by T. Since $\Gamma \subset H_\Gamma = U_\Gamma \cdot T$, we have the inclusion

$$(\bigwedge \mathfrak{u}^*)^T = A^*(U_\Gamma)^{H_\Gamma} \subset A^*(U_\Gamma)^\Gamma = A^*(M_\Gamma).$$

Theorem 4.4 ([2, Theorem 1.8])**.** *This inclusion induces a cohomology isomorphism.*

4.3. **Main results.** Let $\omega \in (\bigwedge \mathfrak{u}^*)^T$ be a symplectic form. Denote

$$\bigwedge_{coE} \mathfrak{u}^* = \{\alpha \in \bigwedge \mathfrak{u}^* | \omega \wedge \alpha = 0\}$$

and

$$\tilde{H}^*((\bigwedge \mathfrak{u}^*)^T) = \{[\alpha] \in H^*((\bigwedge \mathfrak{u}^*)^T) | [\omega] \wedge [\alpha] = 0\}.$$

By Theorem 4.4, we have $\tilde{H}^*((\bigwedge \mathfrak{u}^*)^T) \cong \tilde{H}^*(A^*(M_\Gamma))$.

Lemma 4.5. *For $p \geq n-1$, the linear map $\omega\wedge : (\bigwedge^p \mathfrak{u}^*)^T \to (\bigwedge^{p+2} \mathfrak{u}^*)^T$ is surjective.*

Proof. First we notice that the map $\omega\wedge : \bigwedge^p \mathfrak{u}^* \to \bigwedge^{p+2} \mathfrak{u}^*$ is surjective (see [5, Lemma 2.1]). Since T is a d-group, for $t \in T$ the t-action on $\bigwedge \mathfrak{u}^*$ is diagonalizable (see [2]). Hence we have a decomposition

$$\bigwedge^p \mathfrak{u}^* = A^p \oplus B^p$$

such that A^p is the subspace of t-invariant elements and B^p is its complement. Since the t-action is diagonalizable, we have a basis $\{x_1, \dots, x_{2n}\}$ of $\mathfrak{u}^* \otimes \mathbb{C}$ such that the t-action is represented by a diagonal matrix. Then we have

$$A^p \otimes \mathbb{C} = \langle x_{i_1} \wedge \cdots \wedge x_{i_p} | 1 \le i_1 < \cdots < i_p \le 2n,\, t \cdot (x_{i_1} \wedge \cdots \wedge x_{i_p}) = x_{i_1} \wedge \cdots \wedge x_{i_p} \rangle$$

and

$$\begin{aligned} B^p \otimes \mathbb{C} = \langle x_{i_1} \wedge \cdots \wedge x_{i_p} | 1 \le i_1 < \cdots < i_p \le 2n, \\ t \cdot (x_{i_1} \wedge \cdots \wedge x_{i_p}) = \alpha_{i_1 \dots i_p}(t) x_{i_1} \wedge \cdots \wedge x_{i_p}, \alpha_{i_1 \dots i_p}(t) \neq 1 \rangle. \end{aligned}$$

By $\omega \in (\bigwedge \mathfrak{u}^*)^T$, we have $\omega = \sum a_{kl} x_k \wedge x_l$ such that if $a_{kl} \neq 0$, then $x_k \wedge x_l \in A^p \otimes \mathbb{C}$. Then for $x_{i_1} \wedge \cdots \wedge x_{i_p} \in B^p \otimes \mathbb{C}$ we have

$$\omega \wedge x_{i_1} \wedge \cdots \wedge x_{i_p} = \sum a_{kl} x_k \wedge x_l \wedge x_{i_1} \wedge \cdots \wedge x_{i_p}.$$

If $a_{kl} \neq 0$, we have

$$t \cdot (x_k \wedge x_l \wedge x_{i_1} \wedge \cdots \wedge x_{i_p}) = \alpha_{i_1 \dots i_p}(t) x_k \wedge x_l \wedge x_{i_1} \wedge \cdots \wedge x_{i_p}.$$

Thus $\omega \wedge x_{i_1} \wedge \cdots \wedge x_{i_p} \in B^{p+2} \otimes \mathbb{C}$. By this we have $(\omega \wedge B^p) \subset B^{p+2}$. Since T acts semi-simply on $\bigwedge^p \mathfrak{u}^*$, we consider the decomposition

$$\bigwedge^p \mathfrak{u}^* = (\bigwedge^p \mathfrak{u}^*)^T \oplus C^p$$

such that C^p is a complement of $(\bigwedge^p \mathfrak{u}^*)^T$ for T-action. By the above argument we have $(\omega \wedge C^p) \subset C^{p+2}$. Clearly we have $(\omega \wedge (\bigwedge^p \mathfrak{u}^*)^T) \subset (\bigwedge^{p+2} \mathfrak{u}^*)^T$. Since for $p \ge n-1$ the map $\omega\wedge : \bigwedge^p \mathfrak{u}^* \to \bigwedge^{p+2} \mathfrak{u}^*$ is surjective, we have

$$(\bigwedge^{p+2} \mathfrak{u}^*)^T \oplus C^{p+2} = \omega \wedge \bigwedge^p \mathfrak{u}^* = (\omega \wedge (\bigwedge^p \mathfrak{u}^*)^T) \oplus (\omega \wedge C^p).$$

Thus we have $\omega \wedge (\bigwedge^p \mathfrak{u}^*)^T = (\bigwedge^{p+2} \mathfrak{u}^*)^T$. Hence the lemma follows. $\square$

By this lemma and Proposition 2.2, we have:

Theorem 4.6. *Let Γ be a torsion-free virtually polycyclic group and M_Γ the standard Γ-manifold with a symplectic form ω such that $\omega \in (\bigwedge \mathfrak{u}^*)^T$. Then for $p \ge n$, the inclusion $\Phi : (\bigwedge_{coE} \mathfrak{u}^*)^T \to A^*_{coE}(M_\Gamma)$ induces an isomorphism $\Phi^* : H^*((\bigwedge_{coE} \mathfrak{u}^*)^T) \cong H^*(A^p_{coE}(M_\Gamma))$.*

Remark 1. In [10], the author showed that if there exists $[\omega] \in H^2(M_\Gamma, \mathbb{R})$ such that $[\omega]^{\frac{1}{2}\dim M_\Gamma} \neq 0$, then an invariant form $\omega \in (\bigwedge \mathfrak{u}^*)^T$ which represents the cohomology class $[\omega]$ is a symplectic form on M_Γ. Hence if M_Γ is cohomologically symplectic (i.e. there exists $[\omega] \in H^2(M_\Gamma, \mathbb{R})$ such that $[\omega]^{\frac{1}{2}\dim M_\Gamma} \neq 0$), then M_Γ admits a symplectic form ω such that $\omega \in (\bigwedge \mathfrak{u}^*)^T$.

Corollary 4.7. *Under the same assumption of Theorem 4.6, if U_Γ is abelian, then for $p \ge n$ we have an isomorphism*

$$H^p(A^*_{coE}(M_\Gamma)) \cong \tilde{H}^p(A^*(M_\Gamma)).$$

Proof. If U_Γ is abelian, then the differential of $\bigwedge \mathfrak{u}^*$ is 0. Hence we have

$$H^*(A^*(M_\Gamma)) \cong H^*((\bigwedge \mathfrak{u}^*)^T) = (\bigwedge \mathfrak{u}^*)^T$$

and

$$H^*((\bigwedge_{coE} \mathfrak{u}^*)^T) = (\bigwedge_{coE} \mathfrak{u}^*)^T.$$

This gives

$$\begin{aligned}\tilde{H}^*(A^*(M_\Gamma)) \cong \tilde{H}^*((\bigwedge \mathfrak{u}^*)^T) &= \{\alpha \in (\bigwedge \mathfrak{u}^*)^T | \alpha \wedge \omega = 0\} \\ &= (\bigwedge_{coE} \mathfrak{u}^*)^T = H^*((\bigwedge_{coE} \mathfrak{u}^*)^T).\end{aligned}$$

Hence by the above theorem the corollary follows. □

In [9] the author showed the following theorem.

Theorem 4.8 ([9])**.** *Let Γ be a torsion-free virtually polycyclic group. Then the following two conditions are equivalent:*

(1) $\mathbf{U}_\Gamma$ *is abelian.*

(2) Γ *is a finite extension group of a lattice of a Lie group $G = \mathbb{R}^n \ltimes_\phi \mathbb{R}^m$ such that the action $\phi : \mathbb{R}^n \to \mathrm{Aut}(\mathbb{R}^m)$ is semi-simple.*

Hence we have:

Corollary 4.9. *Under the same assumptions of Theorem* 4.6*, if Γ satisfies the condition* (2) *in Theorem* 4.8*, then for $p \geq n$ we have an isomorphism*

$$H^p(A^*_{coE}(M_\Gamma)) \cong \tilde{H}^p(A^*(M_\Gamma)).$$

Remark 2. In fact by Arapura and Nori's theorem ([1]) a virtually polycyclic group Γ must be virtually abelian if the standard Γ-manifold is Kähler. Therefore G/Γ is finitely covered by a torus and the assumptions of Theorem 4.8 are satisfied. By Arapura and Nori's theorem, if a solvmanifold G/Γ admits a Kähler structure, then G is (I)-type (i.e. for any $g \in G$ all eigenvalues of the adjoint operator Ad_g have absolute value 1). Thus in the above corollary if G is not (I)-type, then M_Γ does not admit a Kähler structure. The author gave such non-Kähler examples in [9].

5. Examples

Example 1. First we give examples of solvmanifolds such that $H^p(A^*_{coE}(M_\Gamma)) \cong \tilde{H}^p(A^*(M_\Gamma))$ by using Corollary 4.9. We notice that if a solvmanifold G/Γ has a symplectic form ω, then we have a closed two-form $\omega_0 \in (\bigwedge \mathfrak{u}^*)^T$ which is homologous to ω and ω_0 is also a symplectic form as we note in Remark 1. Let $G = \mathbb{C} \ltimes_\phi \mathbb{C}^2$ with $\phi(x) = \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix}$. Then it is known that G has a left-invariant symplectic form and a lattice Γ (see [13]). Thus we have a symplectic form $\omega \in (\bigwedge \mathfrak{u}^*)^T$, and by Corollary 4.9 we have an isomorphism $H^p(A^*_{coE}(G/\Gamma)) \cong \tilde{H}^p(A^*(G/\Gamma))$.

Remark 3. G is not completely solvable. In fact the de Rham cohomology of G/Γ varies according to a choice of a lattice Γ. Thus it is not easy to compute the coeffective cohomology of G/Γ by using $\bigwedge \mathfrak{g}^*$.

Remark 4. G is not (I)-type, and hence G/Γ does not admit a Kähler structure.

Example 2. We give an example of a symplectic manifold M_Γ such that the isomorphism $H^p(A^*_{coE}(M_\Gamma)) \cong \tilde{H}^p(A^*(M_\Gamma))$ holds but U_Γ is not abelian. Let $\Gamma = \mathbb{Z} \ltimes_\phi \mathbb{Z}^2$ such that for $t \in \mathbb{Z}$,

$$\phi(t) = \begin{pmatrix} (-1)^t & (-1)^t t \\ 0 & (-1)^t \end{pmatrix}.$$

Then we have $H_\Gamma = \{\pm 1\} \ltimes U_3(\mathbb{R})$ such that

$$(-1) \cdot \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & (-1)z \\ 0 & 1 & (-1)y \\ 0 & 0 & 1 \end{pmatrix}$$

(see [9, Section 7]). The dual space of the Lie algebra $\mathfrak{u}$ of U_Γ is given by $\mathfrak{u}^* = \langle x_1, x_2, x_3 \rangle$ such that the differential is given by

$$dx_1 = dx_2 = 0, \ dx_3 = -x_1 \wedge x_2.$$

The action of $\{\pm 1\}$ on U_Γ is given by

$$(-1) \cdot x_1 = x_1,$$
$$(-1) \cdot x_2 = -x_2, \ (-1) \cdot x_3 = -x_3.$$

Then we have $(\bigwedge \mathfrak{u}^*)^{\{\pm 1\}} = \bigwedge \langle x_1, x_2 \wedge x_3 \rangle$. By this the differential on $(\bigwedge \mathfrak{u}^*)^{\{\pm 1\}}$ is 0. We consider the product $M_\Gamma \times M_\Gamma$ for this Γ. Then by the cochain complex $(\bigwedge \mathfrak{u}^*)^{\{\pm 1\}} \otimes (\bigwedge \mathfrak{u}^*)^{\{\pm 1\}} = \bigwedge \langle x_1, x_2 \wedge x_3 \rangle \otimes \bigwedge \langle y_1, y_2 \wedge y_3 \rangle$ we can compute the de Rham cohomology and coeffective cohomology of $M_\Gamma \times M_\Gamma$, where we denote by y_1, y_2, y_3 the copy of x_1, x_2, x_3. We have a symplectic form

$$\omega = x_1 \wedge y_1 + x_2 \wedge x_3 + y_2 \wedge y_3$$

on $M_\Gamma \times M_\Gamma$. Then we have:

Proposition 5.1. *For $p \geq n$ we have an isomorphism*

$$H^p(A^*_{coE}(M_\Gamma \times M_\Gamma)) \cong \tilde{H}^p(A^*(M_\Gamma \times M_\Gamma)).$$

Proof. Since the differential on $(\bigwedge \mathfrak{u}^*)^{\{\pm 1\}} \otimes (\bigwedge \mathfrak{u}^*)^{\{\pm 1\}}$ is 0 as above, the proposition follows as the proof of Corollary 4.7. □

Remark 5. M_Γ is finitely covered by a quotient of $U_3(\mathbb{R})$ by a lattice. Thus $M_\Gamma \times M_\Gamma$ is finitely covered by the product of such nilmanifolds. The de Rham cohomology and coeffective cohomology of this covering space are computed by $\bigwedge \mathfrak{u}^* \otimes \bigwedge \mathfrak{u}^*$. This space does not satisfy the isomorphism in this proposition. Indeed $x_1 \wedge x_2 \wedge y_2 \wedge y_3$ is coeffective and its coeffective cohomology class is not 0. But we have $d(x_3 \wedge y_2 \wedge y_3) = x_1 \wedge x_2 \wedge y_2 \wedge y_3$, and hence its de Rham cohomology class is 0. Thus we have

$$H^4(A^*_{coE}((U_3(\mathbb{R})/\Gamma') \times (U_3(\mathbb{R})/\Gamma'))) \ncong \tilde{H}^4(A^*((U_3(\mathbb{R})/\Gamma') \times (U_3(\mathbb{R})/\Gamma'))).$$

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References

[1] D. Arapura, M. Nori, Solvable fundamental groups of algebraic varieties and Kähler manifolds. Compositio Math. **116** (1999), no. 2, 173–188. MR1686777 (2000k:14018)
[2] O. Baues, Infra-solvmanifolds and rigidity of subgroups in solvable linear algebraic groups. Topology **43** (2004), no. 4, 903–924. MR2061212 (2005c:57048)
[3] A. Borel, Linear algebraic groups, 2nd enlarged edition, Springer-Verlag (1991). MR1102012 (92d:20001)
[4] T. Bouche, La cohomologie coeffective d'une variété symplectique. Bull. Sci. Math. **114** (1990), no. 2, 115–122. MR1056157 (91i:58005)
[5] M. Fernandez, R. Ibanez, M. de Leon, A Nomizu's theorem for the coeffective cohomology. Math. Z. **226** (1997), no. 1, 11–23. MR1472138 (98k:58008)
[6] M. Fernandez, R. Ibanez, M. de Leon, Coeffective and de Rham cohomologies of symplectic manifolds. J. Geom. Phys. **27** (1998), no. 3-4, 281–296. MR1645044 (99k:53053)
[7] K. Hasegawa, A note on compact solvmanifolds with Kähler structures. Osaka J. Math. **43** (2006), no. 1, 131–135. MR2222405 (2007h:32035)
[8] A. Hattori, Spectral sequence in the de Rham cohomology of fibre bundles. J. Fac. Sci. Univ. Tokyo Sect. I **8** 1960 289–331 (1960). MR0124918 (23:A2226)
[9] H. Kasuya, Formality and hard Lefschetz properties of aspherical manifolds. http://arxiv.org/abs/0910.1175. To appear in Osaka J. Math.
[10] H. Kasuya Cohomologically symplectic solvmanifolds are symplectic. J. Symplectic Geom. 9 (2011), no. 4.
[11] P. Libermann, Ch. M. Marle, Symplectic Geometry and Analytical Mechanics. Reidel/Kluwer, Dordrecht (1987). MR882548 (88c:58016)
[12] G. D. Mostow, On the fundamental group of a homogeneous space, Ann. of Math. (2) **66** (1957), 249-255. MR0088675 (19:561c)
[13] I. Nakamura, Complex parallelisable manifolds and their small deformations. J. Differential Geometry **10** (1975), 85–112. MR0393580 (52:14389)
[14] K. Nomizu, On the cohomology of compact homogeneous spaces of nilpotent Lie groups. Ann. of Math. (2) **59** (1954), 531–538. MR0064057 (16:219c)
[15] M. S. Raghunathan, Discrete Subgroups of Lie Groups, Springer-Verlag, New York, 1972. MR0507234 (58:22394a)

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