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BOUNDED SYMPLECTIC DIFFEOMORPHISMS AND SPLIT FLUX GROUPS

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Abstract. We prove the bounded isometry conjecture of F. Lalonde and L. Polterovich for a special class of closed symplectic manifolds. As a byproduct, it is shown that the flux group of a product of these special symplectic manifolds is isomorphic to the direct sum of the flux group of each symplectic manifold.

1. Introduction

For a closed symplectic manifold (M, ω), the group $\mathrm{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms carries a norm called the Hofer norm. The group $\mathrm{Ham}(M, \omega)$ is a normal subgroup of $\mathrm{Symp}_0(M, \omega)$, the group of symplectic diffeomorphisms, and the Hofer norm is invariant under conjugation by $\mathrm{Symp}_0(M, \omega)$. For a fixed symplectic diffeomorphism ψ, the map $\mathcal{C}_\psi : \mathrm{Ham}(M, \omega) \to \mathrm{Ham}(M, \omega)$ defined by $\mathcal{C}_\psi(h) = \psi \circ h \circ \psi^{-1}$ is an isometry with respect to the Hofer norm. In [5] F. Lalonde and L. Polterovich study the isometries of the group of Hamiltonian diffeomorphisms with respect to the Hofer norm. Based on this they call a symplectic diffeomorphism ψ bounded if the Hofer norm of the commutator $[\psi, h]$ remains bounded as h varies in $\mathrm{Ham}(M, \omega)$. The set of bounded symplectic diffeomorphisms $\mathrm{BI}_0(M, \omega)$ of (M, ω) is a group that contains all Hamiltonian diffeomorphisms.

Lalonde and Polterovich conjectured that $\mathrm{Ham}(M, \omega) = \mathrm{BI}_0(M, \omega)$ for any closed symplectic manifold (M, ω). This problem is known as the bounded isometry conjecture. In [5] they proved the conjecture when the symplectic manifold is a surface of positive genus or is a product of these surfaces. In [4] Lalonde and C. Pestieau proved the conjecture for the product of a closed surface of positive genus and a simply connected manifold. Recently, Z. Han [2] proved the conjecture for the Kodaira–Thurston manifold.

In fact, in [5] Lalonde and Polterovich proved a stronger result than the bounded isometry conjecture. They proved that if an equivalence class of $\mathrm{Symp}_0(M, \omega)/\mathrm{Ham}(M, \omega)$ has an unbounded symplectic diffeomorphism, then there is a strongly unbounded symplectic diffeomorphism in the same class. This is equivalent to the fact that for any nonzero element v of $H^1(M)/\Gamma_M$ there is a strongly unbounded symplectic diffeomorphism with flux v. Here Γ_M stands for the flux group of (M, ω). For the details, see Section 3. Here we prove this stronger result.

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We prove the bounded isometry conjecture for a closed symplectic manifold (M, ω) of dimension $2n$ satisfying the following two conditions:

(a) There are open sets $U_1, \ldots, U_l \subset M$ such that each U_k is symplectomorphic to $\mathbb{T}^{2n} \setminus B(\epsilon_k)$ with the standard symplectic form. Here $\mathbb{T}^{2n}$ is the $2n$–dimensional torus and $B(\epsilon_k)$ is the embedded image of the standard closed ball in $\mathbb{R}^{2n}$ for a sufficiently small $\epsilon_k \geq 0$.

(b) Let $j_k : U_k \to M$ be the inclusion map and $j_{k*} : H^1_c(U_k) \to H^1(M)$ the induced map in cohomology. Then

$$H^1(M) = \sum_{k=1}^{l} j_{k*}(H^1_c(U_k)).$$

A symplectic manifold satisfying the conditions above is said to satisfy *(H)*. Unless otherwise stated, throughout this article cohomology $H^*(\cdot)$ stands for de Rham cohomology and $H^*_c(\cdot)$ stands for de Rham cohomology with compact support.

Theorem 1. *Let (M, ω) be a closed symplectic manifold that satisfies (H). Then*

$$\mathrm{BI}_0(M, \omega) = \mathrm{Ham}(M, \omega).$$

The proof of Theorem 1 is based on the fact that the bounded isometry conjecture holds for the punctured torus $(\mathbb{T}^{2n}_*, \omega_0)$. This was shown in [5] for $n = 1$, but in fact their argument works for all n. For the sake of completeness we prove that $\mathrm{BI}_0(\mathbb{T}^{2n}_*, \omega_0) = \mathrm{Ham}^c(\mathbb{T}^{2n}_*, \omega_0)$ in Proposition 7. Here $\mathrm{Ham}^c(M, \omega)$ stands for the group of Hamiltonian diffeomorphisms of (M, ω) with compact support.

The first example of a closed symplectic manifold satisfying *(H)* is a closed surface (Σ_g, ω) with g embedded punctured tori. Another example is the blowup of the torus $(\mathbb{T}^{2n}, \omega_0)$ at one point, or more generally the blowup of $(\mathbb{T}^{2n}, \omega_0)$ along a simply connected symplectic submanifold. In Section 2, we give some more examples of symplectic manifolds that satisfy *(H)*.

The bounded isometry conjecture holds for a wider class of symplectic manifolds than just those that satisfy *(H)*.

Corollary 2. *Let (M, ω) be a closed symplectic manifold that satisfies (H), and let (N, η) be a closed symplectic manifold such that $H^1(N)$ is trivial or satisfies (H). Then*

$$\mathrm{BI}_0(M \times N, \omega \oplus \eta) = \mathrm{Ham}(M \times N, \omega \oplus \eta).$$

As a consequence of our argument in the proof of the bounded isometry conjecture for this particular class of manifolds, we obtain an interesting result about the flux group. We show that the flux group of a product of two closed symplectic manifolds is isomorphic to the direct sum of the flux group of each manifold. That is, if (M, ω) and (N, η) are symplectic manifolds as in Corollary 2 with flux groups Γ_M and Γ_N, then $\Gamma_{M \times N} \simeq \Gamma_M \oplus \Gamma_N$, where $\Gamma_{M \times N}$ is the flux group of $(M \times N, \omega \oplus \eta)$. When this relation holds, we say that the flux group of $(M \times N, \omega \oplus \eta)$ splits.

For instance, this is well known when we consider copies of $(\mathbb{T}^2, \omega_0)$. In this case, $\Gamma_{\mathbb{T}^2}$ is equal to $H^1(\mathbb{T}^2, \mathbb{Z})$ and a direct calculation shows that $\Gamma_{\mathbb{T}^{2n}} = \Gamma_{\mathbb{T}^2} \oplus \cdots \oplus \Gamma_{\mathbb{T}^2}$ (see [11, Ch. 10] and [14, Ch. 14]). Recall that in [3], J. Kędra gave conditions under which the flux group vanishes and these conditions are compatible with products. For instance, if (M, ω) is aspherical with nonzero Euler characteristic, then by Theorem B of [3], $\Gamma_M \oplus \Gamma_M = \Gamma_{M \times M} = 0$.

When (M, ω) is a closed surface of genus greater than one, the flux group is trivial; when $g = 1$, the flux group equals $\mathbb{Z}^2$. In [5, Remark 4.3.E], Lalonde and Polterovich showed that the flux group splits when the manifold is a product of closed surfaces of positive genus. They achieved this in their study of bounded symplectic diffeomorphisms. Here we closely follow their ideas.

Theorem 3. *Let (M, ω) and (N, η) be closed symplectic manifolds as in Corollary 2. Then the flux group of $(M \times N, \omega \oplus \eta)$ splits: $\Gamma_{M\times N} \simeq \Gamma_M \oplus \Gamma_N$.*

In Section 2 we give an application of Theorem 3 to the fundamental group of $\mathrm{Ham}(S^2 \times \Sigma_g)$. Finally we point out an equivalent statement to that of Theorem 3.

Theorem 4. *Let (M, ω) and (N, η) be closed symplectic manifolds as in Corollary 2. If $\psi \in \mathrm{Symp}_0(M, \omega)$ and $\phi \in \mathrm{Symp}_0(N, \eta)$ are such that $\psi \times \phi$ is a Hamiltonian diffeomorphism of $(M \times N, \omega \oplus \eta)$, then ψ and ϕ are Hamiltonian diffeomorphisms.*

Finally we remark that all the results remain true in the noncompact case as long as one considers diffeomorphisms with compact support.

2. Examples

Example. Consider the torus $(\mathbb{T}^{2n}, \omega_0)$ with its standard symplectic form. Let $(\tilde{\mathbb{T}}^{2n}, \tilde{\omega}_0)$ be its blowup at one point. See [9]. There is a small $\epsilon > 0$ such that the inclusion $\mathbb{T}^{2n} \setminus B(\epsilon) \to \tilde{\mathbb{T}}^{2n}$ is a symplectic embedding. Moreover, the induced map $H^1_c(\mathbb{T}^{2n} \setminus B(\epsilon)) \to H^1(\tilde{\mathbb{T}}^{2n})$ is an isomorphism. Hence, $(\tilde{\mathbb{T}}^{2n}, \tilde{\omega}_0)$ satisfies *(H)* and $\mathrm{BI}_0(\tilde{\mathbb{T}}^{2n}, \tilde{\omega}_0) = \mathrm{Ham}(\tilde{\mathbb{T}}^{2n}, \tilde{\omega}_0)$.

More generally, let N be a simply connected symplectic submanifold of $(\mathbb{T}^{2n}, \omega_0)$. Denote by $(\tilde{\mathbb{T}}^{2n}_N, \tilde{\omega}_0)$ the blowup of $(\mathbb{T}^{2n}, \omega_0)$ along N. Since N is simply connected, the blowup map $\tilde{\mathbb{T}}^{2n}_N \to \mathbb{T}^{2n}$ induces an isomorphism $H^1(\mathbb{T}^{2n}) \to H^1(\tilde{\mathbb{T}}^{2n}_N)$. Therefore, $\mathrm{BI}_0(\tilde{\mathbb{T}}^{2n}_N, \tilde{\omega}_0) = \mathrm{Ham}(\tilde{\mathbb{T}}^{2n}_N, \tilde{\omega}_0)$ by Theorem 1.

Thus, in every dimension we have new examples of symplectic manifolds that satisfy the bounded isometry conjecture. The next example explores some consequences of Theorem 3.

Example. Consider the symplectic embedding $(\mathbb{T}^{2n}, \omega_0) \to (\mathbb{T}^{2(m+n)}, \omega)$ in the last $2n$ coordinates. The symplectic form on the torus is the canonical symplectic form. Thus $\mathbb{T}^{2(m+n)} \setminus \mathbb{T}^{2n} = \mathbb{T}^{2m}_* \times \mathbb{T}^{2n}$. It follows by Corollary 2 that $\mathrm{BI}_0(\mathbb{T}^{2(m+n)} \setminus \mathbb{T}^{2n}, \omega) = \mathrm{Ham}^c(\mathbb{T}^{2(m+n)} \setminus \mathbb{T}^{2n}, \omega)$.

It is also possible to show directly that $(\mathbb{T}^{2(m+n)} \setminus \mathbb{T}^{2n}, \omega)$ satisfies the bounded isometry conjecture. Our arguments in the proof of Proposition 7 apply to this case with no major changes. In fact, condition *(H)* can be weakened by allowing the set U to be symplectomorphic to $\mathbb{T}^{2(m+n)} \setminus \mathbb{T}^{2n}$, and not only symplectomorphic to a punctured torus.

Example. Let (S^2, ω) be the 2–sphere and (Σ_g, η) a Riemann surface of genus $g \geq 1$, each with a symplectic form of total area 1. Recall that $\pi_1(\mathrm{Ham}(S^2, \omega)) \simeq \mathbb{Z}_2$ and $\mathrm{Ham}(\Sigma_g, \eta)$ is simply connected for $g \geq 1$. By [8] and [13] we can say that $\pi_1(\mathrm{Ham}(S^2 \times \Sigma_g, \omega \oplus \eta))$ has an element of order two. We can say more by using Theorem 3.

Since the flux group Γ_{S^2} is trivial, by Theorem 3 it follows that $\Gamma_{S^2\times\Sigma_g} = \Gamma_{\Sigma_g}$. Thus, $\Gamma_{S^2\times\Sigma_g}$ equals $\mathbb{Z}\oplus\mathbb{Z}$ for $g=1$ and is trivial for $g>1$. It follows from the exact sequence (3.1) below that for $g>1$ the inclusion map induces an isomorphism

$$\pi_1(\mathrm{Ham}(S^2\times\Sigma_g,\omega\oplus\eta)) \to \pi_1(\mathrm{Symp}_0(S^2\times\Sigma_g,\omega\oplus\eta)).$$

For $g=1$ we get the exact sequence

$$0 \to \pi_1(\mathrm{Ham}(S^2\times\mathbb{T}^2,\omega\oplus\eta)) \to \pi_1(\mathrm{Symp}_0(S^2\times\mathbb{T}^2,\omega\oplus\eta)) \to \mathbb{Z}\oplus\mathbb{Z} \to 0. \tag{2.1}$$

Let $\mathcal{D}^g$ denote the group of volume-preserving diffeomorphisms of $S^2\times\Sigma_g$ that also preserve the fibers of $S^2\times\Sigma_g \to S^2$. According to D. McDuff [10, Prop. 1.6], for $g=1$, the map $\pi_1(\mathrm{Symp}_0(S^2\times\mathbb{T}^2,\omega\oplus\eta)) \to \pi_1(\mathcal{D}^1)$ is an isomorphism and $g>1$, and the map $\pi_1(\mathrm{Symp}_0(S^2\times\Sigma_g,\omega\oplus\eta)) \to \pi_1(\mathcal{D}^g)$ is surjective. Moreover by [10, Cor. 5.4], $\pi_1(\mathcal{D}^g)\otimes\mathbb{Q}$ has dimension three when $g=1$ and dimension one when $g>1$. Hence, from the exact sequence (2.1) we get the exact sequence

$$\begin{aligned} 0 \to \pi_1(\mathrm{Ham}(S^2\times\mathbb{T}^2,\omega\oplus\eta))\otimes\mathbb{Q} \to \pi_1(\mathrm{Symp}_0(S^2\times\mathbb{T}^2,\omega\oplus\eta))\otimes\mathbb{Q} \\ \to (\mathbb{Z}\oplus\mathbb{Z})\otimes\mathbb{Q} \to 0. \end{aligned}$$

We conclude that the dimension of $\pi_1(\mathrm{Ham}(S^2\times\Sigma_g,\omega\oplus\eta))\otimes\mathbb{Q}$ is at least one for $g\geq 1$.

3. The flux morphism

First a word of warning: if G is a group of diffeomorphisms, we will use ψ to denote an element in $\pi_1(G)$ and also to denote a diffeomorphism. It will be clear from the context what it represents.

Let (M,ω) be a closed symplectic manifold and $\psi=\{\psi_t\}_{0\leq t\leq 1}$ be a loop that represents an element of $\pi_1(\mathrm{Symp}_0(M,\omega))$. The isotopy $\{\psi_t\}$ induces a time-dependent vector field X_t given by the equation

$$\frac{d}{dt}\psi_t = X_t\circ\psi_t.$$

Then the flux morphism $\mathrm{Flux}_M : \pi_1(\mathrm{Symp}_0(M,\omega)) \to H^1(M)$ is defined by

$$\mathrm{Flux}_M(\psi) = \int_0^1 [\iota(X_t)\omega]dt.$$

This map is well defined; that is, it depends only on the homotopy class in $\mathrm{Symp}_0(M,\omega)$ based at the identity and is a group morphism. The image of Flux_M is denoted by Γ_M and is called the *flux group* of (M,ω). The rank of Γ_M is bounded by the first Betti number $b_1(M)$ and is a discrete subgroup of $H^1(M)$ (see [7] and [12]). Moreover, the flux morphism fits into the exact sequence of abelian groups

$$0 \to \pi_1(\mathrm{Ham}(M,\omega)) \to \pi_1(\mathrm{Symp}_0(M,\omega)) \to \Gamma_M \to 0, \tag{3.1}$$

where the first map is induced by inclusion and the next one is the flux morphism.

The flux morphism can also be defined on $\mathrm{Symp}_0(M,\omega)$ rather than on its fundamental group. In this case, for a given symplectic diffeomorphism ψ, one considers a symplectic isotopy that joins 1_M with ψ; this will induce a time-dependent vector field X_t as before. As in the previous case we have the map $\mathrm{Flux}_M : \mathrm{Symp}_0(M,\omega) \to H^1(M)/\Gamma_M$. There is also an exact sequence for this morphism,

$$0 \to \mathrm{Ham}(M,\omega) \to \mathrm{Symp}_0(M,\omega) \to H^1(M)/\Gamma_M \to 0, \tag{3.2}$$

where the first map is inclusion and the last one is the flux morphism just defined. Note that if ψ and ϕ are symplectic diffeomorphisms with the same flux, then by the exact sequence (3.2) there is a Hamiltonian diffeomorphism θ such that $\psi = \phi \circ \theta$. This observation will be used later.

Finally, the flux morphism can also be defined for noncompact symplectic manifolds. In this case one considers symplectic diffeomorphisms with compact support, and the flux morphism takes the form $\mathrm{Flux}_M : \pi_1(\mathrm{Symp}_0^c(M,\omega)) \to H_c^1(M)$, and similarly for the flux defined on the group $\mathrm{Symp}_0^c(M,\omega)$. For more details of the flux morphism, see the books of D. McDuff and D. Salamon [11] and of L. Polterovich [14].

Consider two closed symplectic manifolds (M,ω) and (N,η). Then $(M \times N, \omega \oplus \eta)$, where $\omega \oplus \eta$ stands for $\pi_M^*(\omega) + \pi_N^*(\eta)$, is also a symplectic manifold. The map

$$\Psi : \mathrm{Symp}_0(M,\omega) \times \mathrm{Symp}_0(N,\eta) \to \mathrm{Symp}_0(M \times N, \omega \oplus \eta)$$

given by $\Psi(\psi,\phi) = \psi \times \phi$ is a well-defined group homomorphism. It also follows that $\Gamma_M \oplus \Gamma_N$ is a subgroup of $\Gamma_{M\times N}$, so the induced map $i_0 : H^1(M \times N)/\Gamma_M \oplus \Gamma_N \to H^1(M \times N)/\Gamma_{M\times N}$ is surjective. To prove that $\Gamma_{M\times N} \simeq \Gamma_M \oplus \Gamma_N$, it suffices to show that the map i_0 is injective. We can rephrase this in terms of Hamiltonian diffeomorphisms via the exact sequence of the flux morphism.

Lemma 5. *Theorems* 3 *and* 4 *are equivalent.*

Proof. This follows by analyzing the exact sequence (3.2) of the flux morphism. We use the exact sequence (3.2) for the manifolds $(M,\omega), (N,\eta)$ and $(M \times N, \omega \oplus \eta)$ as in the next diagram, where the rows are exact:

$$\begin{array}{ccccccc}
\cdots & \to & \mathrm{Symp}_0(M,\omega) \oplus \mathrm{Symp}_0(N,\eta) & \longrightarrow & H^1(M \times N)/\Gamma_M \oplus \Gamma_N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
\cdots & \to & \mathrm{Symp}_0(M \times N, \omega \oplus \eta) & \longrightarrow & H^1(M \times N)/\Gamma_{M\times N} & \longrightarrow & 0.
\end{array}$$

Here the horizontal maps are $\mathrm{Flux}_M \oplus \mathrm{Flux}_N$ and $\mathrm{Flux}_{M\times N}$, and the vertical maps are Ψ and i_0. So defined, the diagram commutes and the lemma follows. □

This lemma is the link between the theory of bounded symplectic diffeomorphisms and our question about the splitting of the flux group.

4. Bounded symplectic diffeomorphisms

Recall that the group $\mathrm{Ham}(M,\omega)$ of Hamiltonian diffeomorphisms is a normal subgroup of the group $\mathrm{Symp}(M,\omega)$ of symplectic diffeomorphisms. A symplectic diffeomorphism ψ is called *bounded* if the set

$$\{\|[\psi, f]\| : f \in \mathrm{Ham}(M,\omega)\}$$

is bounded. Here $\|\cdot\|$ stands for the Hofer norm on $\mathrm{Ham}(M,\omega)$. A symplectic diffeomorphism is called *unbounded* if it is not bounded. The set of bounded symplectic diffeomorphisms forms a subgroup of $\mathrm{Symp}(M,\omega)$ and is denoted by $\mathrm{BI}(M,\omega)$.

Since for any $\psi \in \mathrm{Ham}(M,\omega)$ and $f \in \mathrm{Ham}(M,\omega)$, we have $\|[\psi,f]\| \le 2\|\psi\|$, every Hamiltonian diffeomorphism is a bounded diffeomorphism. Thus $\mathrm{Ham}(M,\omega)$ is a subgroup of $\mathrm{BI}(M,\omega)$. Define

$$\mathrm{BI}_0(M,\omega) = \mathrm{BI}(M,\omega) \cap \mathrm{Symp}_0(M,\omega).$$

In this section we generalize the work [5] of F. Lalonde and L. Polterovich, in which a fundamental observation was that $\mathrm{BI}_0(\mathbb{T}^2 \setminus \{pt\}, \omega_0) = \mathrm{Ham}^c(\mathbb{T}^2 \setminus \{pt\}, \omega_0)$.

A symplectic diffeomorphism ψ of (M,ω) is called *strongly unbounded* if for every $c>0$ there is an $f\in \mathrm{Ham}(M,\omega)$ such that the lift of $[\psi,f]$ to $\tilde{M}$ disjoins a ball of capacity equal to c from itself. Here $\tilde{M}$ stands for the universal cover of M.

Recall that the universal cover $(\tilde{M},\tilde{\omega})$ of (M,ω) is also a symplectic manifold; moreover the projection map $\pi\colon \tilde{M}\to M$ satisfies $\pi^*(\omega)=\tilde{\omega}$. Let ψ be a Hamiltonian diffeomorphism of (M,ω) and $H_t\colon M\to\mathbb{R}$ a Hamiltonian function whose time-one flow is ψ. Then $H_t\circ\pi$ generates a Hamiltonian flow on $(\tilde{M},\tilde{\omega})$, with time-one map $\tilde{\psi}$. So defined, $\tilde{\psi}$ is a Hamiltonian diffeomorphism that lifts ψ. According to Z. Han [1, Lemma 2.1], every Hamiltonian diffeomorphism has a unique lift to $(\tilde{M},\tilde{\omega})$.

The concepts of strongly unbounded symplectic diffeomorphisms and lifts of Hamiltonian diffeomorphisms are fundamental in the proof of the bounded isometry conjecture. The reason is that by using them, one can get large lower bounds for the Hofer norm. By the energy-capacity inequality, if ψ is a Hamiltonian diffeomorphism such that $\psi(A)\cap A=\emptyset$ for $A\subset M$, then

$$\frac{1}{2}c_G(A)\le e(A)\le \|\psi\|.$$

Here $e(A)$ is the displacement energy of A and $c_G(A)$ is Gromov's capacity of A (see [6]). However, this inequality is not enough for closed symplectic manifolds, since the capacity $c_G(\cdot)$ is bounded from above. Hence we need to pass to the universal cover of the symplectic manifold, since on this open symplectic manifold there are subsets with arbitrarily large capacity.

Proposition 6 (Prop. 1.5A in [5])**.** *If ψ is a Hamiltonian diffeomorphism of (M,ω) whose unique lift $\tilde{\psi}:\tilde{M}\to\tilde{M}$ disjoins a ball of capacity c from itself, then $\|\psi\|\ge c/2$.*

We will show that the bounded isometry conjecture holds for $\mathbb{T}^{2n}\setminus B(\epsilon_0)$. In dimension two this was proved by Lalonde and Polterovich in [5]. Our proof is just an extension of their arguments.

We review a couple of facts of [5] that we need in the proof of the next proposition. In order to have a clear exposition of the arguments, instead of considering diffeomorphisms of $\mathbb{T}^{2n}\setminus B(\epsilon_0)$, we will consider periodic diffeomorphisms of $\mathbb{R}^{2n}$ minus a small ball centered at every point of $\mathbb{Z}^{2n}$. So let a be a small positive number greater than ϵ_0, and for each $(k_1,l_1,\dots,k_n,l_n)\in\mathbb{Z}^{2n}$, consider the small box $\{(x_1,y_1,\dots,x_n,y_n):|x_j-k_j|\le 3a \text{ and } |y_j-l_j|\le 3a\}$. Denote by W the union of all such boxes as the point $(k_1,l_1,\dots,k_n,l_n)$ varies in $\mathbb{Z}^{2n}$. Let $p:\mathbb{R}\to\mathbb{R}$ be any smooth 1-periodic function such that

$$p=\begin{cases} 0 & \text{on } [0,4a-2\epsilon],[4a+2\epsilon,5a-2\epsilon] \text{ and } [5a+2\epsilon,1]\\ -1 & \text{on } [5a-\epsilon,5a+\epsilon]\\ 1 & \text{on } [4a-\epsilon,4a+\epsilon]\\ \text{monotone} & \text{on the remaining subintervals of } [0,1]. \end{cases}$$

Here ϵ is a positive number so small that the definition of p make sense. Finally we also require that

$$\int_0^1 p(s)ds=0.$$

Proposition 7. *For any nonzero v in $H^1_c(\mathbb{T}^{2n} \setminus B(\epsilon_0))/\Gamma_{\mathbb{T}^{2n}\setminus B(\epsilon_0)}$, there is a symplectic diffeomorphism θ with compact support in $\mathbb{T}^{2n} \setminus B(\epsilon_0)$ that is strongly unbounded and with* $\mathrm{Flux}(\theta) = v$. *In particular* $\mathrm{BI}_0(\mathbb{T}^{2n} \setminus B(\epsilon_0), \omega) = \mathrm{Ham}^c(\mathbb{T}^{2n} \setminus B(\epsilon_0), \omega)$.

Proof. Since $H^1_c(\mathbb{T}^{2n} \setminus B(\epsilon_0)) = \mathbb{R}^{2n}$ we can find generators $e_1, f_1, \ldots, e_n, f_n$ of $H^1_c(\mathbb{T}^{2n} \setminus B(\epsilon_0))$ that are dual to the canonical cycles of the torus. Also let $(a_1, b_1, \ldots, a_n, b_n)$ be a $2n$-tuple of nonnegative real numbers, not all of which are zero. We will define $\psi_i, \phi_i \in \mathrm{Symp}^c_0(\mathbb{T}^{2n} \setminus B(\epsilon_0))$ with flux $a_i e_i$ and $b_i f_i$ respectively.

Let a and W be as above. Consider a smooth 1-periodic function $h_{a_j} : \mathbb{R} \to \mathbb{R}$ such that it is equal to zero on $[0, 1/3]$ and $[2/3, 1]$, is positive otherwise, and satisfies

$$a + \int_0^1 h_{a_j}(s)ds = a_j. \tag{4.1}$$

Similarly for each j we have a function h_{b_j} satisfying the same properties with b_j instead of a_j. Then define the symplectic diffeomorphisms ψ_j and ϕ_j of $(\mathbb{R}^{2n}, \omega_0)$ as

$$\psi_j(x_1, y_1, \ldots, x_n, y_n) = (x_1, \ldots, \quad x_j, \quad y_j + a + h_{a_j}(x_j), \quad \ldots, y_n)$$

and

$$\phi_j(x_1, y_1, \ldots, x_n, y_n) = (x_1, \ldots, \quad x_j + a + h_{b_j}(y_j), \quad y_j, \quad \ldots, y_n)$$

outside W and fix the points close to each $B(\epsilon_0)$. The maps ψ_j correspond to the time-one map of the symplectic flow

$$(x_1, y_1, \ldots, x_n, y_n) \mapsto (x_1, \ldots, \quad x_j, \quad y_j + t(a + h_{a_j}(x_j)), \quad \ldots, y_n)$$

and similarly for ϕ_j.

Basically the maps ψ_j and ϕ_j are translations along the y_j-axes and x_j-axes of $\mathbb{R}^{2n}$ respectively. From equation (4.1) it follows that $\mathrm{Flux}(\psi_j) = a_j$ and $\mathrm{Flux}(\phi_j) = b_j$. Then the flux of $\psi_1 \circ \phi_1 \circ \cdots \circ \psi_n \circ \phi_n$ is equal to $a_1 e_1 + b_1 f_1 + \cdots + a_n e_n + b_n f_n$. Recall that a_j and b_j are assumed to be nonnegative. If a_j is zero, we define ψ_j to be the identity diffeomorphism. If a_j is negative, we proceed as above with $-a_j$ instead of a_j and then $\mathrm{Flux}(\psi_j^{-1}) = a_j$.

We claim that the symplectic diffeomorphism $\theta = \psi_1 \circ \phi_1 \circ \cdots \circ \psi_n \circ \phi_n$ is strongly unbounded. To see this, consider the symplectic isotopy

$$f_t(x_1, y_1, \ldots, x_n, y_n) = (x_1, \quad y_1 + tp(x_1), \ldots, x_n, y_n),$$

where $p : \mathbb{R} \to \mathbb{R}$ is the 1-periodic function defined above. Since p vanishes on $[0, 4a - 2\epsilon]$ and $[5a + 2\epsilon, 1]$, each f_t leaves W fixed pointwise. The zero mean condition on p implies that $\{f_t\}$ is a Hamiltonian isotopy. Since f_t commutes with $\phi_1, \quad \psi_2, \ldots, \psi_n$ and ϕ_n but not with ψ_1, we have $[\theta, f_t] = [\psi_1, f_t]$. Note that $[\theta, f_t] = [\psi_1, f_t]$ is the identity on the last $2n - 2$ coordinates of $\mathbb{R}^{2n}$, and in the (x_1, y_1)-plane it corresponds to the symplectic diffeomorphism g_t constructed in [5], that is,

$$[\theta, f_t] = [\psi_1, f_t] = g_t \times 1_{\mathbb{R}^{2n-2}}.$$

Recall from [5] that g_t disjoins a rectangle B_t whose area is a function of t. Therefore $[\theta, f_t](B_t \times \mathbb{R}^{2n-2}) \cap (B_t \times \mathbb{R}^{2n-2}) = \emptyset$. In $\mathbb{R}^2$ the rectangle B_t is symplectomorphic to a disk of the same area. Since the area of B_t goes to infinity as t goes to infinity, by the energy-capacity inequality the Hofer norm of $[\theta, f_t] = [\psi_1, f_t]$ goes to infinity as t goes to infinity. Hence $\theta \in \mathrm{Symp}^c(\mathbb{T}^{2n} \setminus B(\epsilon_0), \omega)$ is strongly unbounded. $\square$

Remark. It is important to note from the proof of Proposition 7 that the (x_1, y_1)-plane of $\mathbb{R}^{2n}$ and the Hamiltonian isotopy $\{f_t\}$ are not related at all to $v \in H^1_c(\mathbb{T}^{2n} \setminus B(\epsilon_0))/\Gamma_{\mathbb{T}^{2n}\setminus B(\epsilon_0)}$. This observation will be useful when we generalize this result to closed symplectic manifolds that satisfy hypothesis *(H)*.

Before we extend the previous result to symplectic manifolds that satisfy *(H)*, we need the following lemma. It will be used in order to show that the strongly unbounded diffeomorphism θ, defined in the proof of Proposition 7, would remain strongly unbounded on (M, ω) and not only on the open manifold $\mathbb{T}^{2n} \setminus B(\epsilon_0)$.

Lemma 8. *Let (M, ω) be a closed manifold with nontrivial $H^1(M)$. Then there is a symplectic embedding of $((0, \epsilon) \times \mathbb{R}, dx \wedge dy)$ into $(\tilde{M}, \tilde{\omega})$, where $\epsilon > 0$ is small.*

Proof. Since $H^1(M)$ is nontrivial, there is an embedding $i : \mathbb{R} \to \tilde{M}$. Moreover, since $\mathbb{R}$ is contractible, the normal bundle ν is isomorphic to $\mathbb{R}^{2n-1} \times \mathbb{R}$. Put the canonical symplectic form on ν. Then there is a symplectic diffeomorphism between a neighborhood of the zero section of ν and a neighborhood of $i(\mathbb{R})$ in $\tilde{M}$. It follows that $(0, \epsilon) \times \mathbb{R}$ embeds symplectically into $\tilde{M}$. □

Since a symplectic diffeomorphism with compact support in U can be thought of as a symplectic diffeomorphism on M, there is a natural map $\tau : \mathrm{Symp}^c_0(U, \omega) \to \mathrm{Symp}_0(M, \omega)$. This gives rise to the commutative diagram $\mathrm{Flux}_M \circ \tau = j_* \circ \mathrm{Flux}_U$, where $j_* : H^1_c(U) \to H^1(M)$. Hence $j_*(\Gamma_U)$ is a subgroup of Γ_M.

Then from the commutative diagram

$$\begin{array}{ccc} \mathrm{Symp}^c_0(U, \omega) & \xrightarrow{\tau} & \mathrm{Symp}_0(M, \omega) \\ \downarrow{\scriptstyle \mathrm{Flux}_U} & & \downarrow{\scriptstyle \mathrm{Flux}_M} \\ H^1_c(U)/\Gamma_U & \xrightarrow{\tilde{j}_*} & H^1(M)/\Gamma_M \end{array}$$

and Proposition 7 we have the following result.

Proposition 9. *Let (M, ω) be a closed symplectic manifold that satisfies hypothesis (H). Then for any nonzero v in $H^1(M)/\Gamma_M$, there is a symplectic diffeomorphism ψ that is strongly unbounded and $\mathrm{Flux}(\psi) = v$. In particular, Theorem 1 holds: $\mathrm{BI}_0(M, \omega) = \mathrm{Ham}(M, \omega)$.*

Proof. Let v in $H^1(M)/\Gamma_M$. Since (M, ω) satisfies *(H)*, we have that $v = v_1 + \cdots + v_l$, where $v_r \in j_{r,*}(H^1_c(U_r))$. For simplicity assume $l = 1$. Thus there is an open set $U \subset M$ that is symplectomorphic to $\mathbb{T}^{2n} \setminus B(\epsilon_0)$ and v_0 in $H^1_c(U)/\Gamma_U$ nonzero such that $\tilde{j}_*(v_0) = v$ under the inclusion map $j : U \to M$. By Proposition 7 there is $\psi \in \mathrm{Symp}^c_0(U, \omega)$ strongly unbounded and flux equal to v_0.

Consider ψ as a symplectic diffeomorphism in $\mathrm{Symp}_0(M, \omega)$. Thus ψ has flux v. It only remains to show that ψ is a strongly unbounded diffeomorphism of (M, ω). Note that ψ is not necessarily strongly unbounded on (M, ω) since $\|\psi\|_U \geq \|\psi\|_M$.

By Lemma 8, we have a symplectic embedding of $(a, b) \times \mathbb{R}$ into $\tilde{M}$, where (a, b) is a small interval. Recall that the symplectic diffeomorphism ψ is the one from the proof of Proposition 7, except that now we consider it on (M, ω). Thus on $(\tilde{M}, \tilde{\omega})$ we have the same symplectic displacement as before. Hence the same arguments of the proof of Proposition 7 apply in this case. Thus ψ is strongly unbounded in $\mathrm{Symp}_0(M, \omega)$. □

Remark. From this result it follows that $\mathrm{BI}_0(\Sigma_g, \omega) = \mathrm{Ham}(\Sigma_g, \omega)$, for $g \geq 1$. The argument presented here is different from the proof that appears in [5]. But still the heart of our argument is the same as their approach, namely Proposition 7.

For completeness we recall the following result, which we will need later. It corresponds to Lemma 4.2 from [4].

Lemma 10. *Consider (M, ω) and (N, η) to be closed symplectic manifolds. If $\psi \in \mathrm{Symp}_0(M, \omega)$ is strongly unbounded, then $\psi \times \phi$ is unbounded for all $\phi \in \mathrm{Symp}_0(N, \eta)$.*

Proof. Let c be a positive real number. Since ψ is strongly unbounded there is a Hamiltonian diffeomorphism h of (M, ω) such that the lift of $[\psi, h]$ to $\tilde{M}$ disjoins a ball $B^{2n}(c_0)$ of capacity c. Note that $[\psi \times \phi, h \times 1_N] = [\psi, h] \times 1_N$, so the lift $[\psi, h]\tilde{} \times 1_N \ : \tilde{M} \times N \to \tilde{M} \times N$ disjoins $B^{2n}(c_0) \times N$.

Thus by the stable version of the energy-capacity inequality of Lalonde and Pestieau [4], we get

$$c/2 = c(B^{2n}(c_0))/2 \leq e(B^{2n}(c_0) \times N) \leq \|[\psi, h]\tilde{} \times 1_N\| \leq \|[\psi, h] \times 1_N\|.$$

Therefore,

$$\|[\psi \times \phi, h \times 1_N]\| = \|[\psi, h] \times 1_N\| \geq c/2$$

with $h \times 1_N$ a Hamiltonian diffeomorphism. Hence $\psi \times \phi$ is unbounded. □

With this result at hand we can prove the following generalization of [5, Theorem 1.3.C] and of [4, Lemma 4.3].

Theorem 11. *Let (M, ω) be a symplectic manifold that satisfies hypothesis (H), and let (N, η) be any closed symplectic manifold. If $\psi \times \phi \in \mathrm{Symp}_0(M \times N)$ is a bounded symplectic diffeomorphism, then ψ is a Hamiltonian diffeomorphism of (M, ω).*

Proof. Assume that ψ is not a Hamiltonian diffeomorphism. By Theorem 1 we have $\mathrm{BI}_0(M, \omega) = \mathrm{Ham}(M, \omega)$, so ψ is an unbounded symplectic diffeomorphism. Let $v \in H^1(M)/\Gamma_M$ be the flux of ψ. Since v is nonzero, it follows from Proposition 9 that there is $\psi_0 \in \mathrm{Symp}_0(M, \omega)$ that is strongly unbounded and has flux equal to v.

Therefore there exists a Hamiltonian diffeomorphism α of (M, ω) such that $\psi = \psi_0 \circ \alpha$. Hence by Lemma 10 we have that $\psi_0 \times \phi$ is unbounded. Hence also $\psi \times \phi$ is unbounded, which is a contradiction. Therefore ψ is a Hamiltonian diffeomorphism. □

5. Proof of the main results

The proof of Theorem 1 follows from Proposition 9, where we showed that for any v in $H^1(M)/\Gamma_M$ there is an unbounded symplectic diffeomorphism with flux v.

Proof of Corollary 2. By the flux exact sequence we have for any $\psi \in \mathrm{Symp}_0(M \times N, \omega \oplus \eta)$ that there exist $\theta \in \mathrm{Ham}(M \times N, \omega \oplus \eta)$, $\psi_1 \in \mathrm{Symp}_0(M, \omega)$ and $\psi_2 \in \mathrm{Symp}_0(N, \eta)$ such that $\psi = \theta \circ (\psi_1 \times \psi_2)$. Now if ψ is bounded it follows that $\psi_1 \times \psi_2$ is also bounded. Now by Theorem 11, we have that each one of them is a Hamiltonian diffeomorphism; hence $\psi_1 \times \psi_2$ and ψ are also Hamiltonian. □

Proof of Theorem 3. First note that since (M, ω) and (N, η) satisfy *(H)*, by Corollary 2 we have $\mathrm{BI}_0(M \times N, \omega \oplus \eta) = \mathrm{Ham}(M \times N, \omega \oplus \eta)$.

Consider $\psi \in \mathrm{Symp}_0(M, \omega)$ and $\phi \in \mathrm{Symp}_0(N, \eta)$ such that $\psi \times \phi$ is a Hamiltonian diffeomorphism. Thus $\psi \times \phi$ is a bounded symplectic diffeomorphism. Hence from Theorem 11 we have that ψ and ϕ are Hamiltonian diffeomorphisms as well. Therefore from Lemma 5, we have that the flux group of $(M \times N, \omega \oplus \eta)$ splits. $\square$

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