

ON THE WAVE OPERATOR FOR THE GENERALIZED BOUSSINESQ EQUATION

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ABSTRACT. We study the asymptotic behavior of solutions for the generalized Boussinesq equation in a singular framework. We construct a wave operator (inverse scattering) for large profiles \vec{h} belonging to an infinite mass framework based on weak L^p -spaces. Our solutions converge towards the prescribed scattering state with a polynomial rate.

1. INTRODUCTION

In this work we concern ourselves with the generalized Boussinesq equation

$$(1.1) \quad \begin{aligned} u_{tt} - \Delta u + \Delta^2 u + \Delta f(u) &= 0, \quad x \in \mathbb{R}^n, t > 0, \\ u(x, 0) = \phi, \quad u_t(x, 0) &= \psi, \end{aligned}$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(0) = 0$ and there is $C_f > 0$ such that

$$(1.2) \quad |f(a) - f(b)| \leq C_f |a - b| (|a|^{\rho-1} + |b|^{\rho-1}), \quad \forall a, b \in \mathbb{R},$$

for a given $1 < \rho < \infty$. A typical example of (1.2) is $f(u) = u|u|^{\rho-1}$. The equation (1.1) is equivalent to the system of equations

$$(1.3) \quad \begin{cases} u_t = \Delta v, \\ v_t = u - \Delta u - f(u), \end{cases} \quad x \in \mathbb{R}^n, t > 0.$$

We can also write the integral equation for the above system. First, recall that the solution of the linear system associated to (1.3) with initial data $\vec{u}_0 = (u_{0,1}, u_{0,2})$ is given by

$$(1.4) \quad B(t)\vec{u}_0 = \int_{-\infty}^{+\infty} e^{ix\xi} \begin{pmatrix} \cos(t|\xi|\langle\xi\rangle) & -\frac{|\xi|}{\langle\xi\rangle} \sin(t|\xi|\langle\xi\rangle) \\ \frac{\langle\xi\rangle}{|\xi|} \sin(t|\xi|\langle\xi\rangle) & \cos(t|\xi|\langle\xi\rangle) \end{pmatrix} \widehat{\vec{u}}_0(\xi) d\xi,$$

where $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$ and $\widehat{\vec{u}}_0 = (\widehat{u}_{0,1}, \widehat{u}_{0,2})$. In view of Duhamel principle, solutions of (1.1) and (1.3) can be rewritten as

$$(1.5) \quad \vec{u}(t) = B(t)\vec{u}_0 + \int_0^t B(t-t')\vec{f}(\vec{u}(t'))dt',$$

where $\vec{f}(\vec{u}) = (0, f(u))$ and $\vec{u} = (u_1, u_2) = (u, v)$.

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This kind of equation occurs in a large number of physical situations, which has motivated their study through physical and mathematical approaches. Initially, in one space dimension and with opposite sign in the term $\Delta^2 u$, equation (1.1) was employed by Boussinesq [3] to model nonlinear shallow water waves. Later on, further applications appeared by describing dynamics of stretched string, shape-memory alloys, ion-sound waves, and many others (see e.g. [7, 4, 16]).

The issues of existence and asymptotic behavior of solutions for the generalized Boussinesq equation (1.1) have been addressed by several authors. In the framework of Sobolev spaces, we refer the reader to [2, 13, 18, 9, 10] for local or global well-posedness results and [14] for small scattering theory. In [4, 15], the authors proved existence results by employing certain Besov spaces with positive regularity. In these two works, a scattering theory for small solutions was also developed together with some decay rates for the norms involved. The reader is also referred to [5, 17, 19] for existence and scattering results in Sobolev spaces involving other dispersive systems connected to (1.1). Next we briefly explain the scattering problem. Given a nonlinear PDE, the problem consists of finding suitable V such that the solution $u_V(t)$ of the linear problem associated to this PDE with initial data V describes the asymptotic behavior of its global solution with small data ϕ and ψ ; precisely, $\lim_{t \rightarrow +\infty} \|u(t) - u_V(t)\|_X = 0$, where X is a Banach space.

Another interesting question studied for dispersive models is the reciprocal problem of the scattering theory, namely *inverse scattering*, which consists of constructing a solution with a prescribed scattering state. More roughly, for a given profile V (regardless of its size), one looks for a solution of the nonlinear problem $u(t)$, defined for large enough t , such that

$$\lim_{t \rightarrow \infty} \|u(t) - u_V(t)\|_Y = 0,$$

where $u_V(t)$ is the solution of the linear problem with initial data V and Y stands for a suitable Banach space. Solving the latter problem is also known as the construction of a wave operator. Inverse scattering was studied in Sobolev or weighted Sobolev spaces for Schrödinger equations (see e.g. [11, 12]) and for generalized Korteweg-de Vries equations in [6]. By using the integral formulation (1.5), the first author [8] studied inverse scattering through a H_p^s -framework ($s > 0$).

In the above mentioned papers, although the smallness assumption on the profile V has been removed, finite L^p -norm and some positive regularity on V are necessary to obtain some scattering or inverse scattering results. The reason comes from the needed estimates to perform a contraction argument in the corresponding frameworks.

In this paper, our aim is to construct a large data wave operator for the generalized Boussinesq equation (1.1) by means of the integral formulation (1.5) on weak L^p -type spaces. In this kind of space the elements, in general, have infinite L^p -norm, for all $1 \leq p \leq \infty$, and particularly infinite mass, that is $\|\cdot\|_{L^2} = \infty$. Therefore, our framework does not make use of positive regularity, and it contains interesting singular functions, including some with infinite local mass (see Remark 2.3 below). Our main result, given in Theorem 2.2, extends the results in [8] in the sense that the admissible profile class \mathcal{I}_σ^T is strictly larger; see Remark 2.2 below for details. Our solutions go towards the prescribed scattering states with polynomial-time rates.

The approach used here also generalizes those employed in [8, 6] in the sense that it is easy to see that the family of linear operators $\{B(t)\}_{t \in \mathbb{R}}$ given in (1.4) verifies (at least formally) the group property with operation being the composition. Moreover, in view of Lemma 3.1 below, we are dealing with a framework in which the linear group $B(t)$ is not unitary; that is, it does not preserve weak L^p -norms. In the previously mentioned papers the authors dealt with certain Besov and Sobolev norms in which $B(t)$ is a unitary group.

The plan of this paper is the following: In Section 2, we introduce some notation, review facts about Lorentz spaces, and state our results. We provide some linear estimates in Section 3. Finally, the proofs of the results are given in Section 4.

2. NOTATION AND MAIN RESULTS

We start by recalling some definitions and properties on Lorentz spaces. For more details we refer the reader to [1]. The Lorentz space $L^{(p,q)}$ is a Banach space of measurable functions endowed with the norm $\|\cdot\|_{(p,q)}$ given by

$$(2.1) \quad \|f\|_{(p,q)} = \begin{cases} \left(\frac{p}{q} \int_0^\infty \left[t^{\frac{1}{p}} f^{**}(t) \right]^q \frac{dt}{t} \right)^{\frac{1}{q}}, & \text{if } 1 < p < \infty, 1 \leq q < \infty, \\ \sup_{t>0} t^{\frac{1}{p}} f^{**}(t), & \text{if } 1 < p \leq \infty, q = \infty, \end{cases}$$

and

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds \text{ and } f^*(t) = \inf\{s > 0 : m\{x \in \mathbb{R}^n : |f(x)| > s\} \leq t\},$$

where m stands for the Lebesgue measure in \mathbb{R}^n . The weak L^p -spaces correspond to the special case $q = \infty$ (i.e. $L^{(p,\infty)}$) and they are also known as Marcinkiewicz spaces. In view of the continuous inclusions

$$(2.2) \quad L^{(p,1)} \subset L^{(p,q_1)} \subset L^{(p,p)} = L^p \subset L^{(p,q_2)} \subset L^{(p,\infty)},$$

for all $1 \leq q_1 \leq p \leq q_2 \leq \infty$, $L^{(p,\infty)}$ is the largest space in the family $\{L^{(p,q)}\}_{1 \leq q \leq \infty}$. The Holder inequalities also hold in this framework, namely

$$(2.3) \quad \|h\|_{(r,s)} \leq C(r) \|f\|_{(p_1,q_1)} \|g\|_{(p_2,q_2)},$$

where $1 < p_1, p_2 < \infty$, $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$, $s \geq 1$ and $\frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{s}$. From the interpolation point of view, we have $L^{(p,q)} = (L^1, L^\infty)_{1-\frac{1}{p}, q}$ with $1 < p < \infty$ and $1 \leq q \leq \infty$. Moreover, it follows from the Reiteration Theorem that

$$(L^{(p_0,q_0)}, L^{(p_1,q_1)})_{\theta,q} = L^{(p,q)},$$

provided that $1 < p_0 < p_1 < \infty$, $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $1 \leq q_0, q_1, q \leq \infty$.

To state our results we also need to define the following product spaces based on the Lorentz spaces.

Definition 2.1. Let $D^s = (-\Delta)^{s/2}$ and $J^s = (1 - \Delta)^{s/2}$. Consider the Banach space

$$(2.4) \quad X_{(p,d)} = L^{(p,d)} \times D^{-1}J(L^{(p,d)})$$

endowed with the norm

$$\|[g_1, g_2]\|_{X_{(p,d)}} = \max\{\|g_1\|_{(p,d)}, \|DJ^{-1}g_2\|_{(p,d)}\}.$$

Given $0 \leq \sigma < \infty$ and $0 < T < \infty$, define \mathcal{E}_σ^T to be the space of Bochner measurable vectors $\vec{u}(t) = (u_1(t), u_2(t)) \vec{u} : (-\infty, \infty) \rightarrow X_{(\rho+1, \infty)}$ with norm

$$(2.5) \quad \|\vec{u}\|_{\mathcal{E}_\sigma^T} \equiv \sup_{t>T} t^\sigma \|\vec{u}(\cdot, t)\|_{X_{(\rho+1, \infty)}}.$$

We also define the profile space \mathcal{I}_σ^T as the set of vectors $\vec{h} = (h_1, h_2) \in \mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n)$ satisfying

$$(2.6) \quad \|\vec{h}\|_{\mathcal{I}_\sigma^T} = \sup_{t>T} t^\sigma \|B(t)\vec{h}\|_{X_{(\rho+1, \infty)}} < \infty.$$

Henceforth $\gamma(n) = \infty$ if $n = 1, 2$ and $\gamma(n) = \frac{n+2}{n-2}$, otherwise. This parameter seems to have a universal role and appears frequently in the literature related to the Sobolev critical exponent.

Furthermore, we denote $\beta = n(\rho - 1)/2(\rho + 1)$, $\alpha = 1/(\rho - 1) - n/2(\rho + 1)$, and $\rho_0(n) = \frac{n+2+\sqrt{n^2+12n+4}}{2n}$ the positive root of $n\rho^2 - (n + 2)\rho - 2 = 0$. Notice that $\rho_0(n) < \rho$ is equivalent to $\alpha < \beta$.

Now we can give a precise statement of our results.

Theorem 2.1. *Let $\rho_0(n) < \rho < \gamma(n)$ and $\alpha < \sigma < \beta$. For any $\vec{h} = (h_1, h_2) \in L^{(\frac{\rho+1}{\rho}, \infty)} \times L^{(\frac{\rho+1}{\rho}, \infty)}$, there exist $T_0 = T_0(\vec{h}) > 0$ and a solution $\vec{u} \in \mathcal{E}_\sigma^{T_0}$ of the integral equation (1.5) such that*

$$(2.7) \quad \lim_{t \rightarrow \infty} t^\sigma \|\vec{u}(t) - B(t)\vec{h}\|_{X_{(\rho+1, \infty)}} = 0.$$

Moreover, \vec{u} is the unique solution such that

$$\sup_{t>T} t^\sigma \|\vec{u}(t)\|_{X_{(\rho+1, \infty)}} < \infty, \text{ for some } T \geq T_0.$$

Actually, a slight modification on the above theorem allows us to relax the hypotheses on ρ, σ and \vec{h} and to improve the decay given in (2.7). This is the content of the following theorem.

Theorem 2.2. *Let $1 < \rho < \gamma(n)$, $\alpha < \sigma < \infty$, and $\tilde{T}_0 > 0$. For any $\vec{h} = (h_1, h_2) \in \mathcal{I}_\sigma^{\tilde{T}_0}$ there exist $T_0 = T_0(\vec{h}) \geq \tilde{T}_0$ and a solution $\vec{u} \in \mathcal{E}_\sigma^{T_0}$ of the integral equation (1.5) such that*

$$(2.8) \quad \lim_{t \rightarrow \infty} t^\sigma \|\vec{u}(t) - B(t)\vec{h}\|_{X_{(\rho+1, \infty)}} = 0.$$

Moreover, assume that $\vec{v} \in \mathcal{E}_\sigma^{T_0}$ is another solution with profile $\vec{h} \in \mathcal{I}_\sigma^{\tilde{T}_0}$. Then $\vec{u} = \vec{v}$ in $[T, \infty)$, for some $T \geq T_0$.

Below we compare the decay rates obtained in the above theorems with the decay rate of the free Boussinesq equation.

Remark 2.1. Recall that the solution of the free Boussinesq equation with data $[u_0, v_0]$ is given by $B(t)[u_0, v_0]$. For an arbitrary $\vec{h} \in L^{(\frac{\rho+1}{\rho}, \infty)} \times L^{(\frac{\rho+1}{\rho}, \infty)}$, it follows from Lemma 3.1 below that $\|B(t)[u_0, v_0]\|_{X_{(\rho+1, \infty)}} = o(t^{-\sigma})$ holds true only in the case $\sigma < \beta$. Indeed, the latter decay is not verified when $\sigma = \beta$ and \vec{h} is as in Remark 2.3. Therefore, in view of Theorem 2.1, the solution $\vec{u}(t)$ goes to $B(t)\vec{h}$ with the same decay rate of solutions of the free Boussinesq equation. On the other hand, in Theorem 2.2 the constraint $\sigma < \beta$ has been removed by taking the profile $\vec{h} \in \mathcal{I}_\sigma^{\tilde{T}_0}$. However, from an easy-handling point of view, the characterization of elements of $L^{(\frac{\rho+1}{\rho}, \infty)} \times L^{(\frac{\rho+1}{\rho}, \infty)}$ is better than that of $\mathcal{I}_\sigma^{\tilde{T}_0}$.

In the sequel we compare our profile class with that of [8].

Remark 2.2. Recall the notation $\dot{B}_{p,q}^s$ for homogeneous Besov spaces and define the operator $(D_1(\psi))^\wedge = \left(|\xi| \langle \xi \rangle^{-1}\right)^{\frac{(n-2)}{2}} \widehat{\psi}$. The author of [8] considered profiles $\vec{h} \in Y_2^1 \cap Y_{\frac{\rho+1}{\rho}}^1$ with $\rho > 2 + \sqrt{7}$ for $n = 1$, and $\vec{h} \in \mathfrak{B}_{\frac{\rho+1}{\rho},2}^{\frac{\rho-1}{\rho},s}$ with $\rho_0(n) < \rho < \gamma(n)$ for $n > 1$, where $Y_p^s = H_p^s \times D^{-1}J(H_p^s)$, $\mathfrak{B}_{p,q}^s = N_{p,q}^s \times D^{-1}J(N_{p,q}^s)$ and $N_{p,q}^s = (L^p) \cap D_1^{-1}(\dot{B}_{p,q}^s)$. Taking $\alpha < \sigma \leq \beta$, our profile class contains that of [8], since

$$(2.9) \quad Y_{\frac{\rho+1}{\rho}}^1, \mathfrak{B}_{\frac{\rho+1}{\rho},2}^{\frac{\rho-1}{\rho},s} \subset L^{\frac{\rho+1}{\rho}} \times D^{-1}J(L^{\frac{\rho+1}{\rho}}) \subset \mathcal{I}_\sigma^T.$$

Indeed, using the inclusions $L^{\rho+1} \subset L^{(\rho+1,\infty)}$ and $L^{\rho+1} \times D^{-1}J(L^{\rho+1}) \subset X_{(\rho+1,\infty)}$, and afterwards (3.7), we obtain

$$\begin{aligned} \|\vec{h}\|_{\mathcal{I}_\sigma^T} &= \sup_{t>T} t^\sigma \|B(t)\vec{h}\|_{X_{(\rho+1,\infty)}} \leq C \sup_{t>T} t^\sigma \|B(t)\vec{h}\|_{L^{\rho+1} \times D^{-1}J(L^{\rho+1})} \\ &\leq C \left(\sup_{t>T} t^{\sigma-\beta} \right) \max\{\|h_1\|_{L^{\frac{\rho+1}{\rho}}}, \|h_2\|_{L^{\frac{\rho+1}{\rho}}}\} \leq C \|\vec{h}\|_{\mathfrak{B}_{\frac{\rho+1}{\rho},2}^{\frac{\rho-1}{\rho},s}}, \end{aligned}$$

which yields (2.9). A similar analysis holds for the space $Y_{\frac{\rho+1}{\rho}}^1$.

In the next remark we provide examples of infinite mass profile \vec{h} covered by our class defined in (2.6)

Remark 2.3 (Singular profiles with infinite mass). In view of (3.1), if $\alpha < \sigma \leq \beta$, then the continuous inclusion $L^{(\frac{\rho+1}{\rho},\infty)} \times L^{(\frac{\rho+1}{\rho},\infty)} \subset \mathcal{I}_\sigma^{T_0}$ holds true for any $T_0 > 0$. Hence (in particular) the space $L^{(\frac{\rho+1}{\rho},\infty)} \times L^{(\frac{\rho+1}{\rho},\infty)}$ is an admissible profile class for \vec{h} in Theorem 2.2 and contains singular profiles $\vec{h} \notin L_{loc}^2$ and $\vec{h} \notin L^p$ for all $1 \leq p \leq \infty$. To see this, let ν_i, η_j be constants satisfying $\sum_{i=1}^\infty |\nu_i| + \sum_{j=1}^\infty |\eta_j| < \infty$, and consider $\vec{h} = (h_1, h_2)$ such that

$$(2.10) \quad h_1 = \sum_{i=1}^\infty \nu_i P_{k,i}(x - x_i) |x - x_i|^{-k - \frac{n\rho}{\rho+1}} \quad \text{and} \quad h_2 = \sum_{j=1}^\infty \eta_j Q_{s,j}(x - \tilde{x}_j) |x - \tilde{x}_j|^{-s - \frac{n\rho}{\rho+1}},$$

where $x_i, \tilde{x}_j \in \mathbb{R}^n$ and $P_{k,i}(x), Q_{s,j}(x)$ are homogeneous polynomials with degrees k, s , respectively.

Remark 2.4 (Extension to $t = 0$). Due to the high singularity of the spaces that we deal with, it is expected that blow-up will occur when extending the solution from T to $t = 0$. The usual way to perform such an extension is to handle suitable conservative quantities which are available only on Sobolev or Lebesgue spaces. In fact, there are no known conservative quantities involving weak L^p -norms, and it seems to be improbable to obtain those quantities, or, at least, it would be a very difficult task.

In the next theorem, we obtain a stronger convergence than those given in (2.7) and (2.8) by taking \vec{h} in a more restricted space. Note that $X_{(\rho+1,d)} \subset X_{(\rho+1,\infty)}$ for all $1 \leq d \leq \infty$ in view of (2.2).

Theorem 2.3. *Assume that $1 \leq d \leq \infty$ and either the hypotheses of Theorem 2.1 or Theorem 2.2 hold. There exists $T \geq T_0$ such that if*

$$(2.11) \quad \sup_{t>T} t^\sigma \left\| B(t)\vec{h} \right\|_{X_{(\rho+1,d)}} < \infty,$$

then the previous solution $\vec{u} \in \mathcal{E}_\sigma^{T_0}$ of the integral equation (1.5) satisfies

$$(2.12) \quad \lim_{t \rightarrow \infty} t^\sigma \|\vec{u}(t) - B(t)\vec{h}\|_{X_{(\rho+1,d)}} = 0.$$

In particular, condition (2.11) holds true when $\vec{h} \in L^{(\frac{\rho+1}{\rho},d)} \times L^{(\frac{\rho+1}{\rho},d)}$ and $\alpha \leq \sigma \leq \beta$.

Assuming further regularity on \vec{h} , we can also obtain convergence of the derivatives of \vec{u} to those of the scattering state $B(t)\vec{h}$. Precisely, we have the following remark.

Remark 2.5. Let $[\rho]$ be the integer part of ρ , $f \in C^{[\rho]}(\mathbb{R})$, $m \in \mathbb{N}$, and let $k \in (\mathbb{N})^n$ be a multi-index. Let ∇_x^k denote derivatives of order $|k|$. Assume also that $|f^{(l)}(v)| \lesssim |v|^{\rho-l}$ for all integers l such that $0 \leq l \leq \rho$. Let $\mathcal{E}_{\sigma,m}^T$ be the Banach space of Bochner measurable vectors with norm

$$(2.13) \quad \|\vec{u}\|_{\mathcal{E}_{\sigma,m}^T} \equiv \max_{|k| \leq m} \sup_{t>T} t^\sigma \|\nabla_x^k \vec{u}(x,t)\|_{X_{(\rho+1,\infty)}},$$

and $\mathcal{I}_{\sigma,m}^T$ be the set of $\vec{h} \in \mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n)$ satisfying

$$(2.14) \quad \|\vec{h}\|_{\mathcal{I}_{\sigma,m}^T} = \max_{|k| \leq m} \sup_{t>T} t^\sigma \|\nabla_x^k B(t)\vec{h}\|_{X_{(\rho+1,\infty)}} < \infty.$$

Assuming $\vec{h} \in \mathcal{I}_{\sigma,m}^{\tilde{T}_0}$ instead of $\mathcal{I}_{\sigma,0}^{\tilde{T}_0} = \mathcal{I}_\sigma^{\tilde{T}_0}$ and adapting the proofs of Theorems 2.1 and 2.2, one can prove that the previous solution \vec{u} belongs to $\mathcal{E}_{\sigma,m}^{T_0}$ for some $T \geq \tilde{T}_0$, and

$$\lim_{t \rightarrow \infty} t^\sigma \|\nabla_x^k \vec{u}(t) - \nabla_x^k B(t)\vec{h}\|_{X_{(\rho+1,\infty)}} = 0, \text{ for all } |k| \leq m,$$

provided that either ρ is an integer or $|k| \leq \rho - 1$.

3. LINEAR ESTIMATES

In the next lemma we state some linear estimates for the group $B(\cdot)$ defined in (1.4) on Lorentz spaces. From now on, the notation $C > 0$ will denote generic positive constants which may change line by line or even within the same line. We denote by $\|\vec{h}\|_{(p,d)} = \max\{\|h_1\|_{(p,d)}, \|h_2\|_{(p,d)}\}$ the norm in $L^{(p,q)} \times L^{(p,q)}$. In the next lemma we state some linear estimates for the group $B(\cdot)$ defined in (1.4) on Lorentz spaces.

Lemma 3.1. *Let $1 \leq d \leq \infty$, $1 < p \leq 2$, and p' such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then there exists a positive constant $C = C(n, p)$ such that*

$$(3.1) \quad \|B(t)\vec{h}\|_{X_{(p',d)}} \leq C|t|^{-\frac{n}{2}(\frac{2}{p}-1)} \|\vec{h}\|_{(p,d)},$$

for all $\vec{h} \in L^{(p,q)} \times L^{(p,q)}$ and $t \neq 0$.

Proof. Let us first define the multiplier operators $l_1(t)$ and $l_2(t)$ with symbols $\cos(t|\xi| \langle \xi \rangle)$ and $\sin(t|\xi| \langle \xi \rangle)$, respectively. Recall the $L^r - L^{r'}$ estimate (see e.g. [4])

$$(3.2) \quad \|l_i(t)\varphi\|_{L^{r'}} \leq C |t|^{-\frac{n}{2}(\frac{2}{r}-1)} \|\varphi\|_{L^r}, \quad i = 1, 2$$

and the notation $\vec{h} = (h_1, h_2)$. We express $B(t)\vec{h} = (\pi_1 B(t)\vec{h}, \pi_2 B(t)\vec{h})$, where π_i is the i -th projection, $i = 1, 2$. It follows from definition (1.4) and inequality (3.2) that

$$(3.3) \quad \left\| \pi_1 B(t)\vec{h} \right\|_{L^{p'}} \leq C |t|^{-\frac{n}{2}(\frac{2}{p}-1)} \max\{\|h_1\|_{L^p}, \|h_2\|_{L^p}\} \text{ for all } t \neq 0,$$

where $C > 0$ depends only on n, p . Again by definition (1.4), we have

$$DJ^{-1}\pi_2 B(t)\vec{h} = l_2(t)h_1 + DJ^{-1}l_1(t)h_2.$$

Moreover, by L^r -continuity of the operator DJ^{-1} and (3.2), it follows that

$$(3.4) \quad \begin{aligned} \left\| \pi_2 B(t)\vec{h} \right\|_{D^{-1}J(L^{p'})} &= \left\| DJ^{-1}\pi_2 B(t)\vec{h} \right\|_{L^{p'}} \leq \|l_2(t)h_1\|_{L^{p'}} + \|DJ^{-1}l_1(t)h_2\|_{L^{p'}} \\ &\leq \|l_2(t)h_1\|_{L^{p'}} + \|l_1(t)h_2\|_{L^{p'}} \\ &\leq C |t|^{-\frac{n}{2}(\frac{2}{p}-1)} (\|h_1\|_{L^p} + \|h_2\|_{L^p}). \end{aligned}$$

Now, the estimates (3.3) and (3.4) and a real interpolation argument yield

$$(3.5) \quad \left\| \pi_1 B(t)\vec{h} \right\|_{(p',d)} \leq C |t|^{-\frac{n}{2}(\frac{2}{p}-1)} \max\{\|h_1\|_{(p,d)}, \|h_2\|_{(p,d)}\},$$

$$(3.6) \quad \left\| \pi_2 B(t)\vec{h} \right\|_{D^{-1}J(L^{(p',d)})} \leq C |t|^{-\frac{n}{2}(\frac{2}{p}-1)} \max\{\|h_1\|_{(p,d)}, \|h_2\|_{(p,d)}\}$$

for all $t \neq 0$ and $\vec{h} \in L^{(p,d)} \times L^{(p,d)}$. The inequalities (3.5) and (3.6) together are equivalent to (3.1). \square

Remark 3.1. For $1 < p \leq 2$, notice that estimates (3.3) and (3.4) together imply

$$(3.7) \quad \|B(t)\vec{h}\|_{X_{(p',p')}} \leq C |t|^{-\frac{n}{2}(\frac{2}{p}-1)} \|\vec{h}\|_{(p,p)}.$$

We will also need the following numerical result.

Lemma 3.2. *Let $0 < T < \infty$, $1 < \rho < \gamma(n)$, $\beta = n(\rho - 1)/2(\rho + 1)$, $\alpha = 1/(\rho - 1) - n/2(\rho + 1)$ and $\alpha \leq \sigma < \infty$. Denote*

$$I_\sigma(T) = \sup_{t \geq T} t^\sigma \int_t^\infty |t - t'|^{-\beta} (t')^{-\sigma\rho} dt'$$

and

$$H_\sigma(T) = \sup_{t \geq T} t^{\sigma-\beta}.$$

Then $I_\sigma(T) < \infty$, and we have that

$$(3.8) \quad I_\sigma(T) \rightarrow 0 \text{ as } T \rightarrow \infty, \quad \text{for } \alpha < \sigma < \infty,$$

and

$$(3.9) \quad \sup_{T \geq 1} H_\sigma(T) \leq 1, \quad \text{for } \sigma < \beta.$$

Proof. First, it is clear that $1 < \rho < \gamma(n)$ implies $\beta < 1$ and $\alpha > 0$. The change of variable $t' = st$ yields

$$(3.10) \quad t^\sigma \int_t^\infty |t-t'|^{-\beta} (t')^{-\sigma\rho} dt' = t^{\sigma+1-\sigma\rho-\beta} \int_1^\infty (s-1)^{-\beta} (s)^{-\sigma\rho} ds.$$

Note that

$$(3.11) \quad \sigma + 1 - \sigma\rho - \beta \leq \alpha + 1 - \alpha\rho - \beta = 0$$

with strict inequality when $\alpha < \sigma$. On the other hand,

$$(3.12) \quad \int_1^\infty (s-1)^{-\beta} (s)^{-\sigma\rho} ds \leq \int_1^2 (s-1)^{-\beta} ds + \left(\int_2^\infty s^{-\beta-\sigma\rho} ds \right) \sup_{s>2} \left(\frac{s}{s-1} \right)^\beta < \infty,$$

since $\beta < 1$ and $\beta + \sigma\rho \geq \beta + \alpha\rho = 1 + \alpha > 1$. Relations (3.10)–(3.12) immediately yield (3.8).

On the other hand, inequality (3.9) follows easily from the relation $\sigma < \beta$. \square

4. PROOFS OF THEOREMS

In the spirit of inverse scattering theory we must look for a fixed point for the operator

$$\Phi : \vec{w}(t) \longrightarrow - \int_t^\infty B(t-t') \vec{f}(\vec{w}(t')) + B(t') \vec{h} dt'.$$

For the sake of completeness, in the next proposition we prove that this fixed point provides a function \vec{u} satisfying the integral equation (1.5) (see also [8]).

Proposition 4.1. *Let \vec{w} be a fixed point of the operator Φ , and define*

$$(4.1) \quad \vec{u}(t) \equiv B(t) \vec{h} + \vec{w}(t).$$

Then \vec{u} is a solution of (1.5) in the time interval $[T_0, \infty)$.

Proof. Assume that $\vec{w}(t) = - \int_t^\infty B(t-t') \vec{f}(\vec{w}(t')) + B(t') \vec{h} dt'$. It follows from (4.1) and the group property satisfied by $B(t)$ that

$$(4.2) \quad \begin{aligned} B(T_0-t) \vec{w} &= - \int_t^\infty B(T_0-t') \vec{f}(\vec{w}(t')) + B(t') \vec{h} dt' \\ &= \vec{w}(T_0) + \int_{T_0}^t B(T_0-t') \vec{f}(\vec{w}(t')) + B(t') \vec{h} dt' \\ &= \vec{w}(T_0) + \int_{T_0}^t B(T_0-t') \vec{f}(\vec{u}(t')) dt'. \end{aligned}$$

An application of the operator $B(t-T_0)$ in the equality (4.2) yields

$$(4.3) \quad \vec{w}(t) = B(t-T_0) \vec{w}(T_0) + \int_{T_0}^t B(t-t') \vec{f}(\vec{u}(t')) dt'.$$

Next, we add the term $B(t) \vec{h}$ on both sides of (4.3) and afterwards use the group property to obtain

$$(4.4) \quad \begin{aligned} B(t) \vec{h} + \vec{w}(t) &= B(t) \vec{h} + B(t-T_0) \vec{w}(T_0) + \int_{T_0}^t B(t-t') \vec{f}(\vec{u}(t')) dt' \\ &= B(t-T_0) [B(T_0) \vec{h} + \vec{w}(T_0)] + \int_{T_0}^t B(t-t') \vec{f}(\vec{u}(t')) dt'. \end{aligned}$$

Employing (4.1) again, the equality (4.4) becomes

$$\vec{u}(t) = B(t - T_0)\vec{u}(T_0) + \int_{T_0}^t B(t - t')\vec{f}(\vec{u}(t'))dt',$$

which is equivalent to the integral equation (1.5). □

Proof of Theorem 2.1. Let $a > 0$ be a fixed real number and define the following closed subset of a complete metric space:

$$X_T(0, a) = \left\{ \vec{w} \in \mathcal{E}_\sigma^T : \|\vec{w}\|_{\mathcal{E}_\sigma^T} \equiv \sup_{t \geq T} t^\sigma \|\vec{w}(t)\|_{X_{(\rho+1, \infty)}} \leq a \right\},$$

where $T > 0$ will be chosen later. We wish to find $T > 0$ such that Φ is a contraction mapping from $X_T(0, a)$ to itself in the $\|\cdot\|_{\mathcal{E}_\sigma^T}$ -metric. Writing $\vec{v}(t) = (v_1(t), v_2(t)) \equiv \vec{w}(t) + B(t)\vec{h}$, it follows from Lemma 3.1 and the Hölder inequality that

$$\begin{aligned} \|\Phi(\vec{w})(t)\|_{X_{(\rho+1, \infty)}} &\leq C \int_t^\infty |t - t'|^{-\beta} \|v_1(t')\|_{(\rho+1, \infty)}^\rho dt' \\ &\leq C \|\vec{v}\|_{\mathcal{E}_\sigma^T}^\rho \int_t^\infty |t - t'|^{-\beta} (t')^{-\sigma\rho} dt'. \end{aligned}$$

Since the previous inequality holds for all $t \geq T$, we have

$$\|\Phi(\vec{w})\|_{\mathcal{E}_\sigma^T} \leq CI_\sigma(T) \|\vec{v}\|_{\mathcal{E}_\sigma^T}^\rho,$$

where $I_\sigma(T)$ was given in Lemma 3.2. The definition of \vec{v} , Lemma 3.1 and the triangle inequality imply at once that

$$(4.5) \quad \|\Phi(\vec{w})\|_{\mathcal{E}_\sigma^T} \leq C_1 I_\sigma(T) (\|\vec{w}\|_{\mathcal{E}_\sigma^T} + H_\sigma(T) \|\vec{h}\|_{(\frac{\rho+1}{\rho}, \infty)})^\rho,$$

where $H_\sigma(T)$ was given in Lemma 3.2. Next, we recall $\vec{v}(t) \equiv \vec{w}(t) + B(t)\vec{h}$, denote $\vec{r}(t) = (r_1(t), r_2(t)) \equiv \vec{z}(t) + B(t)\vec{h}$ and employ Lemma 3.1 to obtain

$$\|\Phi(\vec{w})(t) - \Phi(\vec{z})(t)\|_{X_{(\rho+1, \infty)}} \leq C \int_t^\infty |t - t'|^{-\beta} \|f(v_1(t')) - f(r_1(t'))\|_{X_{(\frac{\rho+1}{\rho}, \infty)}} dt'.$$

Recalling $(|b_1|^{\rho-1} + |b_2|^{\rho-1}) \leq 2(|b_1| + |b_2|)^{\rho-1}$, invoking (1.2), and again using the Hölder inequality, we estimate

$$\begin{aligned} &\|\Phi(\vec{w}) - \Phi(\vec{z})\|_{\mathcal{E}_\sigma^T} \\ &\leq CI_\sigma(T) \|\vec{w} - \vec{z}\|_{\mathcal{E}_\sigma^T} (\|\vec{v}\|_{\mathcal{E}_\sigma^T}^{\rho-1} + \|\vec{r}\|_{\mathcal{E}_\sigma^T}^{\rho-1}) \\ (4.6) \quad &\leq CI_\sigma(T) \|\vec{w} - \vec{z}\|_{\mathcal{E}_\sigma^T} 2 (\|\vec{v}\|_{\mathcal{E}_\sigma^T} + \|\vec{r}\|_{\mathcal{E}_\sigma^T})^{\rho-1} \\ (4.7) \quad &\leq C_2 I_\sigma(T) \|\vec{w} - \vec{z}\|_{\mathcal{E}_\sigma^T} (\|\vec{w}\|_{\mathcal{E}_\sigma^T} + \|\vec{z}\|_{\mathcal{E}_\sigma^T} + 2H_\sigma(T) \|\vec{h}\|_{(\frac{\rho+1}{\rho}, \infty)})^{\rho-1}. \end{aligned}$$

By Lemma 3.2 we can choose $T_0 > 0$ such that

$$(4.8) \quad \max \left\{ C_1 I_\sigma(T_0) (a + \|\vec{h}\|_{(\frac{\rho+1}{\rho}, \infty)})^\rho, C_2 I_\sigma(T_0) 2^{\rho-1} (a + \|\vec{h}\|_{(\frac{\rho+1}{\rho}, \infty)}) \right\} \leq \min \left\{ a, \frac{1}{2} \right\}.$$

In view of (4.5), (4.7), and (4.8), Φ is a contraction in $X_{T_0}(0, a)$, and therefore it has a unique fixed point, which is denoted by \vec{w} .

Next we show the limit (2.7). By Proposition 4.1, $\vec{u}(t) \equiv B(t)\vec{h} + \vec{w}(t)$ satisfies the integral equation (1.5) in the time interval $[T_0, \infty)$. Since $\|\phi\|_{\mathcal{E}_\sigma^T} \leq \|\phi\|_{\mathcal{E}_\sigma^{T_0}}$ for all $t \geq T_0$, inequalities (4.5) and (4.8) yield

$$\begin{aligned}
 (4.9) \quad \|\vec{w}\|_{\mathcal{E}_\sigma^T} &= \|\Phi(\vec{w})\|_{\mathcal{E}_\sigma^T} \leq C_1 I_\sigma(T) (\|\vec{w}\|_{\mathcal{E}_\sigma^T} + H_\sigma(T) \|\vec{h}\|_{(\frac{\rho+1}{\rho}, \infty)})^\rho \\
 &\leq \frac{I_\sigma(T)}{I_\sigma(T_0)} C_1 I_\sigma(T_0) (a + \|\vec{h}\|_{(\frac{\rho+1}{\rho}, \infty)})^\rho \\
 (4.10) \quad &\leq \frac{I_\sigma(T)}{2I_\sigma(T_0)}.
 \end{aligned}$$

Recalling $0 < \alpha < \sigma$, we conclude that

$$\begin{aligned}
 (4.11) \quad T^\sigma \|\vec{u}(T) - B(T)\vec{h}\|_{X_{(\rho+1, \infty)}} &= T^\sigma \|\vec{w}(T)\|_{X_{(\rho+1, \infty)}} \\
 &\leq \sup_{t \geq T} t^\sigma \|\vec{w}(t)\|_{X_{(\rho+1, \infty)}}
 \end{aligned}$$

$$(4.12) \quad = \|\vec{w}\|_{\mathcal{E}_\sigma^T} \leq \frac{I_\sigma(T)}{2I_\sigma(T_0)} \longrightarrow 0 \text{ as } T \rightarrow \infty.$$

Finally, we deal with the uniqueness assertion. Let $\vec{u}, \vec{v} \in \mathcal{E}_\sigma^{T_0}$ be two solutions satisfying (1.5) and corresponding to the same profile \vec{h} . Write $\vec{w}_1(t) = \vec{u}(t) - B(t)\vec{h}$ and $\vec{w}_2(t) = \vec{v}(t) - B(t)\vec{h}$. It follows from (4.6) that

$$\begin{aligned}
 (4.13) \quad \|\vec{w}_1 - \vec{w}_2\|_{\mathcal{E}_\sigma^T} &= \|\Phi(\vec{w}_1) - \Phi(\vec{w}_2)\|_{\mathcal{E}_\sigma^T} \\
 &\leq C I_\sigma(T) \|\vec{w}_1 - \vec{w}_2\|_{\mathcal{E}_\sigma^T} (\|\vec{u}\|_{\mathcal{E}_\sigma^T} + \|\vec{v}\|_{\mathcal{E}_\sigma^T})^{\rho-1},
 \end{aligned}$$

for $T \geq T_0$. Let $b = \max\{\|\vec{u}\|_{\mathcal{E}_\sigma^{T_0}}, \|\vec{v}\|_{\mathcal{E}_\sigma^{T_0}}\}$ and take $T \geq T_0$ such that

$$(4.14) \quad C I_\sigma(T) 2^{\rho-1} b^{\rho-1} \leq 1/2.$$

In view of (4.13), (4.14) we obtain

$$\|\vec{w}_1 - \vec{w}_2\|_{\mathcal{E}_\sigma^T} \leq \frac{1}{2} \|\vec{w}_1 - \vec{w}_2\|_{\mathcal{E}_\sigma^T},$$

which implies that $\vec{w}_1(t) = \vec{w}_2(t)$ in $[T, \infty)$, and so $\vec{u}(t) = \vec{v}(t)$ in $[T, \infty)$. □

Proof of Theorem 2.2. The proof is parallel to that of Theorem 2.1. Indeed, it is sufficient to replace the terms $\|\vec{h}\|_{(\frac{\rho+1}{\rho}, \infty)}$ and $H_\sigma(T) \|\vec{h}\|_{(\frac{\rho+1}{\rho}, \infty)}$ by $\|\vec{h}\|_{\mathcal{E}_\sigma^T}$ throughout the above proof and note that (4.11), (4.12) imply that

$$T^\sigma \|\vec{u}(T) - B(T)\vec{h}\|_{X_{(\rho+1, \infty)}} \longrightarrow 0 \text{ as } T \rightarrow \infty. \quad \square$$

Proof of Theorem 2.3. From a fixed argument point, we know that $\vec{w}(t)$ is the limit in $X_{T_0}(0, a)$ of the Picard sequence $\{\vec{w}_m(t)\}_{m \geq 1}$

$$(4.15) \quad \vec{w}_1(t) \equiv 0$$

and

$$(4.16) \quad \vec{w}_{m+1}(t) = - \int_t^\infty B(t-t') \vec{f}(\vec{w}_m(t')) + B(t')\vec{h} dt'.$$

First, denote $\|\cdot\|_{\mathcal{F}_\sigma^T} = \sup_{t > T} t^\sigma \|\cdot\|_{X_{(\rho+1, d)}}$ and notice that

$$(4.17) \quad \left\| B(t)\vec{h} \right\|_{\mathcal{F}_\sigma^T} = \sup_{t > T} t^\sigma \|B(t)\vec{h}\|_{X_{(\rho+1, d)}} < \infty.$$

Employing (3.1), the Hölder inequality with $d \neq \infty$, and proceeding as in the proof of (4.9), we obtain

$$\begin{aligned} \|\vec{w}_{m+1}\|_{\mathcal{F}_\sigma^T} &\leq CI_\sigma(T)(\|\vec{w}_m\|_{\mathcal{F}_\sigma^T} + \sup_{t>T} t^\sigma \|B(t)\vec{h}\|_{X_{(\rho+1,d)}})^\rho \\ (4.18) \qquad \qquad \qquad &\leq CI_\sigma(T) \left\| B(t)\vec{h} \right\|_{\mathcal{F}_\sigma^T}^\rho + CI_\sigma(T) \|\vec{w}_m\|_{\mathcal{F}_\sigma^T}^\rho. \end{aligned}$$

In view of

$$(4.19) \qquad \qquad \qquad I_\sigma(T) \rightarrow 0 \text{ as } T \rightarrow \infty,$$

an induction argument (for $m \geq 1$) together with (4.17), (4.18) yields

$$\lim_{t \rightarrow \infty} t^\sigma \|\vec{w}_m(\cdot, t)\|_{X_{(\rho+1,d)}} = 0 \text{ for all } m \geq 1.$$

Moreover, the numerical recurrence inequality (4.18) can be solved provided that $CI_\sigma(T) \left\| B(t)\vec{h} \right\|_{\mathcal{F}_\sigma^T}$ is small enough. Therefore, it follows from (4.19) that there is $C > 0$ such that

$$(4.20) \qquad \qquad \qquad \|\vec{w}_m\|_{\mathcal{F}_\sigma^{T_1}} = \sup_{t>T_1} t^\sigma \|\vec{w}_m\|_{X_{(\rho+1,d)}} \leq C, \text{ for some } T_1 \geq T_0 > 0 \text{ and all } m \geq 1.$$

To conclude the proof, it is sufficient to show that the sequence (4.15), (4.16) is contractive in the norm $\|\cdot\|_{\mathcal{F}_\sigma^T}$ for some $T \geq T_1 > 0$. For that matter, by using (4.20), we estimate

$$\begin{aligned} \|\vec{w}_{m+1} - \vec{w}_m\|_{\mathcal{F}_\sigma^T} &\leq CI_\sigma(T) \|\vec{w}_m - \vec{w}_{m-1}\|_{\mathcal{F}_\sigma^T} \\ &\quad \times (\|\vec{w}_m\|_{\mathcal{F}_\sigma^{T_1}} + \|\vec{w}_{m-1}\|_{\mathcal{F}_\sigma^{T_1}} + \sup_{t>T_1} t^\sigma \|B(t)\vec{h}\|_{X_{(\rho+1,d)}})^{\rho-1} \\ &\leq \frac{1}{2} \|\vec{w}_m - \vec{w}_{m-1}\|_{\mathcal{F}_\sigma^T}, \end{aligned}$$

provided that T is large enough, which yields the desired claim. □

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