RIESZ BASES OF EXPONENTIALS ON MULTIBAND SPECTRA

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(Communicated by Michael T. Lacey)

Abstract. Let $S$ be the union of finitely many disjoint intervals on $\mathbb{R}$. Suppose that there are two real numbers $\alpha, \beta$ such that the length of each interval belongs to $\mathbb{Z}\alpha + \mathbb{Z}\beta$. We use quasicrystals to construct a discrete set $\Lambda \subset \mathbb{R}$ such that the system of exponentials $\{\exp 2\pi i \lambda x, \lambda \in \Lambda\}$ is a Riesz basis in the space $L^2(S)$.

1. Introduction

1.1. Let $S$ be the union of a finite number of bounded intervals on $\mathbb{R}$. We denote by $PW_S$ (after Paley and Wiener) the space of all functions $f \in L^2(\mathbb{R})$ whose Fourier transform

$$\hat{f}(x) = \int_{\mathbb{R}} f(t) e^{2\pi ixt} \, dt$$

vanishes almost everywhere outside of $S$. A discrete set $\Lambda \subset \mathbb{R}$ is called a complete interpolation set for $PW_S$ if the restriction operator $f \mapsto f|_\Lambda$ is a bounded and invertible one from $PW_S$ onto $\ell^2(\Lambda)$. In the context of communication theory this means that $\Lambda$ provides a “stable and non-redundant” sampling of signals with spectrum in $S$.

It is well-known that the complete interpolation property of $\Lambda$ is equivalent to the Riesz basis property of the corresponding exponential system

$$E(\Lambda) = \{\exp 2\pi i \lambda x, \lambda \in \Lambda\}$$

in the space $L^2(S)$.

If $S$ is a single interval, then a complete description of the Riesz bases $E(\Lambda)$ in $L^2(S)$ was given by B. S. Pavlov (1979). Much less is known, however, in the case when $S$ is the union of more than one interval. In fact, it is unknown in general whether an exponential Riesz basis in $L^2(S)$ exists at all. This existence has been established in the following special cases:

(i) $S$ is a finite union of disjoint intervals with commensurable lengths [2,4].

(ii) $S$ is the union of two general intervals [8].

For other results in the subject we refer to the survey paper [4].

In this paper we extend the two results above and prove:

Theorem 1. Let $S$ be the union of finitely many disjoint intervals on $\mathbb{R}$. Suppose that there are two real numbers $\alpha, \beta$ such that the length of each interval belongs to $\mathbb{Z}\alpha + \mathbb{Z}\beta$. Then there is $\Lambda \subset \mathbb{R}$ such that $E(\Lambda)$ is a Riesz basis in $L^2(S)$. 

Received by the editors February 6, 2011 and, in revised form, March 21, 2011.

2010 Mathematics Subject Classification. Primary 42C15, 94A12.

Key words and phrases. Riesz bases, multiband signals, quasicrystals.

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Here $\mathbb{Z}\alpha + \mathbb{Z}\beta$ denotes the set of real numbers of the form $n\alpha + m\beta \ (n, m \in \mathbb{Z})$. The above-mentioned results are thus obtained as special cases of Theorem 1.

1.2. In fact, we will prove the following more general result.

**Theorem 2.** Suppose that the indicator function of a set $S \subset \mathbb{R}$ can be expressed as a linear combination of indicator functions of intervals $I_1, \ldots, I_N$ whose lengths belong to $\mathbb{Z}\alpha + \mathbb{Z}\beta$, that is,

$$1_S(x) = \sum_{j=1}^{N} c_j 1_{I_j}(x), \quad |I_j| \in \mathbb{Z}\alpha + \mathbb{Z}\beta \ (1 \leq j \leq N),$$

where $\alpha, \beta \in \mathbb{R}$. Then there is $\Lambda \subset \mathbb{R}$ such that $E(\Lambda)$ is a Riesz basis in $L^2(S)$.

Sets with the structure (1) form a wider class than unions of disjoint intervals with lengths in $\mathbb{Z}\alpha + \mathbb{Z}\beta$. For example, one may take an interval with “holes” obtained by the removal of disjoint sub-intervals, where the interval and the removed sub-intervals have their lengths in $\mathbb{Z}\alpha + \mathbb{Z}\beta$.

A result of similar type in the periodic setting was obtained in [3].

2. Quasicrystals. Duality

Our approach is inspired by the papers [6, 7] due to Matei and Meyer, who introduced the usage of so-called ‘simple quasicrystals’ in order to construct “universal” sets of sampling or interpolation for $PW_S$ spaces. Here we will use simple quasicrystals to construct complete interpolation sets for spectra $S$ with the structure (1).

Following [6,7] we let $\Gamma$ be a lattice in $\mathbb{R}^2$. Consider the projections $p_1(x, y) = x$ and $p_2(x, y) = y$, and assume that the restrictions of $p_1$ and $p_2$ to $\Gamma$ are injective. Let $\Gamma^*$ be the dual lattice, consisting of all vectors $\gamma^* \in \mathbb{R}^2$ such that $\langle \gamma, \gamma^* \rangle \in \mathbb{Z}$, $\gamma \in \Gamma$.

Let $S$ be the union of disjoint semi-closed intervals,

$$S = \bigcup_{j=1}^{\nu} [a_j, b_j), \quad a_1 < b_1 < \cdots < a_\nu < b_\nu,$$

and let

$$I = [a, b)$$

be a single semi-closed interval. Define

$$\Lambda(\Gamma, I) = \{p_1(\gamma) : \gamma \in \Gamma, \ p_2(\gamma) \in I\},$$

$$\Lambda^*(\Gamma, S) = \{p_2(\gamma^*), \gamma^* \in \Gamma^*, \ p_1(\gamma^*) \in S\}.$$
Lemma 1. The following two properties are equivalent:

(i) $E(\Lambda(\Gamma, I))$ is a Riesz basis in $L^2(S)$;

(ii) $E(\Lambda^*(\Gamma, S))$ is a Riesz basis in $L^2(I)$.

The proof of Lemma 1 is along similar lines as in the paper [7] (Sections 6–7), but in our case there is an additional point concerned with the requirement that the intervals in (2) and (3) should be semi-closed. One may see that this point is indeed significant by keeping in mind that the Riesz basis property of the exponential systems $E(\Lambda(\Gamma, I))$ and $E(\Lambda^*(\Gamma, S))$ is not preserved upon either the addition or removal of any element.

For a proof of the duality lemma in the periodic setting, see [3, Section 2].

3. Proof of Theorem 2

There is no loss of generality in assuming that the numbers $\alpha, \beta$ are linearly independent over the rationals. Moreover, by rescaling we may restrict ourselves to the case when

$$\alpha$$ is an irrational number and $\beta = 1$.

3.1. Define a lattice

$$\Gamma = \{(n(1 + \alpha) - m, m - n\alpha) : n, m \in \mathbb{Z}\},$$

and let $I = [0, \text{mes} S)$ be an interval whose length coincides with the Lebesgue measure of $S$. We will prove that the exponential system $E(\Lambda(\Gamma, I))$ is a Riesz basis in $L^2(S)$. According to Lemma 1 it will be sufficient to show that the system $E(\Lambda^*(\Gamma, S))$ is a Riesz basis in $L^2(I)$.

It is easy to check that the set $\Lambda^*(\Gamma, S)$ may be partitioned as follows:

$$\Lambda^*(\Gamma, S) = \bigcup_{n \in \mathbb{Z}} \Lambda_n, \quad \Lambda_n = (S \cap (n\alpha + \mathbb{Z})) + n$$

(where some of the sets $\Lambda_n$ may be empty). Let $\{s_n\}$ be a sequence of integers such that $s_n - s_{n-1} = \#\Lambda_n$, and choose an enumeration $\{\lambda_j, j \in \mathbb{Z}\}$ of the set $\Lambda^*(\Gamma, S)$ such that

$$\Lambda_n = \{\lambda_j : s_{n-1} \leq j < s_n\} \quad (n \in \mathbb{Z}).$$

In order to prove that $E(\Lambda^*(\Gamma, S))$ is a Riesz basis in $L^2(I)$, it will be sufficient, by a theorem of Avdonin [11], to check that the following three conditions hold:

(a) $\{\lambda_j\}$ is a separated sequence, $\inf_{j \neq k} |\lambda_j - \lambda_k| > 0$.

(b) $\sup_j |\delta_j| < \infty$, where $\delta_j = \lambda_j - j/\text{mes} S$.

(c) There is a constant $c$ and a positive integer $N$ such that

$$\sup_{a \in \mathbb{Z}} \left| \frac{1}{N} \sum_{j=a+1}^{a+N} \delta_j - c \right| < \frac{1}{4 \text{mes} S}.$$

Condition (a) can easily be verified directly from the definition of $\Lambda^*(\Gamma, S)$. 

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3.2. We will next show that (b) holds. Consider the bounded, piecewise constant function
\[ \phi(x) = \sum_{k \in \mathbb{Z}} 1_S(x-k). \]
This function is 1-periodic and hence may be viewed as a function on the circle group \( T = \mathbb{R}/\mathbb{Z} \). The assumptions (1), (2) and (4) now provide the following representation for \( \phi \).

**Lemma 2.** There is a bounded, piecewise linear function \( \psi : T \rightarrow \mathbb{R} \) such that
\[ \phi(x) = \text{mes } S + \psi(x) - \psi(x - \alpha), \quad x \in T. \]
For a proof of Lemma 2 see [3, Lemma 3.2]. Now observe that by (5) we have
\[ s_n - s_{n-1} = \# \Lambda_n = \phi(n\alpha). \]
It follows from (7) and Lemma 2 that
\[ s_n = n \text{ mes } S + \psi(n\alpha) + \text{ const}. \]
Given \( j \) there is \( n = n(j) \) such that \( \lambda_j \in \Lambda_n \) or, equivalently, such that \( s_{n-1} \leq j < s_n \). Then
\[ \delta_j = \lambda_j - \frac{j}{\text{mes } S} = (\lambda_j - n) + \left(n - \frac{s_n}{\text{mes } S}\right) + \frac{s_n - j}{\text{mes } S}. \]
It thus follows from (6), (7) and (8) that \( \sup_j |\delta_j| < \infty \), which confirms condition (b) above.

3.3. It remains to establish (c). In order to show that (6) holds, we will first obtain a simple expression for the sum \( \sum \delta_j \) where \( j \) goes through the interval \( s_{n-1} \leq j < s_n \). Indeed,
\[ \sum_{j=s_{n-1}}^{s_n-1} \delta_j = \sum_{j=s_{n-1}}^{s_n-1} (\lambda_j - n) - \sum_{j=s_{n-1}}^{s_n-1} \left(\frac{j}{\text{mes } S} - n\right) \overset{\text{def}}{=} S_1(n) - S_2(n). \]
We evaluate each one of the sums \( S_1(n), S_2(n) \) separately. First we observe that by (5),
\[ S_1(n) = \sum_{k \in \mathbb{Z}} (n\alpha - k) 1_S(n\alpha - k) \overset{\text{def}}{=} \tau_1(n\alpha). \]
Second, by a direct calculation and using (7) and (8), we find that
\[ S_2(n) = (s_n - s_{n-1}) \left(\frac{s_{n-1} + s_n - 1}{2 \text{mes } S} - n\right) \]
\[ = \frac{\phi(n\alpha) (\psi(n\alpha) - \frac{1}{2} \phi(n\alpha) + \text{ const})}{\text{mes } S} \overset{\text{def}}{=} \tau_2(n\alpha). \]
We conclude that for an appropriately defined function \( \tau : T \rightarrow \mathbb{R} \) (bounded and piecewise continuous) we have
\[ \sum_{j=s_{n-1}}^{s_n-1} \delta_j = \tau(n\alpha), \quad n \in \mathbb{Z}. \]
3.4. Now we can finish the proof of (c) above. Given $a \in \mathbb{Z}$ and a positive (large) integer $N$ there are $n = n(a)$ and $r = r(a, N)$ such that $s_{n-1} \leq a < s_n$, $s_{n+r-1} \leq a + N < s_{n+r}$.

Since the sequence $\{\delta_j\}$ is bounded and due to (9) we have
\[
\sum_{j=a+1}^{a+N} \delta_j = \sum_{j=s_{n-1}}^{s_{n+r-1}} \delta_j + O(1) = \sum_{k=n}^{n+r-1} \tau(k\alpha) + O(1).
\]

The points $\{na\}$ are well-distributed on the circle $\mathbb{T}$ (since $\alpha$ is irrational), and hence
\[
\sum_{k=n}^{n+r-1} \tau(k\alpha) = r \int_{\mathbb{T}} \tau(x) \, dx + o(r), \quad r \to \infty,
\]
uniformly with respect to $n$. Since (7) and (8) imply that $N = r \text{mes} S + O(1)$, we get
\[
\frac{1}{N} \sum_{j=a+1}^{a+N} \delta_j = \frac{1}{\text{mes} S} \int_{\mathbb{T}} \tau(x) \, dx + o(1), \quad N \to \infty,
\]
with the $o(1)$ uniform with respect to $a$. This implies (c) and so Theorem 2 is proved.

4. Remarks

We have constructed Riesz bases of exponentials with real frequencies for multi-band spectra subject to the diophantine condition (1). We would like, though, to comment on one result which is not covered by our theorems above. In the paper [8] existence of such Riesz bases was proved under certain non-discrete conditions on the lengths of the gaps between the intervals. The restrictions obtained there are rather severe; however, the result indicates that diophantine restrictions are not necessarily natural ones in the problem.

We also refer the reader to the paper [5] where the authors construct, for any finite union of intervals, a Riesz basis of exponentials with complex frequencies lying in a horizontal strip along the real axis.

Acknowledgement

The author thanks Kristian Seip for reading an earlier version of this paper.

References


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