THE ZEROS OF CERTAIN LOMMEL FUNCTIONS

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1. Introduction

Lommel’s function \( s_{\mu,\nu}(z) \) is a particular solution of the differential equation \( z^2 y'' + z y' + (z^2 - \nu^2) y = z^{\mu+1} \). Here we present estimates and monotonicity properties of the positive zeros of \( s_{\mu-1/2,1/2}(z) \) when \( \mu \in (0,1) \).

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Theorem 1.2. Set \( w_n := w_n(\mu) := z_n - n\pi \). For \( \mu \in (0, 1) \) we have \( \lim_{n \to \infty} w_n = \mu \pi/2 \) and
\[
(1.1) \quad w_{2n} < w_{2n+2} < \frac{\mu \pi}{2} < w_{2n+1} < w_{2n-1}, \quad n \in \mathbb{N}.
\]
Furthermore, for \( n \in \mathbb{N} \) the functions \( z_{2n-1} \) are strictly increasing in \((0,1)\).

Proposition 1.4 below shows that \( z_{2n} \) is not monotonic in \( \mu \) for any \( n \in \mathbb{N} \).

Our proofs of Theorems 1.1 and 1.2 will be based on the fact that for \( \mu, \nu \in \mathbb{C} \) with \( \Re(\mu + \nu + 1) > 0 \) and \( z \in \mathbb{C}^- := \mathbb{C} \setminus \{z : z \leq 0\} \), one has
\[
(1.2) \quad s_{\mu,\nu}(z) = \frac{\pi}{2} \left[ \frac{2}{\pi z} \sin z \right]_{J_{\mu}(z)}^{J_{\nu}(z)} \int_0^z t^\mu J_\nu(t) \, dt - \int_0^z t^\nu Y_\nu(t) \, dt,
\]
where \( J_\mu(z) \) and \( Y_\nu(z) \) are the usual Bessel functions \( \text{[10, Sec. 10.7]} \). It is well-known that
\[
J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z \quad \text{and} \quad Y_{1/2}(z) = -\sqrt{\frac{2}{\pi z}} \cos z,
\]
and therefore we obtain from (1.2) that for \( \mu > 0 \),
\[
s_{\mu-1/2,1/2}(z) = \frac{1}{z^{1/2}} \int_0^z t^{\mu-1} \sin(z-t) \, dt.
\]

Theorem 1.1 thus takes the following equivalent form.

Theorem 1.3. For all \( \mu \in (0,1) \) the function
\[
F_\mu(z) := \int_0^z t^{\mu-1} \sin(z-t) \, dt
\]
has exactly one zero in each interval \( I_n(\mu) \), \( n \in \mathbb{N} \), and vanishes nowhere else on the positive real axis.

Obviously, \( F_\mu(z) \) is analytic for \( \Re\mu > 0 \) and \( z \in \mathbb{C}^- \). Partial integration shows that
\[
(1.3) \quad F_\mu(z) = \frac{1}{\mu} \int_0^z t^\mu \cos(z-t) \, dt =: \frac{1}{\mu} G_\mu(z).
\]

The following proposition is thus clear.

Proposition 1.4. For \( n \in \mathbb{N} \) we have \( z_n \to n\pi \) as \( \mu \to 0 \) and \( z_{2n-1}(1) = z_{2n}(1) = 2n\pi \).

Our interest in the Lommel functions \( s_{\mu-1/2,1/2}(z) \) stems from the fact that the function \( F_\mu(z) \) plays an important role in the theory of positive trigonometric sums.

In \( \text{[11, V.2.29]} \) it is shown that there is exactly one \( \alpha^* \in (0,1) \) such that \( F_{1-\alpha^*}(\frac{3\pi}{2}) \) vanishes. In \( \text{[2]} \) it is proven that \( \alpha^* \) is such that for all \( \alpha \geq \alpha^* \), \( n \in \mathbb{N} \), and \( x \in (0,\pi) \),
\[
\tau_n(\alpha,x) := 1 + \sum_{k=1}^n \frac{\cos(kx)}{k^\alpha} > 0,
\]
while for \( \alpha < \alpha^* \) the cosine polynomials \( \tau_n(\alpha,x) \) do not have a uniform lower bound on \((0,\pi)\). Nowadays, \( \alpha^* = 0.30844 \ldots \) is called the Littlewood-Salem-Izumi constant \( \text{[1]} \). See also \( \text{[4]} \) and \( \text{[6]} \) for related results and considerations.
In [7] it is claimed that for each $\rho \in (0, 1)$ there is exactly one $\mu^*(\rho) \in (0, 1)$ such that
\begin{equation}
F_{\mu^*(\rho)}((\rho + 1)\pi) = 0.
\end{equation}

It is then shown there that the thus defined function $\mu^*(\rho)$ plays an important role in a conjecture concerning the mapping properties of partial sums of certain univalent functions. In [5] it is proven that this conjecture is equivalent to the fact that for $n \in \mathbb{N}$, $\rho \in (0, 1)$, and $\theta \in (0, \pi)$ one has
\[
\sum_{k=0}^{n} \frac{(\mu)_k}{k!} \sin[(2k + \rho)\theta] > 0
\]
for all $\mu \in (0, \mu^*(\rho)]$ but for no $\mu \in (\mu^*(\rho), 1)$. Here, $(\mu)_k := \mu(\mu + 1) \cdots (\mu + k - 1)$ is the Pochhammer symbol.

It seems that in [7] the proof of the well-definition of $\mu^*(\rho)$ is not completely correct. In order to prove that for each $\rho \in (0, 1)$ there is exactly one $\mu^*(\rho) \in (0, 1)$ such that (1.4) holds, it is claimed in the proof of [7] Lem. 1] that for all $\rho, \mu \in (0, 1)$,
\[
I(\rho, \mu) := \frac{d}{d\mu} F_{\mu}(\rho + 1)\pi = \int_{0}^{(\rho + 1)\pi} t^{\mu - 1} \sin((\rho + 1)\pi - t) \log(t) \, dt > 0.
\]
But for $\rho = 0$ it is easy to see that
\[
I(0, \mu) = \int_{0}^{\pi} t^{\mu - 1} \sin(t) \log(t) \, dt
\]
takes a negative value when $\mu = 0$. Hence, because of continuity, there is an $\epsilon > 0$ such that for all $\mu \in (0, \epsilon)$ there is a $\rho > 0$ for which $I(\rho, \mu) < 0$.

Despite this, the function $\mu^*(\rho)$ is well-defined, as we can show here. It follows readily from Theorem 1.2 and Proposition 1.3 that $z_1$ is a strictly increasing analytic function from $(0, 1)$ onto $(\pi, 2\pi)$. Hence, $\mu^*(\rho)$ is well-defined, and we have $\mu^*(\rho) = z_1^{-1}((\rho + 1)\pi)$. The next proposition is thus clear.

**Proposition 1.5.** For each $\rho \in (0, 1)$ there is exactly one $\mu^*(\rho) \in (0, 1)$ such that $F_{\mu^*(\rho)}((\rho + 1)\pi) = 0$. $\mu^*(\rho)$ is strictly increasing and analytic in $(0, 1)$.

In order to prove our results we will first show a weak form of Theorem 1.3 in the next section. In Section 3 we will then present the proofs of Theorems 1.2 and 1.3. For the proof of Theorem 1.2 we will need some information concerning the set of $(\rho, \mu) \in (0, 1)^2$ for which $I(\rho, \mu)$ is positive. The relevant results will be presented in Section 4.

### 2. A weak form of Theorem 1.3

In this section we will prove the following lemma.

**Lemma 2.1.** For $\mu \in (0, 1)$ the function $F_{\mu}(z)$ has exactly one zero in every interval $(n\pi, (n + 1)\pi)$, $n \in \mathbb{N}$, and vanishes nowhere else on the positive axis.

First, note that, since $t^{\mu - 1}$ is decreasing on $(0, \infty)$ when $\mu \in [0, 1)$, obviously
\begin{equation}
\int_{0}^{n} t^{\mu - 1} \sin(t) \, dt > 0 \text{ when } a > 0 \text{ and } \mu \in [0, 1).
\end{equation}

The next lemma is the key result for the proof of Lemma 2.1.
Lemma 2.2. Suppose \( \mu \in (0,1) \) and \( z^* > 0 \) are such that \( F_\mu(z^*) = 0 \). Then

\[
F'_\mu(z^*) = \int_0^{z^*} t^{\mu-1} \cos(z^* - t) \, dt \neq 0.
\]

Proof. If both \( F'_\mu(z^*) \) and \( F''_\mu(z^*) \) would vanish, then so would

\[
F'_\mu(z^*) + iF''_\mu(z^*) = e^{iz^*} \int_0^{z^*} t^{\mu-1} e^{-it} \, dt.
\]

In particular,

\[
\int_0^{z^*} t^{\mu-1} \sin(t) \, dt = 0
\]

would hold. Since this contradicts (2.1), the proof is complete. \( \square \)

It follows from (2.1) that \( F'_\mu(n\pi) \neq 0 \) and

\[
\text{sgn } F'_\mu(n\pi) = (-1)^{n+1} \quad \text{for } \mu \in (0,1), n \in \mathbb{N}.
\]

Furthermore, it is easy to see that \( F_\mu(z) \) is positive for \( z \in (0, \pi) \) and \( \mu \in (0,1) \).

Now, in order to verify the remaining statements of Lemma 2.1 let \( n \in \mathbb{N} \) and \( \mu^* \in (0,1) \) and suppose that \( F_{\mu^*} \) has exactly \( m \) zeros in the interval \( I := (n\pi, (n+1)\pi) \). Then, because of Lemma 2.2, \( F_{\mu^*} \) has \( m \) simple zeros \( x_1, \ldots, x_m \) in \( I \) and therefore there are \( m \) open subintervals \( J_1, \ldots, J_m \) of \( (0,1) \), all containing \( \mu^* \), and \( m \) differentiable functions \( x_k(\mu), k \in \{1, \ldots, m\} \), that satisfy \( x_k(\mu^*) = x_k \) and \( F_\mu(x_k(\mu)) = 0 \) for \( \mu \in J_k \). Since \( F'_\mu(n\pi) \neq 0 \) for all \( \mu \in (0,1) \) and \( n \in \mathbb{N} \), \( x_k(\mu) \) lies in \( I \) for all \( \mu \in J_k \) and \( k \in \{1, \ldots, m\} \).

If \( k \in \{1, \ldots, m\} \) and \( \sigma \in (0,1) \) is a boundary point of \( J_k \), then \( \lim_{\mu \to \sigma} x_k(\mu) \) exists. Otherwise, because \( x_k(\mu) \) is continuous and satisfies \( x_k(\mu) \in I \) for \( \mu \in J_k \), there is an open non-empty subinterval \( I^* \) of \( I \) such that for every \( z \in I^* \) the set \( x_k^{-1}(\mu) \) contains an infinite number of points that accumulate at \( \sigma \). Consequently, \( F_\sigma(z) = 0 \) for all \( z \in I^* \), a contradiction.

We can therefore assume that \( J_k = (0,1) \) for all \( k \in \{1, \ldots, m\} \). Otherwise, because of what we have just shown, one interval \( J_k \) would have a boundary point \( \sigma \in (0,1) \) for which

\[
F_\sigma \left( \lim_{\mu \to \sigma} x_k(\mu) \right) = 0 = F_\sigma' \left( \lim_{\mu \to \sigma} x_k(\mu) \right).
\]

This would contradict Lemma 2.2.

We have thus proven that if \( F_\mu \) has exactly \( m \) zeros in \( (0,1) \) for one \( \mu \in (0,1) \), then \( F_\mu \) has at least \( m \) zeros in \( (0,1) \) for all \( \mu \in (0,1) \). This readily implies that if \( F_\mu \) has exactly \( m \) zeros in \( (0,1) \) for one \( \mu \in (0,1) \), then \( F'_\mu \) has exactly \( m \) zeros in \( (0,1) \) for all \( \mu \in (0,1) \).

In order to complete the proof of Lemma 2.1 we will now determine a \( \mu \in (0,1) \) for which \( F_\mu \) has exactly one zero in \( I \).

To that end, observe that, because of (1.3), \( F_\mu \) vanishes if, and only if, \( G_\mu \) vanishes. For positive \( z \) the function \( G_0(z) = \sin(z) \) vanishes exactly at the points \( n\pi, n \in \mathbb{N} \), and satisfies \( G_0'(n\pi) = (-1)^n \). Therefore for every \( n \in \mathbb{N} \) there is a differentiable function \( z_n(\mu), \) defined in an open real neighborhood \( U_n \) of the origin, such that

\[
z_n(0) = n\pi, \quad G_\mu(z_n(\mu)) = 0 \quad \text{for all } \mu \in U_n, \quad \text{and}
\]

\[
z'_n(0) = -\frac{d}{d\mu}G_\mu(z_n(0))|_{\mu=0} = (-1)^{n+1} \int_0^{n\pi} \cos(n\pi - t) \log(t) \, dt.
\]
Partial integration gives
\[
\int_0^{\pi n} \cos(n\pi - t) \log(t) \, dt = (-1)^{n+1} \int_0^{\pi n} \frac{\sin(t)}{t} \, dt.
\]
The integral on the right-hand side of this equation is positive because of (2.1), and thus it follows from (2.3) and Hurwitz’s theorem that for every \( n \in \mathbb{N} \) there is a \( \mu_n > 0 \) such that for \( 0 < \mu < \mu_n \) the functions \( G_\mu \) and \( F_\mu \) have exactly one zero in \((n\pi, (n+1)\pi)\). The proof of Lemma 2.1 is complete.

3. Proof of Theorems 1.2 and 1.3

The following result is an easy consequence of Lemma 2.1 and (2.2).

**Proposition 3.1.** For a \( z \in (n\pi, (n+1)\pi) \) the relations
\[
F_\mu(z) > 0 \quad \text{and} \quad F_\mu(z) < 0
\]
hold if, and only if, \( z < z_n \) and \( z > z_n \), respectively, in the case where \( n \) is odd, and if, and only if, \( z > z_n \) and \( z < z_n \), respectively, in the case where \( n \) is even.

Now, let \( n \in \mathbb{N}, n \geq 3 \). Then
\[
0 = \int_0^{z_n} t^{\mu-1} \sin(z_n - t) \, dt
= \int_0^{z_{n-2\pi}} t^{\mu-1} \sin(z_n - t) \, dt + \int_{z_{n-2\pi}}^{z_n} t^{\mu-1} \sin(z_n - t) \, dt.
\]
Since \( t^{\mu-1} \) is decreasing on \((0, \infty)\), it is clear that the integral on the right-hand side of the sum is negative, and therefore
\[
\int_0^{z_{n-2\pi}} t^{\mu-1} \sin(z_n - 2\pi - t) \, dt > 0.
\]
Hence, it follows from Proposition 3.1 that
\[
z_n - 2\pi < z_{n-2} \quad \text{or} \quad z_n - 2\pi > z_{n-2},
\]
depending on whether \( n \) is odd or even.

We have thus shown that, for fixed \( \mu \in (0, 1) \), the sequence \( w_{2n-1} \) is strictly decreasing, while \( w_{2n} \) is strictly increasing. Consequently, since \( w_n \in (0, \pi) \) for all \( n \in \mathbb{N} \), \( w_e = \lim_{n \to \infty} w_{2n} \) and \( w_o = \lim_{n \to \infty} w_{2n-1} \) exist.

In order to prove that \( w_e = w_o = \mu \pi/2 \), recall that the generalized sine and cosine integrals are defined by
\[
\text{Si}(z, \mu) := \int_0^z t^{\mu-1} \sin(t) \, dt \quad \text{and} \quad \text{Ci}(z, \mu) := \int_0^z t^{\mu-1} \cos(t) \, dt,
\]
respectively, and satisfy
\[
\lim_{z \to \infty} \text{Si}(z, \mu) = \sin \left( \frac{\mu \pi}{2} \right) \Gamma(\mu) \quad \text{and} \quad \lim_{z \to \infty} \text{Ci}(z, \mu) = \cos \left( \frac{\mu \pi}{2} \right) \Gamma(\mu)
\]
for \( z \) tending to \( \infty \) on the real axis and \( \mu \in (0, 1) \) [8]. For all \( n \in \mathbb{N} \) we have
\[
0 = (-1)^n F_\mu(z_n) = \sin(w_n) \text{Ci}(z_n, \mu) - \cos(w_n) \text{Si}(z_n, \mu).
\]
Letting \( n \to \infty \), we obtain from (3.1) that both for \( w = w_e \) and \( w = w_o \),
\[
0 = \Gamma(\mu) \sin \left( w - \frac{\mu \pi}{2} \right),
\]
and thus that \( w_e = w_o = \mu \pi/2 \).
Next, observe that, according to the implicit function theorem,
\[ z'_n = -\frac{d}{d\sigma} F_\sigma(z_n)|_{\sigma=\mu}. \]
Since \( F'_\mu(z_n) \) does not change sign in \((0, 1)\), it follows from Proposition 3.1 that in order to prove that \( z_{2n-1} \) is increasing in \((0, 1)\), it will suffice to show
\[ (3.2) \quad \frac{d}{d\sigma} F_\sigma(z_{2n-1})|_{\sigma=\mu} = \int_0^{z_{2n-1}} t^{\mu-1} \log(t) \sin(w_{2n-1}) \, dt > 0 \]
for \( \mu \in (0, 1) \).
To that end, note first that
\[ \int_0^{z_1} t^{\mu-1} \log(t) \sin(t - w_1) \, dt = \int_0^{w_1} t^{\mu-1} \log(t) \sin(t - w_1) \, dt + \int_{w_1}^{z_1} t^{\mu-1} \log(t) \sin(t - w_1) \, dt. \]
Since \( t^{\mu-1} \sin(t - w_1) < 0 \) and \( t^{\mu-1} \sin(t - w_1) > 0 \) for \( t \in (0, w_1) \) and \( t \in (w_1, z_1) \), respectively, we therefore obtain
\[ \int_0^{z_1} t^{\mu-1} \log(t) \sin(t - w_1) \, dt > \log(w_1) F_\mu(z_1) = 0. \]
We have thus shown that \( z_1 \) is a strictly increasing and analytic function in \((0, 1)\). As explained in the paragraph preceding Proposition 3.1, this implies that the function \( \mu^*(\rho) \) is well-defined. We can therefore make use of the inequality
\[ (3.3) \quad \rho \leq \mu^*(\rho), \quad \rho \in (0, 1), \]
which was established in [7].

Now, before we show (3.2) also for \( n = 2, 3, \ldots \), we will first complete the proof of Theorem 1.3.

**Proof of Theorem 1.3.** Because of Proposition 3.1 and what we have shown so far, only the inequality \( F_\mu((2n - 1 + \mu)\pi) \leq 0, \ n \in \mathbb{N}, \) remains to be verified. In fact, since
\[ F_\mu((2n - 1 + \mu)\pi) = F_\mu((1 + \mu)\pi) + \int_{(1+\mu)\pi}^{(2n-1+\mu)\pi} t^{\mu-1} \sin(t - \mu\pi) \, dt \leq F_\mu((1 + \mu)\pi), \]
it suffices to prove that \( F_\mu((1 + \mu)\pi) \leq 0 \) or, equivalently, that \( z_1 \leq (1 + \mu)\pi \).
However, since \( \mu^*(\rho) = z_1^{-1}(\rho + 1)\pi \), the latter is equivalent to (3.3). \( \square \)

**Proof of Theorem 1.2.** It remains to verify (3.2) for \( n \in \{2, 3, \ldots \} \). Because of (3.1) we have
\[ \int_0^x t^{\mu-1} \sin(t - x) \, dt = \Gamma(\mu) \sin\left(\frac{\mu\pi}{2} - x\right), \quad x \in (0, \pi), \]
and thus
\[ \int_0^\infty t^{\mu-1} \log(t) \sin(t - x) \, dt = \Gamma(\mu) \sin\left(\frac{\mu\pi}{2} - x\right) + \frac{\pi}{2} \Gamma(\mu) \cos\left(\frac{\mu\pi}{2} - x\right). \]
Since by Theorem 1.3 \( \mu \pi / 2 < w_{2n-1} \leq \mu \pi \), we find that

\[
J_n := \int_0^\infty t^{\mu-1} \log(t) \sin(t - w_{2n-1}) \, dt > 0, \quad n \in \mathbb{N}.
\]

For \( n \in \mathbb{N} \) define the sequence

\[
D_{n,k} := \int_{\max\{0,w_{2n-1}+(2k-3)\pi\}}^{w_{2n-1}+(2k-1)\pi} t^{\mu-1} \log(t) \sin(t - w_{2n-1}) \, dt, \quad k \in \mathbb{N}.
\]

Then

\[
\sum_{k=1}^\infty D_{n,k} = J_n > 0
\]

and

\[
\sum_{k=1}^n D_{n,k} = \int_0^{z_{2n-1}} t^{\mu-1} \log(t) \sin(t - w_{2n-1}) \, dt.
\]

The proof of the theorem will therefore be complete if we can show that for each \( n \in \{2,3,\ldots\} \) there is a \( k(n) \in \mathbb{N} \) such that \( D_{n,k} \geq 0 \) for \( k \in \{1,\ldots,k(n)\} \) and \( D_{n,k} \leq 0 \) for \( k > k(n) \).

It is easy to check that for every \( \mu \in (0,1) \) there is a \( t_\mu > e \) such that \( t_\mu^{\mu-1} \log(t) \) is positive and increasing in \((1,t_\mu)\) and positive and decreasing in \((t_\mu,\infty)\). Let \( k^* \in \{1,2,3,\ldots\} \) be such that \( t_\mu \in [w_{2n-1}+(2k^*-3)\pi,w_{2n-1}+(2k^*-1)\pi) \). Then it is clear that \( D_{n,k} > 0 \) for \( k \in \{2,\ldots,k^*-1\} \) and \( D_{n,k} < 0 \) for \( k > k^* \). We therefore set \( k(n) = k^* \) if \( D_{n,k^*} \geq 0 \) and \( k(n) = k^* - 1 \) otherwise.

\( D_{n,1} > 0 \) remains to be verified. But since \( \mu \pi / 2 < w_{2n-1} \leq \mu \pi \) for \( \mu \in (0,1) \), this follows readily from Theorem 1.1 below.

We have thus shown that for \( n \in \mathbb{N} \) the functions \( z_{2n-1} \) are strictly increasing in \((0,1)\). While the functions \( z_{2n} \) are not monotonic in \((0,1)\) for any \( n \in \mathbb{N} \) (cf. Proposition 1.4), numerical computation strongly supports the conjecture that the functions \( z_{2n-1} \) and \( z_{2n} \), \( n \in \mathbb{N} \), are convex and concave, respectively, in \((0,1)\). This, however, seems quite hard to prove.

4. ON THE SIGN OF \( I(\rho, \mu) \)

Recall that

\[
I(\rho, \mu) := \frac{d}{d\mu} F_\mu((\rho + 1)\pi) = \int_0^{(\rho+1)\pi} t^{\mu-1} \sin(t - \rho \pi) \log(t) \, dt.
\]

In this section we will determine a large subset of \((\rho, \mu) \in (0,1)^2\) for which \( I(\rho, \mu) \) is positive.

To that end, observe first that the function

\[
p(\mu) := \int_0^\pi t^{\mu-1} \log(t) \sin(t) \, dt
\]

is strictly increasing on \((0,1)\) and satisfies \( p(0) = -0.53 \ldots \) and \( p(1) = 0.64 \ldots \). Consequently, the equation \( p(\mu) = 0 \) has a unique solution \( \mu_0 \) in \((0,1)\).

The numerical value of \( \mu_0 \) is \( \mu_0 = 0.32 \ldots \). Further, let \( M \) be the union of the three sets

\[
M_1 := \{(\rho, \mu) : (10\pi)^{-1} \leq \rho \leq 1, 0 \leq \mu \leq 1\},
M_2 := \{(\rho, \mu) : 0 < \rho < (10\pi)^{-1}, \mu_0 \leq \mu \leq 1\},
M_3 := \{(\rho, \mu) : 0 < \mu \leq 2\rho \leq (5\pi)^{-1}\}.
\]
We will show the following.

**Theorem 4.1.** For all \((\rho, \mu) \in M\) we have \(I(\rho, \mu) > 0\). To each \(\mu \in (0, \mu_0)\) there is a \(\delta_\mu > 0\) such that \(I(\rho, \mu) < 0\) for \(\rho \in (0, \delta_\mu)\).

The second statement of this theorem has essentially been shown in the introduction. Hence, it only remains to verify that \(I(\rho, \mu) > 0\) for \((\rho, \mu) \in M\). To that end, we need some auxiliary lemmas. For \(x \in [0, \pi]\) define

\[
 f(x) := \int_0^x \log(t) \sin(t-x) \, dt \quad \text{and} \quad g(x) := \int_0^\pi \frac{\log(t+x)}{t+x} \sin(t) \, dt.
\]

**Lemma 4.2.** For \(x \in (0, 1)\) we have \(f(x) > 0\).

*Proof.* If \(x \in (0, 1)\), then clearly

\[
 f(x) \geq \log(x) \int_0^x \sin(t-x) \, dt = -\log(x) (1 - \cos(x)) > 0.
\]

\(\square\)

**Lemma 4.3.** The function \(f(x)\) is concave on \((1, \pi)\).

*Proof.* We recall that

\[
 \text{Si}(x) = \int_0^x \frac{\sin(t)}{t} \, dt = \frac{\pi}{2} - \int_x^\infty \frac{\sin(t)}{t} \, dt,
\]

\[
 \text{Ci}(x) = -\int_x^\infty \frac{\cos(t)}{t} \, dt = \gamma + \log(x) - \int_0^x \frac{1 - \cos(t)}{t} \, dt,
\]

where \(\gamma\) is Euler’s constant.

It is then easy to see that

\[
 \int_0^x \log(t) \cos(t) \, dt = \log(x) \sin(x) - \text{Si}(x)
\]

and

\[
 \int_0^x \log(t) \sin(t) \, dt = -\log(x) \cos(x) + \text{Ci}(x) - \gamma.
\]

From these, the relation

\[
 f(x) = -\log(x) \cos(x) (\text{Ci}(x) - \gamma) + \sin(x) \text{Si}(x)
\]

follows. We observe also that

\[
 f''(x) = -f(x) - \log(x).
\]

Therefore

\[
 f''(x) = \cos(x) (-\text{Ci}(x) + \gamma) - \sin(x) \text{Si}(x).
\]

It is clear that the function \(w(x) := -\text{Ci}(x) + \gamma\) is strictly decreasing on \((0, \pi/2)\) and strictly increasing on \((\pi/2, \pi)\); hence \(w(x) \geq w(\pi/2) = 0.1... > 0\). Since \(\text{Si}(x) > 0\) for all \(x > 0\), it readily follows from \(\text{Si}(\pi/2) > 0\) that \(f''(x) < 0\) for \(\pi/2 \leq x \leq \pi\). On the other hand, for \(1 \leq x < \pi/2\) we have

\[
 f''(x) \leq ( -\text{Ci}(1) + \gamma) \cos(x) - \text{Si}(1) \sin(x) < 0.
\]

\(\square\)

**Lemma 4.4.** The function \(g(x)\) is concave on \((0, \pi)\).
Proof. For $x \in (0, \pi)$ we have
\[
g^{(4)}(x) = \int_0^\pi \frac{24 \log(t + x) - 50}{(t + x)^5} \sin(t) \, dt
\leq \left(24 \log(2\pi) - 50\right) \int_0^\pi \frac{\sin(t)}{(t + x)^5} \, dt < 0.
\]
This implies $g''''(x) > g''''(\pi) = 0.011 \ldots > 0$, which, in turn, gives $g''''(x) < g''''(\pi) = -0.0012 \ldots < 0$ for $x \in (0, \pi)$.

We will now prove Theorem 4.1 separately for the three sets $M_1$, $M_2$, and $M_3$. Note that
\[
(4.4) \quad I(\rho, \mu) = \int_0^{\rho\pi} t^{\mu-1} \log(t) \sin(t - \rho\pi) \, dt + \int_{\rho\pi}^{(\rho+1)\pi} t^{\mu-1} \log(t) \sin(t - \rho\pi) \, dt.
\]
Proof of Theorem 4.1 for $(\rho, \mu) \in M_1$. It follows readily from (4.4) that for $(\rho, \mu) \in M_1$,
\[
I(\rho, \mu) \geq f(\rho\pi) + g(\rho\pi).
\]
It will therefore be enough to prove
\[
(4.5) \quad f(x) + g(x) > 0 \quad \text{for} \quad x \in [1/10, \pi].
\]
For $x \in [1/10, 1]$ we use Lemmas 4.2 and 4.3 to obtain
\[
f(x) + g(x) > g(x) > \min\{g(1/10), g(1)\} = g(1/10) = 0.0819 \ldots > 0.
\]
For $x \in (1, \pi]$ Lemmas 4.3 and 4.4 give
\[
f(x) + g(x) > \min\{f(1) + g(1), f(\pi) + g(\pi)\} = f(\pi) + g(\pi) = 0.0169 \ldots > 0.
\]

Proof of Theorem 4.1 for $(\rho, \mu) \in M_2$. Observe that for $\rho \in (0, 1/10\pi)$,
\[
\int_{\rho\pi}^{(\rho+1)\pi} t^{\mu-1} \log(t) \sin(t - \rho\pi) \, dt > 0.
\]
Equation (4.4) therefore implies
\[
I(\rho, \mu) > \int_{\rho\pi}^{(\rho+1)\pi} t^{\mu-1} \log(t) \sin(t - \rho\pi) \, dt = \int_0^\pi (t + \rho\pi)^{\mu-1} \log(t + \rho\pi) \sin(t) \, dt
\geq \int_0^\pi (t + \rho\pi)^{\mu-1} \log(t + \rho\pi) \sin(t) \, dt \geq \int_0^\pi t^{\mu-1} \log(t) \sin(t) \, dt = 0
\]
for $\rho \in [0, 1/10\pi]$. Here we have made use of the fact that, for all $t \in (0, \pi)$, $(t + \rho\pi)^{\mu-1} \log(t + \rho\pi)$ is an increasing function of $\mu$ when $\rho \in (0, 1/10\pi)$ and an increasing function of $\rho$ when $\mu = \mu_0$.

Proof of Theorem 4.1 for $(\rho, \mu) \in M_3$. Since $\rho \in [\pi/2, 1/10\pi]$, we have
\[
\int_0^{(\rho+1)\pi} t^{\mu-1} \log(t) \sin(t - \rho\pi) \, dt > \int_0^1 t^{\mu-1} \log(t) \sin(t - \rho\pi) \, dt.
\]
The integral on the right-hand side of this inequality can be written as
\[ \cos(\rho \pi) \int_0^1 t^{\mu - 1} \log(t) \sin(t) \, dt - \sin(\rho \pi) \int_0^1 t^{\mu - 1} \log(t) \cos(t) \, dt \]
and is thus larger than
\[ \sin(\rho \pi) \int_0^1 t^{\mu - 1} \log(t) \cos(t) \, dt. \]

Applying the estimates \( \sin(t) < t \) and \( \cos(t) > 1 - 2t/\pi, \ t \in (0, \pi/2) \), and calculating the resulting integrals, we find that the term in (4.6) is larger than
\[ \frac{2\mu \pi \sin \frac{\mu \pi}{2} - \pi \mu^2 \cos \frac{\mu \pi}{2} + \sin \frac{\mu \pi}{2} (\mu^2 (\pi - 2) + \pi)}{\pi \mu^2 (\mu + 1)^2}. \]

It is easy to check that \( 2 \sin \frac{\mu \pi}{2} > \mu \cos \frac{\mu \pi}{2} \) for \( \mu \in (0, \frac{1}{\pi}) \), and therefore the proof is complete. \( \Box \)

References


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