COMPLETELY POSITIVE MATRIX NUMERICAL INDEX
ON MATRIX REGULAR OPERATOR SPACES

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Abstract. In the article, we compute the completely positive matrix numerical index of matrix regular operator spaces and show that they take values in the interval \( \left[ \frac{1}{2}, 1 \right] \). Moreover, we show that the dual of a unital operator system has the completely positive matrix numerical index \( \frac{1}{2} \) if its dimension is greater than 1. Furthermore, both \( S_p(H) \) and \( L_p(M) \) have the completely positive matrix numerical index \( 2 - \frac{1}{p} \) if their dimensions are greater than 1, where \( p \in [1, +\infty) \), \( H \) is a Hilbert space and \( M \) is a finite von Neumann algebra.

1. Introduction

The completely positive matrix numerical index \( n_{cb}^+(V) \) of a matrix pre-ordered operator space \( V \) is a constant based on the matrix norm and the matrix order of the space. It was used in [10] to characterize (non-unital) operator systems (in fact non-selfadjoint, non-unital operator systems are considered in [10]). By a (non-unital) operator system, we mean a selfadjoint linear subspace of some \( L(H) \) equipped with the induced matrix order operator space structure. Let us recall the relevant definitions. Let \( V \) be a matrix pre-ordered operator space. We define the completely positive \( n \)-matrix state space
\[
Q^V_n := \{ \varphi \in CB(V, M_n) : \varphi \in CP(V, M_n), \| \varphi \|_{cb} \leq 1 \},
\]
and the completely positive matrix numerical radius of an element \( x \) in \( M_n(V)(n \in \mathbb{N}) \) is defined as
\[
\gamma^+_n(x) := \sup \{ \| \varphi_n(x) \| : \varphi \in Q^V_n, k \in \mathbb{N} \},
\]
as well as the completely positive matrix numerical index of \( V \) is given by
\[
n_{cb}^+(V) := \inf \{ \gamma^+_n(v) : v \in M_n(V), \| v \|_n = 1, n \in \mathbb{N} \}.
\]
Note that \( n_{cb}^+(V) \) is the greatest constant \( t \geq 0 \) such that \( t\| x \| \leq \gamma^+_n(x) \) for every \( x \in M_k(V) \).

Ng shows in [10] Theorem 2.6] that \( n_{cb}^+(V) = 1 \) (respectively, \( n_{cb}^+(V) > 0 \)) if and only if it is an operator system (respectively, there exist a Hilbert space \( H \) and a map \( \Phi : V \rightarrow B(H) \) which is \( n_{cb}^+(V)^{-1} \)-completely isomorphically and complete order isomorphically). Thus, all unital operator systems have completely positive...

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matrix numerical index 1. A matrix pre-ordered operator space \( V \) is called a \textit{quasi-operator system} if \( n_{cb}^+(V) > 0 \). It may be worth reminding the readers that in some papers (see [3, 4, 14]) the term “operator system” will be used for the matrix pre-order operator space with strictly positive completely positive matrix numerical index and its abstract characterization given by Werner [14, Theorem 4.15].

The purpose of this paper is to compute the completely positive numerical index of classical matrix regular operator spaces. First, we show that \( n_{cb}^+(V) \geq \frac{1}{2} \) for any matrix regular operator space \( V \). This implies Karn’s Theorem [7] that every matrix regular operator space can be embedded into \( \mathcal{L}(H) \) 2-completely isomorphically and complete order isomorphically. We prove that a lot of matrix regular operator spaces of dimension greater than one are quasi-operator systems but not operator systems. In fact, we will prove that the predual of a von Neumann algebra and the dual of a unital operator system, with dimension greater than one, have the completely positive numerical index \( \frac{1}{2} \). Moreover, we exhibit that the spaces \( S_p(H) \) and \( L_p(M) \) for \( 1 \leq p < \infty \) have the same completely positive numerical index \( 2^{\frac{1}{p}} \), where \( H \) is a Hilbert space of dimension greater than one and \( M \) is a finite von Neumann algebra such that \( \dim(M) > 1 \). This implies that for a matrix regular operator space \( V \), the completely positive numerical index \( n_{cb}^+(V) \) can be any number in the interval \( [\frac{1}{2}, 1] \). Finally, we show that \( n_{cb}^+(CB(V, W)) \geq \frac{1}{2} n_{cb}^+(W) \), where \( V \) is a matrix regular operator space and \( W \) is a matrix pre-order operator space. In this case, \( CB(V, W) \) is a quasi-operator system when \( W \) is a quasi-operator system.

2. Preliminaries

A complex involutive vector space \( V \) is called a \textit{matrix pre-ordered vector space} if for each \( n \in \mathbb{N} \) there is a set \( M_n(V)_+ \subseteq M_n(V)_{sa} \) so that

(a) \( M_n(V)_+ \oplus M_m(V)_+ \subseteq M_{n+m}(V)_+ \) for all \( m, n \in \mathbb{N} \),
(b) \( \gamma^* M_n(V)_+ \gamma \subseteq M_n(V)_+ \) for each \( m, n \in \mathbb{N} \) and all \( \gamma \in M_{m,n} \).

A matrix pre-ordered vector space \( V \) is called a \textit{matrix pre-ordered operator space} if \( V \) is an operator space, its involution is an isometry on \( M_n(V) \) and the cones \( M_n(V)_+ \) are closed for all \( n \in \mathbb{N} \). Following W. Werner [14] we say that \( V \) is a \textit{matrix ordered operator space} if \( M_n(V)_+ \cap -M_n(V)_+ = \{0\} \) for all \( n \in \mathbb{N} \). In fact, if \( V_+ \cap -V_+ = \{0\} \), then \( M_n(V)_+ \cap -M_n(V)_+ = \{0\} \) for all \( n \in \mathbb{N} \) (see [14, Remark 2.2 (ii)]).

Let \( \Phi : V \to W \) be a linear map between two matrix pre-ordered vector spaces \( V \) and \( W \), and define \( \Phi^* \) by \( \Phi^*(v) := \Phi(v^*)^* \). We say that \( \Phi : V \to W \) is \textit{positive} if \( \Phi^* = \Phi \) and \( \Phi(V_+) \subseteq W_+ \). We let \( \Phi_n : M_n(V) \to M_n(W) \) be defined by \( \Phi_n((x_{ij})) := (\Phi(x_{ij})) \) and we call \( \Phi \) \textit{completely positive} if \( \Phi_n \) is positive for all \( n \in \mathbb{N} \). We denote the set of completely positive mappings from \( V \) to \( W \) by \( \text{CP}(V, W) \). An injective completely positive mapping \( \Phi \) is called a \textit{complete order monomorphism} if for all \( n \in \mathbb{N} \), \( \Phi_n(M_n(V)_+) = \Phi_n(M_n(V)) \cap M_n(W)_+ \).

Let \( V \) and \( W \) be matrix ordered operator spaces. We set \( M_n(CB(V, W)) := CB(V, M_n(W)) \cap \text{CP}(V, M_n(W)) \) as well as \( M_n(CB(V, W))_{sa} := \{ \varphi \in CB(V, M_n(W)) : \varphi = \varphi^* \} \).
We claim that a matrix pre-ordered operator space

\[ \exists \]

Proof. System.

Then \( \text{CB}(V, W) \) is a matrix pre-ordered operator space (see [12] Proposition 3.1). In particular, if \( W = \mathbb{C} \) we will use the symbol \( V' \) to indicate \( \text{CB}(V, \mathbb{C}) \). In this case, \( M_n(V)'_+ = \text{CB}(V, M_n)'_+ \).

**Definition 2.1.** A matrix ordered operator space is called a matrix regular (or matrical Riesz) operator space if for each \( n \in \mathbb{N} \) and for all \( v \in M_n(V)_{sa} \),

(a) \( u \in M_n(V)_{sa} \) and \( -u \leq v \leq u \) imply that \( \|v\|_n \leq \|u\|_n \);

(b) \( \|v\|_n < 1 \) implies that there exists \( u \in M_n(V)_{sa} \) such that \( \|u\|_n < 1 \) and \( -u \leq v \leq u \).

Note that examples of matrix ordered regular operator spaces include all unital operator systems and their duals, \( C^\star \)-algebras, the preduals of von Neumann algebras, the Schatten class spaces and the commutative \( L_p \) spaces [13].

The following result is due to the work of W. J. Schreiner [12].

**Proposition 2.2.** Let \( V \) be a matrix ordered operator space. Then the following are equivalent:

(a) \( V \) is a matrix regular operator space.

(b) \( V' \) is a matrix regular operator space.

(c) For each \( n \in \mathbb{N} \) and for all \( v \in M_n(V) \), \( \|v\|_n < 1 \) if and only if there exist \( u_1, u_2 \in M_n(V)_+ \), \( \|u_1\|_n < 1 \) and \( \|u_2\|_n < 1 \) such that \( \left( \begin{array}{c} u_1 \\ u_2 \end{array} \right) \in M_{2n}(V)_+ \).

3. Computing the Completely Positive Matrix Numerical Index

We will give another proof of Karn’s Theorem [7]. Our proof here is quite distinct and much simpler.

**Theorem 3.1.** If \( V \) is a matrix regular operator space, then \( n_{cb}^+(V) \geq \frac{1}{2} \).

Proof. It follows from Proposition 2.2 that \( V' \) is a matrix regular operator space, and hence given any \( f \in M_n(V) \) with \( \|f\| < 1 \), there exist \( f_1, f_2 \in Q_n^V \) such that \( \frac{1}{2} \left( f_1, f_2 \right) \in Q_n^V \). Thus for any \( x \in M_n(V) \) we have that \( \|x\| \leq 2 \gamma_n^+(x) \). This means that \( n_{cb}^+(V) \geq \frac{1}{2} \). \( \square \)

**Corollary 3.2.** Let \( V \) be the predual of a von Neumann algebra, the dual of a unital operator system, the Schatten class space \( S_p \) or the commutative \( l_p \) space for \( 1 < p < \infty \). If \( \dim(V) > 1 \), then \( V \) is a quasi-operator system but not an operator system.

Proof. We claim that a matrix pre-ordered operator space \( V \) satisfying

\[ \exists \ x_1, x_2 \in V_+ \text{ such that } \|x_1 - x_2\| > \max\{\|x_1\|, \|x_2\|\} \]

is not an operator system. Indeed, suppose that \( V \) is an operator system. Then there exist a Hilbert space \( H \) and a completely isometric complete order monomorphism \( \Phi : V \to \mathcal{L}(H) \). Set \( T_1 = \Phi(x_1) \) and \( T_2 = \Phi(x_2) \). We see that

\[ \|T_1 - T_2\| \leq \max\{\|T_1\|, \|T_2\|\} = \max\{\|x_1\|, \|x_2\|\}. \]

This contradicts the fact that

\[ \|T_1 - T_2\| = \|x_1 - x_2\| > \max\{\|x_1\|, \|x_2\|\}. \]

Notice that the predual of a von Neumann algebra with dimension greater than one satisfies condition (1) by Jordan Decomposition, and so the dual of a unital operator system by [1] Theorem 4, the Schatten class \( S_p \) space (with \( x_1 = E_{1,1} \) and
$x_2 = E_{2,2}$ and the commutative $l_p$ space (with $x_1 = e_1$ and $x_2 = e_2$) are all not operator systems. On the other hand, since these vector spaces are matrix regular operator spaces, they are quasi-operator systems by Theorem 3.1.

In view of the examples given in the last corollary, we will compute the completely positive numerical index on these classical matrix regular operator spaces.

**Theorem 3.3.** Let $M$ be a von Neumann algebra with predual $M_*$, and let $W$ be a unital operator system. If $M$ and $W$ have dimension greater than one, then

$$n_{cb}^+(M_*) = n_{cb}^+(W) = \frac{1}{2}.$$  

**Proof.** Let $V = M_*$ or $W'$. Then there exist $x_1, x_2 \in V_+$ such that

$$\|x_1\| = \|x_2\| = \frac{1}{2}\|x_1 - x_2\|.$$  

In fact, if $V = M_*$, we can find $f, g \in M_+^*$ such that $f(p) = 1$ and $g(I - p) = 1$, where $p \in M$ is a non-trivial projection (i.e., $p \neq 0, I$). Now $f_1, f_2 \in M_+^*$ are defined by setting:

$$f_1(x) := f(pxp), \ f_2(x) := g((I - p)x(I - p)) \quad \text{for any} \ x \in M.$$  

By [8, Theorem 3.3.3],

$$\|f_1\| = \|f_2\| = f_1(I) = f_2(I) = 1,$$

and so

$$\|f_1 - f_2\| \leq \|f_1\| + \|f_2\| = 2.$$  

Moreover, $u := 2p - I$ is a selfadjoint unitary in $M$ and

$$\langle f_1 - f_2, u \rangle = f_1(p) + f_2(I - p) = 2.$$  

On the other hand, it is assumed that $V = W'$. We can pick a non-zero linear functional $f \in W'$ with $f(e) = 0$. Using a similar proof of [1, Theorem 4], we can find a positive linear functional $F$ on $W \times W$ such that

$$F(e, e) = \|f\|, \ F(e, -e) = f(e) = 0 \quad \text{and} \quad f(x) = F(x, 0) - F(0, x).$$  

Let $f_1(x) := F(x, 0)$ and $f_2(x) := F(0, x)$. Then $f_1$ and $f_2$ are positive linear functionals on $W$ such that $f = f_1 - f_2$. Finally, it is evident that

$$\|f_1\| + \|f_2\| = f_1(e) + f_2(e) = F(e, e) = \|f\|$$

and that

$$\|f_1\| - \|f_2\| = f_1(e) - f_2(e) = F(e, -e) = f(e) = 0.$$  

Thus $\|f_1\| = \|f_2\| = \frac{1}{2}\|f\|$.

Next, we will prove that $n_{cb}^+(V) \leq \frac{1}{2}$. Set $x = x_1 - x_2$ and $x_1, x_2 \in V_+$ satisfying

$$\|x_1\| = \|x_2\| = \frac{1}{2}\|x\|.$$  

Then for each $n \in \mathbb{N}$ and $\varphi \in Q_n^V$,

$$\|\varphi(x)\| = \|\varphi(x_1) - \varphi(x_2)\| \leq \max\{\|\varphi(x_1)\|, \|\varphi(x_2)\|\} \leq \frac{1}{2}\|x\|.$$  

It follows that $\gamma_+^+(x) \leq \frac{1}{2}\|x\|$, and hence $n_{cb}^+(V) \leq \frac{1}{2}$. Since $V$ is a matrix regular operator space, Theorem 3.1 implies that $n_{cb}(V) = \frac{1}{2}$. \hfill $\Box$
For computing the matrix numerical index on the Schatten class spaces $S_p(H)$ and the commutative $L_p$ spaces, we need to introduce some more results. Let $V$ be a matrix pre-ordered operator space. For each $x \in M_n(V)$, the modified numerical radius is defined by

$$\nu_n(x) := \sup \left\{ |f\left( \begin{array}{cc} 0 & x \\ x^* & 0 \end{array} \right)| : f \in Q_1^{M_{kn}(V)} \right\}.$$  

**Lemma 3.4.** Let $V$ be a matrix pre-ordered operator space. Then

$$n_{cb}^+(V) = \inf \{ \nu_n(x) : x \in M_n(V), \|x\|_n = 1, n \in \mathbb{N} \} .$$

**Proof.** Let $n \in \mathbb{N}$, $x \in M_n(V)$. By [10] Lemma 2.4 (b) for each $f \in Q_1^{M_{kn}(V)}$, there exist $k \leq 2n$, $\varphi \in Q_k^V$ as well as a unit vector $\eta \in \mathbb{C}^{kn}$ satisfying

$$f(x) = \langle \varphi_{2n}(x)\eta, \eta \rangle \text{ for each } v \in M_{2n}(V).$$

Then we obtain $\nu_n(x) \leq \gamma_n(x)$. Conversely, for each $\varphi \in Q_{m^n}$, there exists a unit vector $\xi \in \mathbb{C}^{m^n}$ such that

$$\|\varphi_n(x)\| = |\langle \varphi_{2n}(x)\xi, \xi \rangle|.$$

Now, consider the functional $f \in Q_1^{M_{kn}(V)}$ given by

$$f(v) := \langle \varphi_{2n}(v)\xi, \xi \rangle \text{ for any } v \in M_{2n}(V).$$

It is clear that

$$f\left( \begin{array}{cc} 0 & x \\ x^* & 0 \end{array} \right) = \langle \varphi_{2n}(x)\xi, \xi \rangle,$$

and thus $\nu_n(x) \geq \gamma_n(x)$. We have proved that $\nu_n(x) = \gamma_n(x)$ for all $n \in \mathbb{N}$, $x \in M_n(V)$. By the definition of $n_{cb}^+(V)$, we conclude that

$$n_{cb}^+(V) = \inf \{ \nu_n(x) : x \in M_n(V), \|x\|_n = 1, n \in \mathbb{N} \} .$$

If $M$ is a semi-finite von Neumann algebra, then there exists a normal faithful semi-finite trace $\tau$ on $M$. In this case, the non-commutative $L_p$-space $L_p(M)$ is defined to be the norm closure

$$L_p(M, \tau) := cl\{ x \in M, \tau(|x|^p) < \infty \} \|\cdot\|_p,$$

with the norm given by

$$\|x\|_p := (\tau((x^*x)^{\frac{p}{2}}))^{\frac{2}{p}}.$$

The matrix order is given by the positive cones $L_p(M_n(M))^+$, and the operator space structure is given by the complex interpolation

$$M_n(L_p(M)) := \left( M_n(M), M_n(M_n^{op}) \right)_\frac{1}{2} .$$

It follows from Lemma [5,3] and [3] Theorem 2.6 that $n_{cb}^+(L_p(M)) \geq 2^{-\frac{1}{p}}$.

**Theorem 3.5.** Let $H$ be Hilbert space and $M$ be a finite von Neumann algebra. If $H$ and $M$ have dimension greater than one, then for $1 \leq p < \infty$,

$$n_{cb}^+(S_p(H)) = n_{cb}^+(L_p(M)) = 2^{-\frac{1}{p}}.$$

In particular, if $(\Omega, \Sigma, \mu)$ is a $\sigma$-finite measure space such that $\dim(L_\infty(\mu)) > 1$, then $n_{cb}^+(L_p(\mu)) = 2^{-\frac{1}{p}}$. 

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Proof. Let $N$ be $\mathcal{L}(H)$ or $M$. Then $N$ has two non-zero orthogonal projections $P, Q \in L_p(N)$. Putting $x = P/\|P\|_p - Q/\|Q\|_p$, we get $\|x\|_p^p = 2$ and

$$\|\varphi(x)\| \leq \max\{\varphi\left(\frac{P}{\|P\|_p}\right), \varphi\left(\frac{Q}{\|Q\|_p}\right)\} \leq 1$$

for all $n \in \mathbb{N}$, $\varphi \in Q^{L_p}_{n}(N)$. It follows that $\gamma_1^+(x) \leq 1$ and thus

$$n_{cb}^+(L_p(N)) \leq \frac{\gamma_1^+(x)}{\|x\|_p} \leq 2^{-\frac{1}{p}}.$$ 

Since $N = \mathcal{L}(H)$ or $M$ is a semi-finite von Neumann algebra, we have

$$n_{cb}^+(L_p(N)) \geq 2^{-\frac{1}{p}}.$$ 

It is known that an abelian von Neumann algebra is a finite von Neumann algebra. This completes the proof. □

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