HOMOMORPHISMS OF VECTOR BUNDLES ON CURVES AND PARABOLIC VECTOR BUNDLES ON A SYMMETRIC PRODUCT

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Abstract. Let $S^n(X)$ be the symmetric product of an irreducible smooth complex projective curve $X$. Given a vector bundle $E$ on $X$, there is a corresponding parabolic vector bundle $V_{E^*}$ on $S^n(X)$. If $E$ is nontrivial, it is known that $V_{E^*}$ is stable if and only if $E$ is stable. We prove that

$$H^0(S^n(X), \text{Hom}_{\text{par}}(V_{E^*}, V_{F^*})) = H^0(X, F \otimes E^*) \oplus (H^0(X, F) \otimes H^0(X, E^*)).$$

As a consequence, the map from a moduli space of vector bundles on $X$ to the corresponding moduli space of parabolic vector bundles on $S^n(X)$ is injective.

1. Introduction

Let $X$ be an irreducible smooth complex projective curve. Fixing an integer $n \geq 2$, let $S^n(X)$ be the $n$–fold symmetric product of $X$. Let $D \subset S^n(X)$ be the reduced irreducible divisor parametrizing nonreduced effective divisors of $X$ of length $n$. Let

$q_1 : S^n(X) \times X \rightarrow S^n(X) \quad \text{and} \quad q_2 : S^n(X) \times X \rightarrow X$

be the natural projections. The tautological hypersurface on $S^n(X) \times X$ will be denoted by $\Delta$. Given a vector bundle $E$ on $X$, define the vector bundle

$$\mathcal{F}(E) := q_1_*(\mathcal{O}_\Delta \otimes \mathcal{O}_{S^n(X) \times X} q_2^*E) \rightarrow S^n(X).$$

This vector bundle $\mathcal{F}(E)$ has a natural parabolic structure over the divisor $D$; the parabolic weights are 0 and $1/2$. (See [BL] for the construction of the parabolic structure.) This parabolic vector bundle will be denoted by $V_{E^*}$.

The parabolic vector bundle $V_{E^*}$ is semistable if and only if the vector bundle $E$ is semistable [BL, Lemma 1.2]. If $E$ is not the trivial vector bundle, then $V_{E^*}$ is stable if and only if $E$ is stable [BL, Theorem 1.3].

Therefore, a morphism from a moduli space of vector bundles on $X$ to a moduli space of parabolic vector bundles on $S^n(X)$ is obtained by sending any $E$ to $V_{E^*}$.

Our aim here is to show that the above morphism is injective.

For two parabolic vector bundles $V_*$ and $W_*$ on $S^n(X)$ with $D$ as the parabolic divisor and underlying vector bundles $V$ and $W$ respectively, let $\text{Hom}_{\text{par}}(V_*, W_*)$ be the vector bundle on $S^n(X)$ defined by the sheaf of homomorphisms from $V$ to $W$ preserving the parabolic structures.
We prove the following (see Corollary 3.4):

**Theorem 1.1.** Let $E$ and $F$ be stable vector bundles over $X$ with
\[
\frac{\text{degree}(E)}{\text{rank}(E)} = \frac{\text{degree}(F)}{\text{rank}(F)}.
\]
Then
\[
H^0(S^n(X), \mathcal{H}om_{\text{par}}(V_{E^*}, V_{F^*})) = 0
\]
if $E \neq F$, and
\[
\dim H^0(S^n(X), \mathcal{H}om_{\text{par}}(V_{E^*}, V_{F^*})) = 1
\]
if $E = F$.

In fact we show that for any vector bundles $E$ and $F$ on $X$,
\[
H^0(S^n(X), \mathcal{H}om_{\text{par}}(V_{E^*}, V_{F^*})) = H^0(X, F \otimes E^\vee) \oplus (H^0(X, F) \otimes H^0(X, E^\vee)).
\]
(See Theorem 3.3.)

2. Invariants of the tensor product

Let $X$ be an irreducible smooth projective curve defined over $\mathbb{C}$. Take any integer $n \geq 2$. For any $i \in \{1, \cdots, n\}$, let
\[
p_i : X^n \longrightarrow X
\]
be the projection to the $i$-th factor. The group of permutations of $\{1, \cdots, n\}$ will be denoted by $\Sigma(n)$. There is a natural action of it on $X^n$,
\[
X^n \times \Sigma(n) \longrightarrow X^n,
\]
that permutes the factors. If $V_0$ is a vector bundle on $X$, the above action of $\Sigma(n)$ on $X^n$ has a natural lift to an action of $\Sigma(n)$ on the vector bundle
\[
\mathcal{V}_0 := \bigoplus_{i=1}^n p_i^* V_0 \longrightarrow X^n
\]
which simply permutes the factors in the direct sum.

Take two algebraic vector bundles $V$ and $W$ over $X$. Define
\[
\mathcal{V} := \bigoplus_{i=1}^n p_i^* V \quad \text{and} \quad \mathcal{W} := \bigoplus_{i=1}^n p_i^* W.
\]
As noted above, $\mathcal{V}$ and $\mathcal{W}$ are equipped with an action of $\Sigma(n)$. The Künneth formula says that
\[
H^0(X^n, \mathcal{V} \otimes \mathcal{W}) = \bigoplus_{i,j=1}^n H^0(X^n, p_i^* V \otimes p_j^* W).
\]
Using the projection formula, we have
\[
H^0(X^n, p_i^* V \otimes p_i^* W) = H^0(X, V \otimes W),
\]
and if $i \neq j$, then
\[
H^0(X^n, p_i^* V \otimes p_j^* W) = H^0(X, V) \otimes H^0(X, W).
\]
Using these we get an embedding
(2.5)
\[ \Phi : H^0(X, V \otimes W) \oplus (H^0(X, V) \otimes H^0(X, W)) \longrightarrow \bigoplus_{i,j=1}^{n} H^0(X^n, p_i^*V \otimes p_j^*W) \]
that sends any \( s \in H^0(X, V \otimes W) \) to
\[ \bigoplus_{i=1}^{n} p_i^*s \in \bigoplus_{i=1}^{n} H^0(X^n, p_i^*(V \otimes W)) \subset \bigoplus_{i,j=1}^{n} H^0(X^n, p_i^*V \otimes p_j^*W) \]
and sends any \( u \otimes t \in H^0(X, V) \otimes H^0(X, W) \) to
\[ \sum_{(i,j) \in [1,n] \times [1,n]; i \neq j} (p_i^*u) \otimes (p_j^*t) \in \bigoplus_{i,j=1; i \neq j}^{n} H^0(X^n, p_i^*V \otimes p_j^*W) \subset \bigoplus_{i,j=1}^{n} H^0(X^n, p_i^*V \otimes p_j^*W). \]

The actions of \( \Sigma(n) \) of \( V \) and \( W \) together produce a linear action of \( \Sigma(n) \) on \( H^0(X^n, V \otimes W) \). Let
\[ H^0(X^n, V \otimes W)^{\Sigma(n)} \subset H^0(X^n, V \otimes W) \]
be the space of invariants.

**Proposition 2.1.** The homomorphism \( \Phi \) in (2.5) is an isomorphism of
\[ H^0(X, V \otimes W) \oplus (H^0(X, V) \otimes H^0(X, W)) \]
with \( H^0(X^n, V \otimes W)^{\Sigma(n)} \).

**Proof.** From the construction of \( \Phi \) it follows immediately that
\[ \Phi(H^0(X, V \otimes W) \oplus (H^0(X, V) \otimes H^0(X, W))) \subset H^0(X^n, V \otimes W)^{\Sigma(n)}. \]

Also, \( \Phi \) is clearly injective.

Consider the decomposition of the vector bundle
(2.6)
\[ V \otimes W = \bigoplus_{i=1}^{n} p_i^*(V \otimes W) \oplus \bigoplus_{i,j=1; i \neq j}^{n} (p_i^*V \otimes p_j^*W) \]
into a direct sum of two vector bundles. Clearly, the action of \( \Sigma(n) \) on \( V \otimes W \) leaves the two direct summands
(2.7)
\[ \bigoplus_{i=1}^{n} p_i^*(V \otimes W) \quad \text{and} \quad \bigoplus_{i,j=1; i \neq j}^{n} (p_i^*V \otimes p_j^*W) \]
in (2.6) invariant.

Since the second subbundle in (2.6) is \( \Sigma(n) \)--invariant, the subspace
(2.8)
\[ \bigoplus_{i,j=1; i \neq j}^{n} H^0(X^n, (p_i^*V \otimes p_j^*W)) \subset H^0(X^n, V \otimes W) \]
is left invariant by the action of \( \Sigma(n) \) on \( H^0(X^n, V \otimes W) \).
Let \( \mathcal{A} \) be the complex vector space of dimension \( n^2 - n \) given by the space of all functions
\[
\alpha : \{1, \cdots, n\} \times \{1, \cdots, n\} \rightarrow \mathbb{C}
\]
such that \( \alpha(i, i) = 0 \) for all \( i \in [1, n] \). The permutation action of \( \Sigma(n) \) on \( \{1, \cdots, n\} \) produces an action of \( \Sigma(n) \) on \( \mathcal{A} \). Consider the \( \Sigma(n) \)-invariant sub-space
\[
\bigoplus_{i,j=1;i\neq j}^{n} H^0(X^n, (p_i^* V) \otimes (p_j^* W))
\]
in (2.8). From (2.4) it follows that
\[
(2.9) \quad \bigoplus_{i,j=1;i\neq j}^{n} H^0(X^n, (p_i^* V) \otimes (p_j^* W)) = A^{\Sigma(n)} \otimes H^0(X, V) \otimes H^0(X, W),
\]
where \( (\bigoplus_{i,j=1;i\neq j}^{n} H^0(X^n, (p_i^* V) \otimes (p_j^* W)))^{\Sigma(n)} \) and \( A^{\Sigma(n)} \) are the spaces of invariants.

The space of invariants \( A^{\Sigma(n)} \) is generated by the function
\[
\rho : [1, n] \times [1, n] \rightarrow \mathbb{C}
\]
defined by \( (i, j) \mapsto 1 - \delta^i_j \), where \( \delta^i_j = 0 \) if \( i \neq j \) and \( \delta^i_i = 1 \) for all \( i \). This follows from Burnside’s theorem (see [La] p. 648 for Burnside’s theorem). Therefore, we have
\[
(2.10) \quad A^{\Sigma(n)} = \mathbb{C} \cdot \rho = \mathbb{C}.
\]

From (2.9) and (2.10) we conclude that
\[
(2.11) \quad \bigoplus_{i,j=1;i\neq j}^{n} H^0(X^n, (p_i^* V) \otimes (p_j^* W)) = H^0(X, V) \otimes H^0(X, W).
\]

In view of (2.9) and (2.11),
\[
(2.12) \quad H^0(X^n, V \otimes W)^{\Sigma(n)} = \bigoplus_{i=1}^{n} H^0(X^n, p_i^* (V \otimes W))^{\Sigma(n)} \oplus (H^0(X, V) \otimes H^0(X, W)).
\]

Let \( \mathcal{B} \) be the complex vector space of dimension \( n \) given by the space of all functions
\[
\{1, \cdots, n\} \rightarrow \mathbb{C}.
\]
Let \( \mathcal{B}_0 = \mathbb{C} \subset \mathcal{B} \) be the line defined by the constant functions. The group \( \Sigma(n) \) has a natural action on \( \mathcal{B} \). From (2.12)
\[
(2.13) \quad H^0(X^n, V \otimes W)^{\Sigma(n)} = (\mathcal{B}^{\Sigma(n)} \otimes H^0(X, V \otimes W)) \oplus (H^0(X, V) \otimes H^0(X, W)).
\]

It can be shown that
\[
\mathcal{B}^{\Sigma(n)} = \mathcal{B}_0,
\]
where \( \mathcal{B}_0 \) is defined above. Indeed, this is an immediate corollary of Burnside’s theorem mentioned above. Therefore, from (2.13),
\[
H^0(X^n, V \otimes W)^{\Sigma(n)} = H^0(X, V \otimes W) \oplus (H^0(X, V) \otimes H^0(X, W)).
\]
This completes the proof of the proposition. \( \square \)
3. Homomorphisms of vector bundles and parabolic vector bundles

Let

\[ f : X^n \rightarrow X^n/\Sigma(n) =: S^n(X) \]

be the quotient map to the symmetric product of \( X \). Let \( E \rightarrow X \) be a vector bundle. The action of \( \Sigma(n) \) on the vector bundle

\[ \mathcal{E} := \bigoplus_{i=1}^{n} p^*_i E \]

produces an action of \( \Sigma(n) \) on the direct image \( f_\ast \mathcal{E} \); the morphism \( f_\ast \mathcal{E} \rightarrow S^n(X) \) is \( \Sigma(n) \)--equivariant with \( \Sigma(n) \) acting trivially on \( S^n(X) \). The invariant direct image

\[ \mathcal{V}_E := (f_\ast \mathcal{E})^{\Sigma(n)} \subset f_\ast \mathcal{E} \]

is a locally free \( \mathcal{O}_{S^n(X)} \)--module. Using the action of \( \Sigma(n) \) on \( \mathcal{E} \), a parabolic structure on the vector bundle \( \mathcal{V}_E \) is constructed (see [BL, Section 3]). This parabolic vector bundle will be denoted by \( \mathcal{V}_{E_*} \). We will now quickly recall the description of \( \mathcal{V}_{E_*} \).

Let

\[ D \subset S^n(X) \]

be the reduced irreducible divisor parametrizing all \((z_1, \cdots, z_n)\) such that not all \( z_i \) are distinct. The parabolic divisor for \( \mathcal{V}_{E_*} \) is \( \bar{D} \subset X^n \) defined by

\[ \bar{D} = \{ (z_1, \cdots, z_n) : z_i \neq z_j \text{ for some } i, j \in [1, n] \}. \]

So, \( f(\bar{D}) = D \). The action of \( \Sigma(n) \) on \( \mathcal{E} \) preserves the coherent subsheaf \( \mathcal{E} \otimes \mathcal{O}_{X^n}(-\bar{D}) \). Define the invariant direct image

\[ \mathcal{V}'_E := (f_\ast (\mathcal{E} \otimes \mathcal{O}_{X^n}(-\bar{D})))^{\Sigma(n)} \subset f_\ast (\mathcal{E} \otimes \mathcal{O}_{X^n}(-\bar{D})). \]

Clearly,

\[ \mathcal{V}'_E \subset \mathcal{V}_E. \]

The parabolic bundle \( \mathcal{V}_{E_*} \) is defined as follows: \( (\mathcal{V}_E)_{1/2} = \mathcal{V}'_E \) and \( (\mathcal{V}_E)_0 = \mathcal{V}_E \) (see [MY]). Therefore, the quasi–parabolic filtration is a 1–step filtration, and it is constructed from \( \mathcal{V}'_E \); the parabolic weights are 1/2 and 0.

Let \( F \) be a vector bundle over \( X \). Define the vector bundles

\[ \mathcal{F} := \bigoplus_{i=1}^{n} p^*_i F \quad \text{and} \quad \mathcal{V}_F := (f_\ast \mathcal{F})^{\Sigma(n)}. \]

Let \( \mathcal{V}_{F_*} \) be the parabolic vector bundle on \( S^n(X) \), with \( \mathcal{V}_F \) as the underlying vector bundle and parabolic structure over \( D \), obtained by substituting \( F \) for \( E \) in the above construction of \( \mathcal{V}_{E_*} \). Define \( \mathcal{V}'_F \subset \mathcal{V}_F \) as done in (3.4). Let

\[ \mathcal{H}om_{\mathcal{par}}(\mathcal{V}_{E_*}, \mathcal{V}_{F_*}) \subset \mathcal{H}om(\mathcal{V}_{E_*}, \mathcal{V}_{F_*}) \]

be the sheaf of homomorphisms compatible with the parabolic structures [MY], [MS]. We recall that a section \( T \) of \( \mathcal{H}om(\mathcal{V}_{E_*}, \mathcal{V}_{F_*}) = \mathcal{V}_{F_*} \otimes (\mathcal{V}_{E_*})^\vee \) defined over an open subset \( U \subset S^n(X) \) lies in \( \mathcal{H}om_{\mathcal{par}}(\mathcal{V}_{E_*}, \mathcal{V}_{F_*}) \) if and only if

\[ T(\mathcal{V}'_E|_U) \subset \mathcal{V}'_F. \]
Define the vector bundle

\[(3.8)\quad W_{E,F} := \bigoplus_{i,j=1;i\neq j}^{n} p_{i}^{*}F \otimes p_{j}^{*}E^{\vee} \to X^{n}.\]

The actions of \(\Sigma(n)\) on \(E^{\vee}\) and \(F\) together define an action of \(\Sigma(n)\) on \(W_{E,F}\). Define the vector bundle

\[(3.9)\quad \mathcal{W}_{E,F} := (f_{*}W_{E,F})^{\Sigma(n)} \to S^{n}(X).\]

**Lemma 3.1.** Let \( \mathcal{V}_{F \otimes E^{\vee}} \) be the vector bundle on \( S^{n}(X) \) obtained by substituting \( F \otimes E^{\vee} \) for \( E \) in the construction of \( \mathcal{V}_{E} \). There is a canonical injective morphism of \( \mathcal{O}_{S^{n}(X)} \)-modules

\[H : \mathcal{V}_{F \otimes E^{\vee}} \oplus \mathcal{W}_{E,F} \to \mathcal{H}om_{\text{par}}(\mathcal{V}_{E}, \mathcal{V}_{F^{s}}),\]

where \( \mathcal{W}_{E,F} \) and \( \mathcal{H}om_{\text{par}}(\mathcal{V}_{E}, \mathcal{V}_{F^{s}}) \) are defined in (3.9) and (3.7) respectively.

**Proof.** Consider \( \mathcal{E} \) and \( \mathcal{F} \), defined in (3.2) and (3.6) respectively, equipped with action of \( \Sigma(n) \). From the constructions of \( \mathcal{V}_{F \otimes E^{\vee}} \) and \( \mathcal{W}_{E,F} \) it follows that

\[\mathcal{V}_{F \otimes E^{\vee}} \oplus \mathcal{W}_{E,F} = (f_{*}(\mathcal{F} \otimes E^{\vee}))^{\Sigma(n)}.\]

Note that \( \mathcal{F} \otimes E^{\vee} = (\bigoplus_{i=1}^{n} p_{i}^{*}(F \otimes E^{\vee})) \oplus W_{E,F} \), where \( W_{E,F} \) is constructed in (3.8).

Take any nonempty Zariski open subset \( U \subset S^{n}(X) \). Let

\[(3.10)\quad \phi : \mathcal{E}|_{f^{-1}(U)} \to \mathcal{F}|_{f^{-1}(U)}\]

be a homomorphism which intertwines the actions of \( \Sigma(n) \) on \( \mathcal{E}|_{f^{-1}(U)} \) and \( \mathcal{F}|_{f^{-1}(U)} \), where \( f \) is the quotient map in (3.1). Let

\[\tilde{D}_{U} := \tilde{D} \cap f^{-1}(U)\]

be the divisor on \( f^{-1}(U) \). Let

\[(3.11)\quad \tilde{\phi} := \phi \otimes \text{Id} : \mathcal{E}|_{f^{-1}(U)} \otimes \mathcal{O}_{f^{-1}(U)}(-\tilde{D}_{U}) \to \mathcal{F}|_{f^{-1}(U)} \otimes \mathcal{O}_{f^{-1}(U)}(-\tilde{D}_{U})\]

be the homomorphism, where \( \text{Id} \) is the identity automorphism of \( \mathcal{O}_{f^{-1}(U)}(-\tilde{D}_{U}) \).

The restriction of \( \phi \) to the subsheaf

\[\mathcal{E}|_{f^{-1}(U)} \otimes \mathcal{O}_{f^{-1}(U)}(-\tilde{D}_{U}) \subset \mathcal{E}|_{f^{-1}(U)}\]

clearly coincides with \( \tilde{\phi} \).

Since the action of \( \Sigma(n) \) on \( X^{n} \) leaves \( \tilde{D}_{U} \) invariant, we get an action of \( \Sigma(n) \) on \( \mathcal{O}_{f^{-1}(U)}(-\tilde{D}_{U}) \). The actions of \( \Sigma(n) \) on \( \mathcal{E}|_{f^{-1}(U)} \) and \( \mathcal{O}_{f^{-1}(U)}(-\tilde{D}_{U}) \) together produce an action of \( \Sigma(n) \) on \( \mathcal{E}|_{f^{-1}(U)} \otimes \mathcal{O}_{f^{-1}(U)}(-\tilde{D}_{U}) \). Similarly, \( \mathcal{F}|_{f^{-1}(U)} \otimes \mathcal{O}_{f^{-1}(U)}(-\tilde{D}_{U}) \) is equipped with an action of \( \Sigma(n) \). Since \( \phi \) in (3.10) is \( \Sigma(n) \)-equivariant, it follows immediately that the homomorphism \( \tilde{\phi} \) in (3.11) is also \( \Sigma(n) \)-equivariant. Consequently, \( \phi \) produces a section of \( \mathcal{H}om_{\text{par}}(\mathcal{V}_{E}, \mathcal{V}_{F^{s}}) \) over \( U \).

Therefore, we have a homomorphism of \( \mathcal{O}_{S^{n}(X)} \)-modules

\[(3.12)\quad H : \mathcal{V}_{F \otimes E^{\vee}} \oplus \mathcal{W}_{E,F} \to \mathcal{H}om_{\text{par}}(\mathcal{V}_{E}, \mathcal{V}_{F^{s}})\]

that sends any section \( \phi \) of

\[(f_{*}(\mathcal{F} \otimes E^{\vee}))^{\Sigma(n)} = \mathcal{V}_{F \otimes E^{\vee}} \oplus \mathcal{W}_{E,F}\]
over some open subset $U$ to the section of
\[ \mathcal{H}om_{\text{par}}(\mathcal{V}_{E^*}, \mathcal{V}_{F^*})|_U \]
constructed above from $\phi$.

**Proposition 3.2.** Take two vector bundles $E$ and $F$ on $X$. The homomorphism
\[ \tilde{H} : H^0(S^n(X), \mathcal{V}_{F \otimes E'}) \oplus H^0(S^n(X), \mathcal{W}_{E,F}) \rightarrow H^0(S^n(X), \mathcal{H}om_{\text{par}}(\mathcal{V}_{E^*}, \mathcal{V}_{F^*})) \]
given by $H$ in Lemma 3.1 is an isomorphism.

**Proof.** Since $H$ is injective, the corresponding homomorphism
\[ \tilde{H} : H^0(S^n(X), \mathcal{V}_{F \otimes E'}) \oplus H^0(S^n(X), \mathcal{W}_{E,F}) \rightarrow H^0(S^n(X), \mathcal{H}om_{\text{par}}(\mathcal{V}_{E^*}, \mathcal{V}_{F^*})) \]
is also injective. So to prove that $\tilde{H}$ is an isomorphism, it suffices to show that
\[
\dim H^0(S^n(X), \mathcal{H}om_{\text{par}}(\mathcal{V}_{E^*}, \mathcal{V}_{F^*})) \leq \dim H^0(S^n(X), \mathcal{V}_{F \otimes E'})
+ \dim H^0(S^n(X), \mathcal{W}_{E,F}).
\]
(3.13)

From the construction of the vector bundle $\mathcal{H}om_{\text{par}}(\mathcal{V}_{E^*}, \mathcal{V}_{F^*})$ in (3.7) it follows that
\[ \mathcal{H}om_{\text{par}}(\mathcal{V}_{E^*}, \mathcal{V}_{F^*}) \subset (f_*(\mathcal{F} \otimes \mathcal{E}'))^{\Sigma(n)} \subset f_*(\mathcal{F} \otimes \mathcal{E}'), \]
where $\mathcal{E}$ and $\mathcal{F}$ are constructed in (3.2) and (3.6) respectively. Consequently,
\[ H^0(S^n(X), \mathcal{H}om_{\text{par}}(\mathcal{V}_{E^*}, \mathcal{V}_{F^*})) \subset H^0(X^n, f_*(\mathcal{F} \otimes \mathcal{E}'))^{\Sigma(n)}. \]

Hence setting $V$ and $W$ in Proposition 2.1 to be $F$ and $E'$ respectively we conclude that
\[
H^0(S^n(X), \mathcal{H}om_{\text{par}}(\mathcal{V}_{E^*}, \mathcal{V}_{F^*})) \subset H^0(X^n, f_*(\mathcal{F} \otimes \mathcal{E}'))^{\Sigma(n)}
= H^0(X, F \otimes E') \oplus (H^0(X, F) \otimes H^0(X, E')).
\]
(3.14)

On the other hand,
\[
H^0(X, F \otimes E') \oplus (H^0(X, F) \otimes H^0(X, E'))
\subset H^0(S^n(X), \mathcal{V}_{F \otimes E'}) \oplus H^0(S^n(X), \mathcal{W}_{E,F}).
\]
(3.15)

Indeed, $H^0(X, F \otimes E') \subset H^0(S^n(X), \mathcal{V}_{F \otimes E'})$ and
\[ H^0(X, F) \otimes H^0(X, E') \subset H^0(S^n(X), \mathcal{W}_{E,F}). \]

Combining (3.14) and (3.15),
\[ H^0(S^n(X), \mathcal{H}om_{\text{par}}(\mathcal{V}_{E^*}, \mathcal{V}_{F^*})) \subset H^0(S^n(X), \mathcal{V}_{F \otimes E'}) \oplus H^0(S^n(X), \mathcal{W}_{E,F}). \]

Therefore, we conclude that the inequality in (3.13) holds. This completes the proof
of the proposition. \[\square\]

**Theorem 3.3.** There is a canonical isomorphism
\[ H^0(S^n(X), \mathcal{H}om_{\text{par}}(\mathcal{V}_{E^*}, \mathcal{V}_{F^*})) \sim H^0(X, F \otimes E') \oplus (H^0(X, F) \otimes H^0(X, E')). \]

**Proof.** From Proposition 3.2
\[
\dim H^0(S^n(X), \mathcal{H}om_{\text{par}}(\mathcal{V}_{E^*}, \mathcal{V}_{F^*})) = \dim H^0(S^n(X), \mathcal{V}_{F \otimes E'})
+ \dim H^0(S^n(X), \mathcal{W}_{E,F}).
\]
and from (3.15),
\[ \dim H^0(X, F \otimes E^\vee) + \dim(H^0(X, F) \otimes H^0(X, E^\vee)) \leq \dim H^0(S^n(X), V_{F \otimes E^\vee}) + \dim H^0(S^n(X), W_{E, F}). \]
Consequently,
\[ \dim H^0(X, F \otimes E^\vee) + \dim(H^0(X, F) \otimes H^0(X, E^\vee)) \leq \dim H^0(S^n(X), \mathcal{H}om_{\text{par}}(V_{E_*}, V_{F_*})). \]
On the other hand,
\[ \dim H^0(S^n(X), \mathcal{H}om_{\text{par}}(V_{E_*}, V_{F_*})) \leq \dim H^0(X, F \otimes E^\vee) + \dim(H^0(X, F) \otimes H^0(X, E^\vee)) \]
(see (3.14)). Combining these we conclude that
\[ h^0(S^n(X), \mathcal{H}om_{\text{par}}(V_{E_*}, V_{F_*})) = h^0(X, F \otimes E^\vee) + h^0(X, F) \cdot h^0(X, E^\vee). \]
Therefore, the subspace
\[ H^0(S^n(X), \mathcal{H}om_{\text{par}}(V_{E_*}, V_{F_*})) \]
in (3.14) coincides with the ambient space
\[ H^0(X, F \otimes E^\vee) \oplus (H^0(X, F) \otimes H^0(X, E^\vee)). \]

**Corollary 3.4.** Let $E$ and $F$ be stable vector bundles over $X$ with
\[ \text{degree}(E)/\text{rank}(E) = \text{degree}(F)/\text{rank}(F). \]
Then
\[ H^0(S^n(X), \mathcal{H}om_{\text{par}}(V_{E_*}, V_{F_*})) = 0 \]
if $E \neq F$, and
\[ \dim H^0(S^n(X), \mathcal{H}om_{\text{par}}(V_{E_*}, V_{F_*})) = 1 \]
if $E = F$.

**Proof.** If degree$(F) \leq 0$, then $H^0(X, F) = 0$ because $F$ is stable. If degree$(F) > 0$, then $H^0(X, E^\vee) = 0$ because $E^\vee$ is stable with degree$(E^\vee) < 0$.
Therefore, $H^0(X, F) \otimes H^0(X, E^\vee) = 0$. Hence the corollary follows from Theorem 3.3. □

**References**


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