A NUMBER THEORETIC QUESTION ARISING
IN THE GEOMETRY OF PLANE CURVES
AND IN BILLIARD DYNAMICS

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Abstract. We prove that if \( \rho \neq 1/2 \) is a rational number between zero and one, then there is no integer \( n > 1 \) such that
\[
n \tan(\pi \rho) = \tan(n\pi \rho).
\]
This proves a conjecture due to E. Gutkin which he formulated in connection with mathematical billiards. It also may be viewed as a rigidity result for the circle in the theory of bicycle curves.

1. Introduction

A famous problem of Ulam asks if the round ball is the only compact surface that is able to float in a liquid in neutral equilibrium in any orientation. A related problem is to classify which regular infinite cylinders are able to float in equilibrium in any orientation; in this case the relevant question is which convex plane curves can be the cross section of such a cylinder.

This second problem has recently been studied in two seemingly different contexts: plane geometry and mathematical billiards. In both cases, the classification problem leads naturally to the following number theoretic question: for given \( \rho \in (0, 1) \), does there exist an integer \( n > 1 \) satisfying \( n \tan(\pi \rho) = \tan(n \pi \rho) \)? (To the author’s knowledge, the equation \( n \tan(\pi \rho) = \tan(n \pi \rho) \) first appeared in [5].)

We will briefly describe how this question arises.

1.1. Bicycle curves. A closed, convex, unit speed plane curve \( \Gamma : S^1 \to \mathbb{R}^2 \) of perimeter length \( 2\pi \) is called a bicycle curve (of rotation number \( \rho \)) if \( \|\Gamma(t+\rho) - \Gamma(t)\| \) is constant for all \( t \) (see the end note in [7] for a list of papers dealing with bicycle curves). A theorem of Auerbach [1] shows that \( \Gamma \) is a bicycle curve if and only if its image is the cross section of an infinite cylinder that floats in neutral equilibrium in any orientation if and only if it is a bicycle curve (although he does not use this terminology and to the author’s knowledge, the connection was first pointed out in [7]).

In attempting to classify the noncircular bicycle curves, Tabachnikov [7] studied conditions under which the circle can be infinitesimally perturbed as a bicycle curve of rotation number \( \rho \). He obtained a theorem [7, Theorem 7] which says that the circle admits a nontrivial infinitesimal deformation as a smooth plane bicycle curve...
of rotation number \( \rho \) if and only if \( \rho \) is a root of the equation \( n \tan(\pi \rho) = \tan(n\pi \rho) \) for some integer \( n \geq 2 \). In light of our Theorem 1, the circle is rigid as a bicycle curve of any rational rotation number (except \( \rho = 1/2 \) where any curve of constant width is a bicycle curve).

We remark that a similar trigonometric equation was obtained by Tabachnikov to determine the rigidity of the polygonal analog of a bicycle curve (an \((n,k)\)-bicycle polygon: see [7]). In [2], R. Connelly and B. Cskós studied solutions to the equation and were able to classify the first-order flexible bicycle polygons.

1.2. Mathematical billiards. In [3], E. Gutkin studied the billiard map inside the cross section of an infinite cylinder that floats in neutral equilibrium in any orientation. He showed that if \( \Gamma \) is the boundary of a regular, noncircular billiard table, then a necessary condition for \( \Gamma \) to have a constant angle caustic is that \( n \tan(\pi \rho) = \tan(n\pi \rho) \) for some integer \( n > 1 \) and some \( \rho \in \mathbb{R} \).

He further obtained a conditional theorem [3 Theorem 3] (which now follows from Theorem 1) that if \( \Gamma \) is a regular billiard table (that is not a curve of constant width) that has a caustic of constant type, then the restriction of the billiard map to this caustic is an irrational rotation.

2. Statement of results

Our main result is the following.

**Theorem 1.** If \( \rho \in (0,1) \cap \mathbb{Q} \setminus \{1/2\} \), then there is no integer \( n > 1 \) such that
\[
n \tan(\pi \rho) = \tan(n\pi \rho).
\]

We obtain Theorem 1 as a consequence of the following two lemmas.

**Lemma 2.** Suppose \( \rho \in (0,1) \setminus \{1/2\} \). If there exists an integer \( n > 1 \) such that
\[
(1) \quad n \tan(\pi \rho) = \tan(n\pi \rho),
\]
then
\[
(2) \quad \frac{\sin((n-1)\pi \rho)}{\sin((n+1)\pi \rho)} = \frac{n-1}{n+1}.
\]

**Lemma 3.** If \( \rho \in (0,1) \cap \mathbb{Q} \setminus \{1/2\} \) and \( k, m \in \mathbb{Z} \) are such that \( \sin(m\pi \rho) \neq 0 \), then
\[
\frac{\sin(k\pi \rho)}{\sin(m\pi \rho)}
\]
is either \(-1, 0, 1\) or irrational.

**Proof of Theorem 1.** By Lemma 2, any such \( n, \rho \) would have to satisfy (2). Since \( n > 1 \) we know
\[
\frac{n-1}{n+1} \notin \{-1,0,1\},
\]
and so the pair \( k := n-1, m := n+1 \) would contradict Lemma 3 (that \( \sin((n+1)\pi \rho) \neq 0 \) follows from (2)). \( \square \)
3. Proof of Lemma 2

The proof of this lemma is elementary, and the result was already known (see [5], for example). We include it only for completeness.

For $z \in \mathbb{C} \setminus \{\frac{(2k+1)\pi}{2} : k \in \mathbb{Z}\}$,

$$\tan(z) = \frac{i(e^{-iz} - e^{iz})}{e^{-iz} + e^{iz}}.$$ 

By assumption $|\tan(\pi \rho)| < \infty$, so if $n$ satisfies (1), then $|\tan(n \pi \rho)| < \infty$. So our original equation can be rewritten as

$$n i(e^{-i\pi \rho} - e^{i\pi \rho})(e^{-i\pi \rho} + e^{i\pi \rho}) = i(e^{-i\pi \rho} - e^{i\pi \rho})(e^{-i\pi \rho} + e^{i\pi \rho}),$$

which further simplifies to

$$(n - 1)(e^{-i(n+1)\pi \rho} - e^{i(n+1)\pi \rho}) = (n + 1)(e^{-i(n-1)\pi \rho} - e^{i(n-1)\pi \rho}).$$

Since $n > 1$ we know $(n - 1)(n + 1) \neq 0$, so if $e^{-i(n+1)\pi \rho} - e^{i(n+1)\pi \rho} = 0$, then $e^{-i(n-1)\pi \rho} - e^{i(n-1)\pi \rho} = 0$. But if $e^{-i(n-1)\pi \rho} = e^{i(n-1)\pi \rho}$, then

$$e^{-i(n+1)\pi \rho} = e^{i(n-1)\pi \rho} e^{-2i\pi \rho} = e^{i(n-1)\pi \rho} e^{-2i\pi \rho} \neq e^{i(n-1)\pi \rho} e^{2i\pi \rho} = e^{i(n+1)\pi \rho},$$

which is a contradiction (here we used $\rho \neq \frac{1}{2}$). So we can divide in (3) to get

$$\frac{n - 1}{n + 1} = \frac{e^{-i(n-1)\pi \rho} - e^{i(n-1)\pi \rho}}{e^{-i(n+1)\pi \rho} - e^{i(n+1)\pi \rho}} = \frac{\frac{e^{i(n-1)\pi \rho} - e^{i(n+1)\pi \rho}}{2i}}{\frac{e^{i(n+1)\pi \rho} - e^{-i(n+1)\pi \rho}}{2i}} = \frac{\sin((n - 1)\pi \rho)}{\sin((n + 1)\pi \rho)}.$$

\[\square\]

4. Proof of Lemma 3

In this section we set $\omega_n := e^{2\pi i/n}$ to be a primitive $n^{th}$ root of unity and $\mathbb{Q}(\omega_n)$ the $n^{th}$ cyclotomic field. We need the following two well-known facts (e.g. [6]):

1. $|\mathbb{Q}(\omega_n) : \mathbb{Q}| = \phi(n)$ (Euler’s $\phi$-function);
2. $\mathbb{Q}(\omega_n) = \text{Span}\{1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}\}$.

4.1. Basic strategy of the proof. We want to show that if $\rho = \frac{p}{q}$ and $\sin(k\pi \rho) = \lambda \sin(m\pi \rho)$, then $\lambda \in \{-1, 0, 1\}$. Since $\sin(k\pi \rho), \sin(m\pi \rho) \in \mathbb{Q}(\omega_{2q})$, our main tool is the following simple (but useful) observation.

**Lemma 4.** Let $\mathcal{B}$ be a basis for a finite dimensional $\mathbb{Q}$-vector space $V$ and suppose $u, v \in V$ are vectors whose coordinates (relative to $\mathcal{B}$) all come from the set $\{-1, 0, 1\}$. If $u = \lambda v$ for some $\lambda \in \mathbb{Q}$, then $\lambda \in \{-1, 0, 1\}$.

We will prove Lemma 3 by (explicitly) constructing a basis $\mathcal{B}$ for $\mathbb{Q}(\omega_{2q})$ in which, for every integer $\ell$, $i\mathcal{B}(\omega_{2q}^\ell)$ is of the type described in Lemma 4.
4.2. Motivating case. Here we will prove Lemma 3 in the special case that the
denominator of $\rho$ is an odd prime (see Corollary 6). This subsection is not necessary
to prove Lemma 3 but is included to demonstrate the main idea of the proof.

Lemma 5. If $n$ is an odd prime and we define
\[
A_n := \{ \Re(\omega_n), \Re(\omega_n^2), \ldots, \Re(\omega_n^{\phi(n)/2}) \},
\]
\[
B_n := \{ i\Im(\omega_n), i\Im(\omega_n^2), \ldots, i\Im(\omega_n^{\phi(n)/2}) \},
\]
then $A_n \cup B_n$ is a basis for $\mathbb{Q}(\omega_n)$ over $\mathbb{Q}$.

Proof. First note that $\omega_n^k \in \text{Span}(A_n \cup B_n)$ for every $k = 1, 2, \ldots, n$ (sum the
geometric series $\omega_n + \omega_n^2 + \cdots + \omega_n^{n-1} = -1$ to see the $k = n$ case). Next $|A_n \cup B_n| = n - 1 = \phi(n)$ (since $n$ is prime). Any spanning set with $\phi(n)$ elements is a basis. \qed

Since $|\mathbb{Q}(\omega_n) : \mathbb{Q}(\omega)| = \phi(2n) = \phi(n) = |\mathbb{Q}(\omega_n) : \mathbb{Q}|$, $A_n \cup B_n$ is also a basis for $\mathbb{Q}(\omega_{2n})$ over $\mathbb{Q}$.

Corollary 6 (Lemma 3 for an odd prime). If $n$ is an odd prime and $k_1, k_2 \in \mathbb{Z}$ are
such that $\sin \left( \frac{k\pi}{n} \right) \neq 0$, then
\[
\sin \left( \frac{k_1\pi}{n} \right) = \frac{\sin \left( \frac{k\pi}{n} \right)}{\sin \left( \frac{k_2\pi}{n} \right)}
\]
is either $-1, 0, 1$ or irrational.

Proof. If $\sin \left( \frac{k_1\pi}{n} \right) = 0$, we are done. Otherwise find $0 < k_1 \leq k_2 \leq \frac{n-1}{2}$ such that
\[
i \sin \left( \frac{k_1\pi}{n} \right) = \pm i\Im(\omega_{2n}^{k_1}),
\]
\[
i \sin \left( \frac{k_2\pi}{n} \right) = \pm i\Im(\omega_{2n}^{k_2}).
\]

If $\frac{k_1}{n}$ is a rational number, say $i\Im(\omega_{2n}^{k_1}) = i\lambda \Im(\omega_{2n}^{k_2})$ for some $\lambda \in \mathbb{Q}$, then $\lambda \in \{-1, 0, 1\}$ by Lemma 3 (applied to the basis constructed in Lemma 5). \qed

4.3. Prime powers. Here we generalize Lemma 5 to the case where the
denominator of $\rho$ is a prime power.

Lemma 7. If $n = p^k$ is an odd prime power and we define (for integers $t$)
\[
A_{p^k} := \{ \Re(\omega_{p^k}^t) : 1 \leq t \leq \phi(n)/2 \},
\]
\[
B_{p^k} := \{ i\Im(\omega_{p^k}^t) : 1 \leq t \leq \phi(n)/2 \},
\]
then $A_n \cup B_n$ is a basis for $\mathbb{Q}(\omega_n)$ over $\mathbb{Q}$. Moreover for any integer $t$, all coefficients
of the vectors $\Re(\omega_{p^k}^t)$ and $i\Im(\omega_{p^k}^t)$ (with respect to this basis) are contained in the set
\{-1, 0, 1\}.

Proof. The set $A_n \cup B_n$ contains exactly $\phi(n) = p^{k-1}(p-1)$ many elements, so to
prove the first claim it suffices to show that $\text{Span}\{A_n \cup B_n\} = \mathbb{Q}(\omega_n)$ or simply
that $\omega_{p^k}^t \in \text{Span}\{A_n \cup B_n\}$ for every integer $t \in [0, p^k - 1]$.

It is immediate that if $1 \leq t \leq \frac{p^{k-1}(p-1)}{2}$, then $\omega_{p^k}^{\pm t} \in \text{Span}\{A_n \cup B_n\}$ (recall that
$\Re(\omega_{p^k}^{-t}) = \Re(\omega_{p^k}^t)$ and $\Im(\omega_{p^k}^{-t}) = -\Im(\omega_{p^k}^t)$). That is, the only integers $t \in [1, p^k - 1]$
for which we have not yet verified that $\omega_{p^k}^t \in \text{Span}\{A_n \cup B_n\}$ are those satisfying
\[
\frac{p^{k-1}(p-1)}{2} < t < \frac{p^{k-1}(p-1)}{2} + p^{k-1}.
\]
If $0 < s < p^{k-1}$, then there is precisely one $r \in [0, (p - 1)/2)$ so that
\[ t_s := rp^{k-1} + s \in \left( \frac{p^{k-1}(p - 1)}{2}, \frac{p^{k-1}(p - 1)}{2} + p^{k-1} \right). \]

On the other hand, for any $s$,\n\[ \omega_s^{p^{k-1} + s} + \omega_s^{p^{k-1} + s} + \cdots + \omega_s^{p^{k-1} + s} = 0 \]
(divide both sides by $\omega_s^{p^{k-1}}$ and sum the geometric series). In light of (3) and our previous verification that $\omega_n^t \in \text{Span}\{A_n \cup B_n\}$ for every $t \equiv s \pmod{p^{k-1}}$ (except $t_s$ itself), we see $\omega_n^t \in \text{Span}\{A_n \cup B_n\}$. It remains only to see that $1 = \omega_n^0 \in \text{Span}\{A_n \cup B_n\}$. This follows from (3) with $s = 0$ and the observation that $rp^{k-1} \not\in (\frac{p^{k-1}(p - 1)}{2}, \frac{p^{k-1}(p - 1)}{2} + p^{k-1})$ for any integer $r$.

The second claim follows from the previous two paragraphs. \hfill $\square$

**Lemma 8.** If $n = 2^k$ and we define (for integers $t$)
\[ A_{2^k} := \{ \Re(\omega_n^t) : 0 \leq t < 2^{k-2} \}, \]
\[ B_{2^k} := \{ i\Im(\omega_n^t) : 0 < t < 2^{k-2} \}, \]
then $A_n \cup B_n$ is a basis for $\mathbb{Q}(\omega_n)$ over $\mathbb{Q}$. Moreover for any integer $t$, all coefficients of the vectors $\Re(\omega_n^t)$ and $i\Im(\omega_n^t)$ (with respect to this basis) are contained in the set $\{-1, 0, 1\}$.

**Proof.** As before, the number of elements in $A_{2^k} \cup B_{2^k}$ is $2^{k-1} = \phi(2^k)$ so it suffices to see that $\omega_n^t \in \text{Span}(A_{2^k} \cup B_{2^k})$ for every $0 \leq t < 2^k$.

It is immediate that $\omega_n^t \in \text{Span}(A_{2^k} \cup B_{2^k})$ for $0 \leq t < 2^{k-2}$ (note that $\Re(\omega_n^{2^{k-2}}) = \Im(\omega_n^0) = 0$). This describes the set of all $(2^k)^{th}$ roots of unity in the positive quadrant in $\mathbb{C}$. The set of all roots of $x^{2^k} - 1$ is symmetric about the real and imaginary axes, so $\omega_n^t \in \text{Span}(A_{2^k} \cup B_{2^k})$ for every $t$.

Again, the second claim is by construction. \hfill $\square$

**4.4. General result.** Finally we are ready to prove Lemma 3 in full generality. We begin with some notation: if $S_1, S_2, \ldots, S_n$ are (nonempty) subsets of $\mathbb{C}$, define $S_1S_2 \cdots S_n := \{ \alpha_1\alpha_2 \cdots \alpha_n \in \mathbb{C} : \alpha_i \in S_i \}$.

**Lemma 9.** If $p_1^{j_1}p_2^{j_2} \cdots p_k^{j_k}$ is the prime factorization of an integer $n$ (for ease of notation, set $q_i := p_i^{j_i}$) and we define, for $1 \leq i \leq k$ and $0 \leq j < 1$,
\[ D_i^j := \begin{cases} A_{q_i} & \text{if } j = 0, \\ B_{q_i} & \text{if } j = 1 \end{cases} \]
and set
\[ D_n := \bigcup_{(j_1, \ldots, j_k) \in \{0,1\}^k} D_1^{j_1} \cdots D_k^{j_k}, \]
then $D_n$ is a basis for $\mathbb{Q}(\omega_n)$ over $\mathbb{Q}$. Moreover for any integer $t$, all coefficients of the vectors $\Re(\omega_n^t)$ and $i\Im(\omega_n^t)$ (with respect to this basis) are contained in the set $\{-1, 0, 1\}$.

**Proof.** It is immediate that $D_n$ is a basis for $\mathbb{Q}(\omega_n)$, but it remains to see that all coefficients (with respect to the basis $D_n$) of the vectors $\Re(\omega_n^t)$ and $i\Im(\omega_n^t)$ are in the set $\{-1, 0, 1\}$. 

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Suppose $0 \leq t < n$ and write
\[ t = s_1(n/q_1) + s_2(n/q_2) + \cdots + s_k(n/q_k), \]
where $0 \leq s_i < q_i$ for $i = 1, 2, \ldots, k$. Then $\omega_n^t = \omega_{q_1}^{s_1} \omega_{q_2}^{s_2} \cdots \omega_{q_k}^{s_k}$. Write
\[
\Re(\omega_{q_i}^{s_i}) = \lambda_{i,1} \Re(\omega_{q_i}) + \lambda_{i,2} \Re(\omega_{q_i}^2) + \cdots + \lambda_{i,\phi(q_i)/2} \Re(\omega_{q_i}^{\phi(q_i)/2}),
\]
\[
\Im(\omega_{q_i}^{s_i}) = \mu_{i,1} \Im(\omega_{q_i}) + \mu_{i,2} \Im(\omega_{q_i}^2) + \cdots + \mu_{i,\phi(q_i)/2} \Im(\omega_{q_i}^{\phi(q_i)/2}),
\]
where $\lambda_{i,j}, \mu_{i,j} \in \{-1, 0, 1\}$ (possible by Lemmas 7 and 8). Then $\Re(\omega_n^t)$ and $i\Im(\omega_n^t)$ are both sums of expressions of the form
\[ \pm \delta_1 \delta_2 \cdots \delta_k G_1 G_2 \cdots G_k, \]
where $G_i \in A_{q_i} \cup B_{q_i}$ and $\delta_i \in \{\lambda_{i,1}, \ldots, \lambda_{i,\phi(q_i)/2}, \mu_{i,1}, \ldots, \mu_{i,\phi(q_i)/2}\}$. The result follows.

Proof of Lemma 3. Let $\rho \in (0, 1) \cap \mathbb{Q} \setminus \{1/2\}$ and suppose $k, m \in \mathbb{Z}$ are such that $\sin(m\pi \rho) \neq 0$. If $\sin(k\pi \rho)/\sin(m\pi \rho) = \lambda \in \mathbb{Q}$, then, by Lemmas 4 and 9, $\lambda \in \{-1, 0, 1\}$.

References

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