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COMPACT OPERATORS IN TRO'S

G. ANDREOLAS

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ABSTRACT. We give a geometric characterization of the elements of a ternary ring of operators (or simply, TRO) that can be represented as compact operators by a faithful representation of the TRO.

1. INTRODUCTION

A *ternary ring of operators* (or simply, TRO) between Hilbert spaces H_2 and H_1 is a norm closed subspace $\mathcal{V}$ of $\mathcal{B}(H_2, H_1)$ which is closed under the triple product

$$\mathcal{V} \times \mathcal{V} \times \mathcal{V} \ni (x, y, z) \mapsto xy^* z \in \mathcal{V}.$$

A TRO $\mathcal{V} \subseteq \mathcal{B}(H_2, H_1)$ is called a w*-TRO if it is w*-closed (equivalently, weak operator closed, or strong operator closed) in $\mathcal{B}(H_2, H_1)$. TRO's were first introduced by Hestenes [9] and since then they have been studied by many authors. In general, a TRO $\mathcal{V}$ can be identified with the off-diagonal corner (at the (1,2) position) of its *linking* C^*-algebra

$$A(\mathcal{V}) = \begin{pmatrix} \mathcal{C} & \mathcal{V} \\ \mathcal{V}^* & \mathcal{D} \end{pmatrix} \subseteq \mathcal{B}(H_1 \oplus H_2),$$

where $\mathcal{C}$ and $\mathcal{D}$ are the C^*-algebras generated by $\mathcal{V}\mathcal{V}^*$ and $\mathcal{V}^*\mathcal{V}$ respectively.

If $\mathcal{S}$ is a nonempty subset of the unit ball of a normed space $\mathcal{X}$, then the *contractive perturbations* of $\mathcal{S}$ are defined as

$$\mathrm{cp}(\mathcal{S}) = \{x \in \mathcal{X} \quad \|x \pm s\| \leq 1 \ \forall s \in \mathcal{S}\}.$$

It is clear that $\mathcal{S}_1 \subseteq \mathcal{S}_2$ implies $\mathrm{cp}(\mathcal{S}_1) \supseteq \mathrm{cp}(\mathcal{S}_2)$. Also, an element x of the unit ball of $\mathcal{X}$ is an extreme point if and only if $\mathrm{cp}(\{x\}) = \{0\}$. We shall write $\mathrm{cp}(x)$ instead of $\mathrm{cp}(\{x\})$.

One may define contractive perturbations of higher order by using the recursive formula $\mathrm{cp}^{n+1}(\mathcal{S}) = \mathrm{cp}\left(\mathrm{cp}^n(\mathcal{S})\right)$, $n \in \mathbb{N}$. It is clear that $\mathrm{cp}(\mathcal{S})$ is a norm-closed convex subset of the closed unit ball of $\mathcal{X}$. One can also verify that $\mathcal{S} \subseteq \mathrm{cp}^2(\mathcal{S})$; from this it follows that $\mathrm{cp}^3(\mathcal{S}) = \mathrm{cp}(\mathcal{S})$. The second contractive perturbations were introduced in [2]. In [2] it is proved that the set of the second contractive perturbations of an element a of a C^*-algebra $\mathcal{A}$ is compact in the norm topology if and only if there exists a faithful representation ϕ of $\mathcal{A}$ such that $\phi(a)$ is a compact operator. Further study was conducted in [1], [3], [4] and [11]. We shall see that this characterization is not valid for the elements of a TRO.

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In this work we characterize the elements of a TRO that are represented as compact operators by a faithful representation of the TRO, in terms of the size of their contractive perturbations. We show that there exists a faithful representation ϕ of the TRO $\mathcal{V}$ that maps an element a of the unit ball of $\mathcal{V}$ to a compact operator if and only if the set of its second contractive perturbations is weakly compact. It follows from [2] and our result that for an element a of a C^*-algebra $\mathcal{A}$ the set $\mathrm{cp}^2(a)$ is compact if and only if it is weakly compact or, equivalently, there exists a faithful representation π of $\mathcal{A}$ such that $\pi(a)$ is a compact operator.

Ylinen proved in [16] and [17] that for an element a of a C^*-algebra $\mathcal{A}$ the operator $x \to axa$ on $\mathcal{A}$ is compact if and only if it is weakly compact or, equivalently, there exists a faithful representation π of $\mathcal{A}$ such that $\pi(a)$ is a compact operator. We obtain an analogous result for the operator $x \to ax^*a$ on a TRO.

Notation. Throughout, we adopt the following notation: H_1 and H_2 are Hilbert spaces, $\mathcal{B}(H_2, H_1)$ the space of all bounded linear operators $H_2 \to H_1$ and $\mathcal{K}(H_2, H_1)$ the space of all compact operators $H_2 \to H_1$. In particular, $\mathcal{B}(H_1) = \mathcal{B}(H_1, H_1)$ and $\mathcal{K}(H_1, H_1) = \mathcal{K}(H_1)$. $\mathcal{V}$ is a TRO that is a subspace of $\mathcal{B}(H_2, H_1)$. Let $\mathcal{X}$ be a Banach space, $\mathcal{Y} \subseteq \mathcal{X}$ a subspace and $a \in \mathcal{Y}$. Then by $\mathrm{cp}^n_{\mathcal{Y}}(a)$ we denote the set of the n-th contractive perturbations of a computed with respect to $\mathcal{Y}$. If r is a positive number, then by $\mathcal{X}_r$ we denote the closed ball of center 0 and radius r. Let x, y be elements of a Hilbert space H. We denote by $x \otimes y$ the rank one operator on H defined by

$$(x \otimes y)(z) = \langle z, x\rangle y.$$

2. Preliminaries

Let $\mathcal{V}$ and $\mathcal{W}$ be two TRO's. A linear map $\phi : \mathcal{V} \to \mathcal{W}$ is called a *TRO-homomorphism* if it preserves the ternary product

$$\phi(xy^*z) = \phi(x)\phi(y)^*\phi(z)$$

for all $x, y, z \in \mathcal{V}$. If, in addition, ϕ is an injection from $\mathcal{V}$ onto $\mathcal{W}$, we call ϕ a *TRO-isomorphism* from $\mathcal{V}$ onto $\mathcal{W}$. A TRO-homomorphism ϕ from a TRO $\mathcal{V}$ into the set of all bounded operators from one Hilbert space to another is called a *representation* of $\mathcal{V}$. We will say that a representation $\phi : \mathcal{V} \to \mathcal{B}(H_2, H_1)$ is a *faithful* representation of $\mathcal{V}$ if ϕ is injective. It was shown in [8, Proposition 3.4] that every faithful TRO-representation is an isometry.

Proposition 2.1. *Let H_1 and H_2 be Hilbert spaces, $\mathcal{V} \subseteq \mathcal{B}(H_2, H_1)$ a TRO and $A(\mathcal{V})$ its linking algebra. If a is in the unit ball of $\mathcal{V}$, then $\mathrm{cp}^2_{A(\mathcal{V})}(a) \subseteq \mathrm{cp}^2_{\mathcal{V}}(a)$.*

Proof. First we note that if $\mathcal{E}$ is a Banach space and $b \in \mathcal{E}$ has the property $\|x + b\| \leq 1$ for all $x \in \mathcal{E}$ with $\|x\| \leq 1$, then $b = 0$. Indeed, if the above property holds for some $b \neq 0$, then taking $x = b/\|b\|$ we have $\|b/\|b\| + b\| \leq 1$, which implies that $\|b\| = 0$. This yields a contradiction.

Let

$$\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in \mathrm{cp}^2_{A(\mathcal{V})}(a).$$

We can easily see that

$$\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \in \mathrm{cp}_{A(\mathcal{V})}(a)$$

for every $x \in \mathrm{cp}_{\mathcal{V}}(a)$ and $y \in \mathcal{V}^*$ with $\|y\| \leq 1$. So, it follows directly that $b_2 \in \mathrm{cp}^2_{\mathcal{V}}(a)$, while from the remark at the beginning of the proof it follows that $b_3 = 0$. We have that $\begin{pmatrix} b_1 & b_2 \\ y & b_4 \end{pmatrix}$ is a contraction for every $y \in \mathcal{V}^*$ with $\|y\| \leq 1$. Thus, if $\eta \in H_1$, we have that

$$\|b_1\eta\|^2 + \|y\eta\|^2 \leq \|\eta\|^2 \tag{1}$$

for all $y \in \mathcal{V}^*$ with $\|y\| \leq 1$. Since the strong*-topology of $\mathcal{V}^*$ is finer than its strong topology, it follows from [8, Theorem 3.6 (Kaplansky density theorem)] that the inequality (1) holds for all y in the closed unit ball of $\overline{\mathcal{V}^*}^{w^*}$. Therefore, for all partial isometries $y \in \overline{\mathcal{V}^*}^{w^*}$, we have $0 \leq b_1^*b_1 + y^*y \leq 1$. Denoting by p_y the domain projection y^*y of a partial isometry $y \in \overline{\mathcal{V}^*}^{w^*}$, it follows that $0 \leq b_1^*b_1 \leq 1 - p_y$. Multiplying by p_y, we deduce that $p_y b_1^* b_1 p_y = 0$ or $b_1 p_y = 0$. Let Π be the set of all partial isometries of $\overline{\mathcal{V}^*}^{w^*}$ and p the orthogonal projection onto the closed linear span of the subspaces $\{p_y(H_2)\}_{y\in\Pi}$. Then we have proved that $b_1 p = 0$. On the other hand, we can see that $b_1 p^{\perp} = 0$, since b_1 is in the C^*-algebra generated by $\mathcal{V}\mathcal{V}^*$ and $\overline{\mathcal{V}^*}^{w^*}$ is generated by its partial isometries [8, Theorem 3.2]. Hence, we have proved that $b_1 = 0$. By symmetry, we obtain $b_4 = 0$. Thus, we showed that each element of $\mathrm{cp}^2_{A(\mathcal{V})}(a)$ is in $\mathcal{V}$. The fact that $\mathrm{cp}^2_{A(\mathcal{V})}(a) \subseteq \mathrm{cp}^2_{\mathcal{V}}(a)$ is immediate. $\square$

Remark 2.2. The containment in the last proposition may be strict. We shall give an example. Let H_1 and H_2 be Hilbert spaces with $\dim H_1 = \infty$ and $\dim H_2 < \infty$ and $u : H_2 \to H_1$ an isometry. Let $\mathcal{V} = \mathcal{B}(H_2, H_1)$. Since u is an extreme point of $\mathcal{V}$ [18], the set $\mathrm{cp}^2_{\mathcal{V}}(u)$ is equal to the unit ball of $\mathcal{V}$. Now, $A(\mathcal{V}) = \mathcal{B}(H_1 \oplus H_2)$ and it follows from [2, Corollary 2.4] that $\mathrm{cp}^2_{A(\mathcal{V})}(u)$ is compact. Hence, the inclusion $\mathrm{cp}^2_{A(\mathcal{V})}(u) \subset \mathrm{cp}^2_{\mathcal{V}}(u)$ is strict. Considering the identity representation of $\mathcal{V}$ in this example, one can see that the implication $(i) \Rightarrow (ii)$ of [2, Theorem 2.2] does not hold for TRO's.

Remark 2.3. It is known that the linking algebra $A(\mathcal{V})$ is just the C^*-envelope $C^*_e(\mathcal{V})$ of the TRO $\mathcal{V}$. Therefore, the inclusion in Proposition 2.1 in the case of an operator space $\mathcal{O}$ would be $\mathrm{cp}^2_{C^*_e(\mathcal{O})}(a) \subseteq \mathrm{cp}^2_{\mathcal{O}}(a)$. Now, we shall see that this inclusion does not hold in operator spaces in general.

Let H be an infinite dimensional Hilbert space,

$$\mathcal{O} = \left\{ \begin{bmatrix} \lambda\, Id & a \\ 0 & \mu\, Id \end{bmatrix} \ : \ a \in \mathcal{K}(H), \lambda, \mu \in \mathbb{C} \right\}.$$

The C^*-algebra generated by $\mathcal{O}$ in $\mathcal{B}(H \oplus H)$ is

$$C^*_{\mathcal{B}(H\oplus H)}(\mathcal{O}) = \left\{ \begin{bmatrix} \lambda\, Id + a & b \\ c & \mu\, Id + d \end{bmatrix} \ : \ a, b, c, d \in \mathcal{K}(H), \lambda, \mu \in \mathbb{C} \right\}.$$

If $\mathcal{I}$ is a proper ideal of $C^*_{\mathcal{B}(H\oplus H)}(\mathcal{O})$, then $\mathcal{I}$ contains $\mathcal{K}(H \oplus H)$, the compact operators on $H \oplus H$ and, consequently, the quotient space $C^*_{\mathcal{B}(H\oplus H)}(\mathcal{O})/\mathcal{I}$ is finite dimensional. Therefore, $C^*_e(\mathcal{O}) = C^*_{\mathcal{B}(H\oplus H)}(\mathcal{O})$. Then if we consider the operator

$$s = \begin{bmatrix} Id & 0 \\ 0 & 0 \end{bmatrix},$$

we see that

$$\mathrm{cp}^2_{C_e^*(\mathcal{O})}(s) = \left\{ \begin{bmatrix} \lambda\, Id + x & 0 \\ 0 & 0 \end{bmatrix} : \ \lambda \in \mathbb{C},\, x \in \mathcal{K}(H),\, \|\lambda\, Id + x\| \leq 1 \right\}$$

and that

$$\mathrm{cp}^2_{\mathcal{O}}(s) = \left\{ \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} : \ \lambda \in \mathbb{C},\, |\lambda| \leq 1 \right\}.$$

Proposition 2.4. *Let H_1 and H_2 be Hilbert spaces with $\dim H_1 = \infty$ and $\dim H_2 = \infty$ and $\mathcal{V} \subseteq \mathcal{B}(H_2, H_1)$ a TRO. Let $a = \sum_{i=1}^{\infty} \lambda_i u_i \in \mathcal{V}$ be a norm one compact operator, where $\{u_i\}_{i=1}^{\infty}$ are finite rank partial isometries such that $u_i u_j^* = 0$, $u_i^* u_j = 0$ for $i \neq j$ and $\{\lambda_i\}_{i=1}^{\infty}$ is a sequence of positive numbers decreasing to 0. Let $e_k = \sum_{i=1}^{k} u_i^* u_i$ and $f_k = \sum_{i=1}^{k} u_i u_i^*$. If x is any contraction in $\mathcal{V}$, then*

(1) $\|a \pm (1-\lambda_k) f_k^{\perp} x e_k^{\perp}\| \leq 1$,
(2) *e_k and f_k are in the C^*-algebra generated by $\mathcal{V}^*\mathcal{V}$ and $\mathcal{V}\mathcal{V}^*$ respectively,*
(3) $f_k^{\perp} x e_k^{\perp} \in \mathcal{V}$.

Proof. (1) Let y be a contraction in $\mathcal{V}$. From [12] we know that

$$\|a \pm (\mathbf{1} - |a^*|)^{1/2} y (\mathbf{1} - |a|)^{1/2}\| \leq 1.$$

Simple computations show that

$$\left\| a \pm \left(\sum_{i=1}^{\infty} (1-\lambda_i)^{1/2} u_i u_i^* + f^{\perp} \right) y \left(\sum_{j=1}^{\infty} (1-\lambda_j)^{1/2} u_j^* u_j + e^{\perp} \right) \right\| \leq 1,$$

where $e = [\{u_i^* u_i\}_{i\in\mathbb{N}}]$ and $f = [\{u_i u_i^*\}_{i\in\mathbb{N}}]$. Now setting

$$y = \left(\sum_{i=k+1}^{\infty} \frac{(1-\lambda_k)^{1/2}}{(1-\lambda_i)^{1/2}} u_i u_i^* + (1-\lambda_k)^{1/2} f^{\perp} \right) x \\ \times \left(\sum_{j=k+1}^{\infty} \frac{(1-\lambda_k)^{1/2}}{(1-\lambda_j)^{1/2}} u_j^* u_j + (1-\lambda_k)^{1/2} e^{\perp} \right),$$

where $x \in \mathcal{V}$ is a contraction, we get the result.

(2) Assume that $\lambda_1 = 1$. We define a sequence $(a_i)_{i\in\mathbb{N}}$ in $\mathcal{V}$, where $a_1 = a$ and $a_n = a_{n-1} a_{n-1}^* a_{n-1}$. Simple computations show that $\lim_{n\to\infty} a_n = u_1$ is in $\mathcal{V}$. That means that $a - u_1 = \sum_{i=2}^{\infty} \lambda_i u_i$ is in $\mathcal{V}$ and using the same argument for $a - u_1$ we deduce $u_2 \in \mathcal{V}$ and continuing in the above fashion, we inductively get $u_n \in \mathcal{V}$ for all $n \in \mathbb{N}$. Hence, $u_n^* \in \mathcal{V}^*$ for all $n \in \mathbb{N}$. It follows that $e_k = \sum_{i=1}^{k} u_i^* u_i = (\sum_{l=1}^{k} u_l^*)(\sum_{m=1}^{k} u_m) \in \mathcal{V}^*\mathcal{V}$ and $f_k = \sum_{i=1}^{k} u_i u_i^* = (\sum_{l=1}^{k} u_l)(\sum_{m=1}^{k} u_m^*) \in \mathcal{V}\mathcal{V}^*$.

(3) Since $x \in \mathcal{V}$, $x e_k \in \mathcal{V}$, $f_k x \in \mathcal{V}$ and $f_k x f_k \in \mathcal{V}$, it follows that $f_k^{\perp} x e_k^{\perp} = (\mathbf{1} - f_k) x (\mathbf{1} - e_k) = x - x e_k - f_k x + f_k x e_k$ is in $\mathcal{V}$. □

Proposition 2.5. *Let H_1 and H_2 be Hilbert spaces with $\dim H_1 = \infty$ and $\dim H_2 = \infty$ and $\mathcal{V} \subseteq \mathcal{B}(H_2, H_1)$ a TRO. If $\mathcal{C}$ is the TRO that consists of all compact operators of $\mathcal{V}$, then $\mathrm{cp}^2_{\mathcal{V}}(a) \subseteq \mathrm{cp}^2_{\mathcal{C}}(a)$ for all $a \in \mathcal{C}$.*

Proof. Let $a \in \mathcal{C}$. It suffices to show that $\mathrm{cp}^2_{\mathcal{V}}(a) \subseteq \mathcal{K}(H_2, H_1)$. We shall show that if $x \in \mathcal{V} \backslash \mathcal{C}$, then $x \notin \mathrm{cp}^2_{\mathcal{V}}(a)$. Since the operator x is not compact, there exists an

$\varepsilon > 0$ such that for all finite rank projections f, e on H_1, and H_2 respectively, the inequality

$$\|f^{\perp} x e^{\perp}\| > \varepsilon$$

holds, where $f^{\perp} = \mathbf{1} - f$ and $e^{\perp} = \mathbf{1} - e$. Given that the operator a is compact, there exists a unique sequence of positive numbers $(\lambda_i)_{i\in\mathbb{N}}$ decreasing to 0 and a sequence $\{u_i\}_{i=1}^{\infty}$ of finite rank partial isometries with $u_i u_j^* = 0$, $u_i^* u_j = 0$ for $i \neq j$ such that

$$a = \sum_{i=1}^{\infty} \lambda_i u_i.$$

Let $e_k = \sum_{i=1}^{k} u_i^* u_i$ and $f_k = \sum_{i=1}^{k} u_i u_i^*$ for all $k \in \mathbb{N}$. From Proposition 2.4 we know that if y is any contraction in $\mathcal{V}$, then $(1-\lambda_k) f_k^{\perp} y e_k^{\perp} \in \mathrm{cp}_{\mathcal{V}}(a)$. Thus, it suffices to find a $k \in \mathbb{N}$ and a contraction $y \in \mathcal{V}$ such that $\|x \pm (1-\lambda_k) f_k^{\perp} y e_k^{\perp}\| > 1$.

We choose k so that $\lambda_k < \varepsilon$, set $x_k = f_k^{\perp} x e_k^{\perp}$ and $y = x_k/\|x_k\|$. The following computations complete the proof:

$$\begin{aligned}
\|x + (1-\lambda_k) f_k^{\perp} y e_k^{\perp}\| &\geq \|x_k + (1-\lambda_k) y\| \\
&= \left\| x_k + (1-\lambda_k) \frac{x_k}{\|x_k\|} \right\| = \|x_k\| \left| 1 + (1-\lambda_k) \frac{1}{\|x_k\|} \right| \\
&= \|x_k\| + (1-\lambda_k) > \varepsilon + 1 - \varepsilon = 1.
\end{aligned}$$

□

3. The main results

We have seen in Remark 2.2 that the characterization given in [2, Theorem 2.2] does not hold for TRO's. In this section we shall show that there exists a faithful representation ϕ of the TRO $\mathcal{V}$ that maps an element $a \in \mathcal{V}_1$ to a compact operator if and only if the set $\mathrm{cp}^2_{\mathcal{V}}(a)$ is weakly compact. This is one of the main results of this work.

Note that if π is a faithful representation of a TRO $\mathcal{V}$, we can identify $\mathcal{V}$ with $\pi(\mathcal{V})$.

Lemma 3.1. *Let a be a non-compact selfadjoint operator in $\mathcal{B}(H)_1$. Then there exists $\varepsilon > 0$ and an infinite dimensional projection p on H such that $\mathcal{B}(p(H))_{\varepsilon^2/2} \subseteq a\mathcal{B}(H)_{1/2}a$.*

Proof. Let us assume that a is a non-compact selfadjoint contraction. We shall denote by E the unique spectral measure relative to $(\sigma(a), H)$ such that $a = \int z dE$, where z is the inclusion map of $\sigma(a)$ in $\mathbb{C}$. From [7, Proposition 4.1] there exists an $\varepsilon > 0$ such that the projection $p = E(\{z \in \sigma(a) : |z| > \varepsilon\})$ is infinite dimensional. Denote by a_p the operator in $\mathcal{B}(p(H))$ such that $a_p(h) = ap(h) = pa(h)$ for all $h \in p(H)$. The operator a_p is invertible. Let us assume that the operator T is in $p\mathcal{B}(H)_{\varepsilon^2/2}p = \mathcal{B}(p(H))_{\varepsilon^2/2}$. Then

$$\|(a_p)^{-1} T (a_p)^{-1}\| \leq \|(a_p)^{-1}\|^2 \|T\| \leq \frac{1}{\varepsilon^2}\frac{\varepsilon^2}{2} = \frac{1}{2}.$$

Therefore,

$$T = a_p((a_p)^{-1} T (a_p)^{-1}) a_p \in a_p \mathcal{B}(p(H))_{1/2} a_p \subseteq ap\mathcal{B}(H)_{1/2}pa.$$

So,

$$\mathcal{B}(p(H))_{\varepsilon^2/2} = p\mathcal{B}(H)_{\varepsilon^2/2}p \subseteq ap\mathcal{B}(H)_{1/2}pa \subseteq a\mathcal{B}(H)_{1/2}a.$$

□

Proposition 3.2. *Let a be a contractive operator on a Hilbert space H. Then the operator a is compact if and only if the set $\mathrm{cp}^2_{\mathcal{B}(H)}(a)$ is weakly compact.*

Proof. The forward implication is trivial from [2, Corollary 2.4].

Conversely, suppose that the operator a is non-compact. The polar decomposition of a has the following form:

$$a = v|a|,$$

where v is a partial isometry, such that $v^*v|a| = |a|$ and $\mathrm{dom}(v) = \overline{|a|(H)}$. From Lemma 3.1 we know that there exists $\varepsilon > 0$ and an infinite dimensional projection p such that

$$vp\mathcal{B}(H)_{\varepsilon^2/2}p \subseteq v|a|\mathcal{B}(H)_{1/2}|a|.$$

Therefore, the following inclusions hold:

$$\mathcal{B}(p(H), vp(H))_{\varepsilon^2/2} = vp\mathcal{B}(H)_{\varepsilon^2/2}p \subseteq v|a|\mathcal{B}(H)_{1/2}|a|$$
$$= v|a|\mathcal{B}(H)_{1/2}v^*v|a| \subseteq v|a|\mathcal{B}(H)_{1/2}v|a| = a\mathcal{B}(H)_{1/2}a \subseteq \mathrm{cp}^2_{\mathcal{B}(H)}(a).$$

The last inclusion follows from [2, Proposition 1.2]. Since v is a non-compact partial isometry, $\mathcal{B}(p(H), vp(H))_{\varepsilon^2/2}$ is not weakly compact [7, Chapter V, Theorem 4.2]. The proof is complete. □

Theorem 3.3. *Let $\mathcal{A}$ be a C^*-algebra and $a \in \mathcal{A}_1$. Then there exists a faithful representation ϕ of $\mathcal{A}$ such that $\phi(a)$ is a compact operator if and only if $\mathrm{cp}^2(a)$ is a weakly compact set.*

Proof. The forward implication follows from [2, Theorem 2.2].

Conversely assume that $\phi(a)$ is a non-compact operator for all faithful representations ϕ of $\mathcal{A}$. Let $\{(\phi_i, H_i)\}$ be a maximal family of pairwise inequivalent irreducible representations of $\mathcal{A}$ and let ϕ be the reduced atomic representation $(\sum_{i\in I} \oplus\phi_i, \sum_{i\in I} \oplus H_i)$. Since all ϕ_i are irreducible representations, the SOT-closure of $\phi(\mathcal{A})$ equals $\sum_{i\in I} \oplus\mathcal{B}(H_i)$. Kaplansky's Density Theorem shows that $\phi(\mathcal{A}_1)$ is SOT-dense in $(\sum_{i\in I} \oplus\mathcal{B}(H_i))_1$ and so $\phi(a)\phi(\mathcal{A}_{1/2})\phi(a)$ is SOT-dense in

$$\sum_{i\in I} \oplus\phi_i(a)\mathcal{B}(H_i)_{1/2}\phi_i(a).$$

However, [2, Proposition 1.2] shows that $\phi(a)\phi(\mathcal{A}_{1/2})\phi(a)$ is contained in the set $\mathrm{cp}^2(\phi(a))$, which is a SOT-closed set. Thus

$$\left(\sum_{i\in I} \oplus\phi_i(a)\mathcal{B}(H_i)_{1/2}\phi_i(a)\right) \subseteq \mathrm{cp}^2(\phi(a)).$$

The operator $\phi(a)$ is not compact, since the reduced atomic representation is faithful. Thus, there are two cases.

Assume first that there exists an $i_o \in I$ such that $\phi_{i_o}(a)$ is a non-compact operator on H_{i_o}. Therefore, from the proof of Proposition 3.2 there exists an infinite dimensional projection $p \in \mathcal{B}(H_{i_o})$, a non-compact partial isometry v and an $\varepsilon > 0$ such that $\mathcal{B}(p(H_{i_o}), vp(H_{i_o}))_{\varepsilon^2/2} \subseteq \phi_{i_o}(a)\mathcal{B}(H_{i_o})_{1/2}\phi_{i_o}(a)$. It follows that $\mathcal{B}(p(H_{i_o}), vp(H_{i_o}))_{\varepsilon^2/2} \oplus \sum_{i\in I-\{i_o\}} \oplus\phi_i(a)\mathcal{B}(H_i)_{1/2}\phi_i(a) \subseteq \mathrm{cp}^2(a)$. Therefore the set $\mathrm{cp}^2(\phi(a))$ is not weakly compact since $\mathcal{B}(p(H_{i_o}), vp(H_{i_o}))_{\varepsilon^2/2}$ is not a weakly compact set.

Assume now that $\phi_i(a)$ is compact for all $i \in I$. Since $\phi(a)$ is not compact there exists an $\varepsilon > 0$ such that the set $\{i \in I : \|\phi_i(a)\| \geq \varepsilon\}$ is infinite. Then the set

$\sum_{i\in I} \oplus\phi_i(a)\mathcal{B}(H_i)_{1/2}\phi_i(a)$ is not compact since it contains a copy of an l^∞ ball [7, Chapter V, Theorem 4.2]. This completes the proof, as the last set is contained in $\mathrm{cp}^2(a)$. □

Remark 3.4. Let $\mathcal{A}$ be a C^*-algebra and $a \in \mathcal{A}_1$. Then by the theorem above and [2, Theorem 2.2], the following assertions are equivalent:

(1) There exists a faithful representation (ϕ, H) of $\mathcal{A}$ so that $\phi(a)$ is a compact operator.
(2) The set $\mathrm{cp}^2(a)$ is norm compact.
(3) The set $\mathrm{cp}^2(a)$ is weakly compact.

Let $\phi : \mathcal{V} \to \mathcal{B}(H_2, H_1)$ be a representation of a TRO $\mathcal{V}$ and $K_1 \subseteq H_1$ and $K_2 \subseteq H_2$ closed subspaces. A pair of subspaces (K_2, K_1) is said to be *ϕ-invariant* if $\phi(\mathcal{V})K_2 \subseteq K_1$ and $\phi(\mathcal{V})^*K_1 \subseteq K_2$. The representation ϕ is said to be *irreducible* if $(0,0)$ and (H_2, H_1) are the only ϕ-invariant pairs.

Two representations $\phi_i : \mathcal{V} \to \mathcal{B}(H_1^i, H_2^i)$ of $\mathcal{V}$, $i = 1, 2$ are said to be unitarily equivalent if there are unitary operators $U_i : H_i^1 \to H_i^2$, $i = 1, 2$ such that $\phi_1(x) = U_2^*\phi_2(x)U_1$, for all $x \in \mathcal{V}$.

Let $(\phi_i)_{i\in I}$ be a maximal family of pairwise inequivalent irreducible representations of $\mathcal{V}$, $\phi_i : \mathcal{V} \to \mathcal{B}(H_{2,i}, H_{1,i})$. Their direct sum $\phi = \sum_{i\in I} \oplus\phi_i$ is the reduced atomic representation of $\mathcal{V}$. It follows from [5, Lemma 3.5] that an irreducible representation of a TRO is the restriction of an irreducible representation of its linking algebra. Therefore, the reduced atomic representation of a TRO $\mathcal{V}$ is the restriction of the reduced atomic representation of its linking algebra $A(\mathcal{V})$.

Theorem 3.5. *Let $\mathcal{V}$ be a TRO and $a \in \mathcal{V}_1$. The following are equivalent:*

(1) $\mathrm{cp}^2_{\mathcal{V}}(a)$ *is a weakly compact set.*
(2) *There exists a faithful representation π of $\mathcal{V}$ such that $\pi(a)$ is a compact operator.*
(3) $\phi(a)$ *is a compact operator where ϕ is the reduced atomic representation of $\mathcal{V}$.*

Proof. First we show that (1) is equivalent to (2). Suppose that the set $\mathrm{cp}^2_{\mathcal{V}}(a)$ is weakly compact. From Proposition 2.1 we know that $\mathrm{cp}^2_{A(\mathcal{V})}(a) \subseteq \mathrm{cp}^2_{\mathcal{V}}(a)$ and therefore the set $\mathrm{cp}^2_{A(\mathcal{V})}(a)$ is weakly compact. Now, by Theorem 3.3, there exists a faithful representation π of $\mathcal{V}$ that maps a to a compact operator.

Conversely, suppose that π is a faithful representation of $\mathcal{V}$ such that $\pi(a)$ is a compact operator. We may assume that both H_1 and H_2 are infinite dimensional Hilbert spaces. Identifying $\mathcal{V}$ with $\pi(\mathcal{V})$, Proposition 2.5 states that $\mathrm{cp}^2_{\mathcal{V}}(a) \subseteq \mathcal{V} \cap \mathcal{K}(H_2, H_1) \subseteq \mathcal{K}(H_1 \oplus H_2)$. The set $\mathrm{cp}^2_{\mathcal{V}}(a)$ is WOT-closed. Since the relative w* and WOT-topologies on the closed unit ball of $\mathcal{B}(H_1 \oplus H_2)$ coincide, [13, Theorem 4.2.4.], $\mathrm{cp}^2_{\mathcal{V}}(a)$ is a w*-closed set. From the Banach-Alaoglu theorem we deduce that $\mathrm{cp}^2_{\mathcal{V}}(a)$ is a w*-compact set. By [10, Proposition 10.4.3], the weak topology on $\mathcal{K}(H_1 \oplus H_2)$ coincides with the relative w*-topology on $\mathcal{K}(H_1 \oplus H_2)$ and therefore $\mathrm{cp}^2_{\mathcal{V}}(a)$ is a weakly compact set.

Obviously (3) implies (2) since ϕ is a faithful representation. So, we only need to show that (1) implies (3). From Proposition 2.1 we know that $\mathrm{cp}^2_{A(\mathcal{V})}(a) \subseteq \mathrm{cp}^2_{\mathcal{V}}(a)$. Therefore the set $\mathrm{cp}^2_{A(\mathcal{V})}(a)$ is weakly compact and from Theorem 3.3,

$\rho(a)$ is a compact operator, where ρ is the reduced atomic representation of $A(\mathcal{V})$ [2, Theorem 2.2]. The operator $\phi(a)$ is compact since $\phi = \rho|_{\mathcal{V}}$. □

Statement (3) of Theorem 3.5 ensures that the elements of $\mathcal{V}$ that are mapped to a compact operator by a faithful representation of $\mathcal{V}$ form a subTRO.

Remark 3.6. Let $\mathcal{A}$ be a C^*-algebra which acts on a Hilbert space H and contains $\mathcal{K}(H)$, the set of compact operators on H. If $a \in \mathcal{A}_1$, the following assertions are equivalent:

(1) a is a compact operator.
(2) The set $\mathrm{cp}^2(a)$ is norm compact.
(3) The set $\mathrm{cp}^2(a)$ is SOT-compact.
(4) The set $\mathrm{cp}^2(a)$ is weakly compact.

Proof. From [2, Corollary 2.4.], we know that (1) and (2) are equivalent.

That (2) implies (3) is obvious.

Now we show that (3) implies (1). The following arguments are similar to those of [2, Lemma 2.1]. Since $\mathrm{cp}^2(a)$ is SOT-compact and $a(\mathcal{K}(H))_{1/2}a \subseteq \mathrm{cp}^2(a)$, the set $a(\mathcal{K}(H))_{1/2}a$ is SOT-precompact. Let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in H. Without loss of generality we may assume that $\|f_n\| \leq 1/2$, for all $n \in \mathbb{N}$. Let e be a unit vector in $(\ker a^*)^{\perp}$. For every $n \in \mathbb{N}$, let $x_n = e \otimes f_n$. Then $ax_na = a^*e \otimes af_n$. Since $a(\mathcal{K}(H))_{1/2}a$ is a SOT-precompact set, the sequence $\{(a^*e \otimes af_n)(h)\}_{n\in\mathbb{N}}$ has a convergent subsequence for every $h \in H$. Thus, the sequence $\{\langle h, a^*e\rangle af_n\}_{n\in\mathbb{N}}$ has a convergent subsequence and therefore $\{af_n\}_{n\in\mathbb{N}}$ has a convergent subsequence. Hence, a is a compact operator.

Obviously (1) implies (4). So, it suffices to see that (4) implies (1). Let us assume that a is a non-compact operator in $\mathcal{A}$. Following the arguments of the proof of Proposition 3.2 we can easily see that there exists an $\varepsilon > 0$, an infinite dimensional projection p and a non-compact partial isometry v on $p(H)$ such that $\mathcal{K}(p(H), vp(H))_{\varepsilon^2/2} \subseteq \mathrm{cp}^2_{\mathcal{A}}(a)$. Since the ball $\mathcal{K}(p(H), vp(H))_{\varepsilon^2/2}$ is not weakly compact, the set $\mathrm{cp}^2_{\mathcal{A}}(a)$ is not weakly compact. □

The following example shows that the compactness of an element u of a TRO does not imply the SOT-compactness of $\mathrm{cp}^2(u)$.

Example 3.7. Let $\mathcal{V}$ be the TRO $\mathcal{B}(H_2, H_1)$, where H_1 is an infinite dimensional Hilbert space and H_2 a one dimensional Hilbert space. The unit ball of $\mathcal{B}(H_2, H_1)$ is not SOT-compact. Indeed, if $\{e \otimes f_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{B}(H_2, H_1)$, where e is a unit vector of H_2 and $\{f_n\}$ an orthonormal sequence of H_1, then the sequence $\{(e\otimes f_n)(e)\} = \{\langle e, e\rangle f_n\} = \{f_n\}$ does not have a convergent subsequence. Consider an isometry $u \in \mathcal{V}$. Then u is compact and $\mathrm{cp}^2(u) = \mathcal{V}_1$ is not a SOT-compact set.

The set $\mathrm{cp}^2(a)$ of the remark above is always WOT-compact since the WOT-topology of $\mathcal{B}(H)$ coincides with its w*-topology on its closed unit ball. Therefore, the WOT-compactness of $\mathrm{cp}^2(a)$ cannot be equivalent with the statements of Remark 3.6.

Vala introduced the notion of compactness in a normed algebra in [15]. He defined an element a of a normed algebra to be compact if the mapping $x \to axa$ is compact.

Definition 3.8. A linear mapping $u : \mathcal{V} \to \mathcal{V}$ is called a *weakly compact operator* on $\mathcal{V}$ if $\{u(x) : \|x\| \leq 1\}$ is relatively weakly compact in $\mathcal{V}$.

We shall use the following theorem. It was proved by K. Ylinen in [16] and [17].

Theorem 3.9. *Let a be an element of the C^*-algebra $\mathcal{A}$. The following conditions are equivalent:*

(1) *There exists a faithful representation ϕ that maps a to a compact operator.*
(2) *The operator $u : \mathcal{V} \to \mathcal{V}$, $u(x) = axa$ is compact.*
(3) *The operator $u : \mathcal{V} \to \mathcal{V}$, $u(x) = axa$ is weakly compact.*

Bunce and Chu in [6] establish several theorems classifying compact and weakly compact JB^*-triples. A JB^*-triple $\mathcal{A}$ is said to be (weakly) compact if the antilinear operator $x \to \{axa\}$ is (weakly) compact for each $a \in \mathcal{A}$, where $\{\ \}$ denotes the ternary product. It follows from [6, Theorem 3.6] that a TRO $\mathcal{V}$ is isomorphic to a subTRO of $\mathcal{K}(H)$ for some Hilbert space H if and only if the mapping $a \to ax^*a$ is compact or equivalently weakly compact, for all $a \in \mathcal{V}$. The next theorem characterizes the compact elements of a TRO $\mathcal{V}$.

Theorem 3.10. *Let a be an element of a TRO $\mathcal{V}$. The following conditions are equivalent:*

(1) *There exists a faithful representation π that maps a to a compact operator.*
(2) *The operator $u : \mathcal{V} \to \mathcal{V}$, $u(x) = ax^*a$ is compact.*
(3) *The operator $u : \mathcal{V} \to \mathcal{V}$, $u(x) = ax^*a$ is weakly compact.*

Proof. First we show that (2) implies (1). Let $u : \mathcal{V} \to \mathcal{V}$, $u(x) = ax^*a$ be a compact operator. Then the extension of u to the linking algebra $A(\mathcal{V})$ of $\mathcal{V}$ is compact as well, since

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ax_3a \\ 0 & 0 \end{pmatrix} \in \mathcal{V},$$

where $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in A(\mathcal{V})$. It follows that the operator $\tilde{u} : A(\mathcal{V}) \to A(\mathcal{V})$, $\tilde{u}(x) = axa$ is compact. From [16] there exists a faithful representation π of $A(\mathcal{V})$ such that $\pi(a)$ is a compact operator.

Now we show the implication (1)⇒(2). Suppose there exists an isometric representation π of $\mathcal{V}$ on a Hilbert space H so that $\pi(a)$ is a compact operator on H. Then (see [14]) the map $u_1 : \mathcal{B}(H) \to \mathcal{B}(H)$, $u_1(x) = \pi(a)x\pi(a)$ is compact. Obviously, the map $u_2 : \mathcal{B}(H) \to \mathcal{B}(H)$, $u_2(x) = \pi(a)x^*\pi(a)$ is compact as well. Therefore, the restriction of u_2 to $\pi(\mathcal{V})$ is a compact operator. Since π is an isometry the result follows.

That (1) implies (3) can be readily verified.

Applying the arguments at the beginning of this proof and Theorem 3.9 we deduce that (3) implies (1). □

Remark 3.11. Let $\mathcal{V}$ be a TRO. It follows from Remark 2.2 and Theorem 3.5 that the weak compactness of $\mathrm{cp}^2_{\mathcal{V}}(a)$ does not imply its norm compactness. On the other hand, we would like to note that the norm compactness and weak compactness of the mapping $u : \mathcal{V} \to \mathcal{V}$, $u(x) = ax^*a$ are equivalent.

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Department of Mathematics, University of the Aegean, 832 00 Karlovassi, Samos, Greece

E-mail address: gandreolas@aegean.gr