CENTRAL VALUES OF THE SYMMETRIC SQUARE $L$-FUNCTIONS

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Abstract. We establish a sharp bound for the square mean of the central values of the symmetric square $L$-functions associated to holomorphic cusp forms of level 1, as the weight $k$ varies in the short interval $[K, K + K^{1/2 + \epsilon}]$.

1. Introduction

Let $S_k(\Gamma_0(1))$ denote as usual the space of holomorphic cusp forms of weight $k$ with respect to the modular group $\Gamma_0(1)$, and let $H_k$ be the normalized Hecke basis for $S_k(\Gamma_0(1))$, where the normalization means that the first Fourier coefficient of each basis element $f \in H_k$ is 1.

Each $f \in H_k$ has the Fourier expansion

$$f(z) = \sum_{n \geq 1} a_f(n)n^{(k-1)/2}e(nz), \ a_f = 1,$$

and the associated $L$-function

$$L(s, f) = \sum_{n \geq 1} a_f(n)n^{-s} = \prod_p (1 - \alpha_p p^{-s})^{-1}(1 - \beta_p p^{-s})^{-1}, \ \Re(s) > 1,$$

admits analytic continuation to the whole complex plane $\mathbb{C}$ and satisfies the functional equation

$$(2\pi)^{-s/2} \Gamma\left(\frac{k-1}{2} + s\right)L(s, f) = i^k (2\pi)^{1-s} \Gamma\left(\frac{k-1}{2} + 1 - s\right)L(1-s, f).$$

The symmetric square $L$-function associated to $f$ is defined by

$$L(s, \text{sym}^2(f)) = \prod_p (1 - \alpha_p^2 p^{-s})^{-1}(1 - \alpha_p \beta_p p^{-s})^{-1}(1 - \beta_p^2 p^{-s})^{-1}, \ \Re(s) > 1,$$

and by the works of Shimura \[7\] and Gelbart-Jacquet \[4\], it admits analytic continuation to the whole complex plane $\mathbb{C}$ and satisfies the functional equation

$$\Lambda(s, \text{sym}^2(f)) = \pi^{-3s/2} \Gamma\left(\frac{s + 1}{2}\right) \Gamma\left(\frac{s + k - 1}{2}\right) \Gamma\left(\frac{s + k}{2}\right)L(s, \text{sym}^2(f))$$

$$= \Lambda(1-s, \text{sym}^2(f)).$$

Write

$$L_\infty(s) = L_\infty(s, \text{sym}^2(f)) = \pi^{-3s/2} \Gamma\left(\frac{s + 1}{2}\right) \Gamma\left(\frac{s + k - 1}{2}\right) \Gamma\left(\frac{s + k}{2}\right).$$

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and note that
\[ L(s, \text{sym}^2(f)) = \zeta(2s) \sum_{n \geq 1} a_f(n^2) n^{-s}, \quad \Re(s) > 1. \]

To study the central \( L \)-value \( L(1/2, \text{sym}^2(f)) \), the following approximate functional equation is a standard device (see, for example, [5, Lemma 2.2] for a detailed proof).

**Lemma 1** ([5]). We have
\[ L(1/2, \text{sym}^2(f)) = 2 \sum_{n \geq 1} \frac{a_f(n^2)}{n^{1/2}} V_k(n), \]
where
\[ V_k(\xi) = \frac{1}{2\pi i} \int_{(\sigma)} \frac{L_\infty(1/2 + y)}{L_\infty(1/2)} \zeta(1 + 2y) \xi^{-y} \frac{dy}{y} \quad (\sigma > 0), \]
and it satisfies the following bounds:
\[ V_k(\xi) \ll A \left( \frac{k}{\xi} \right)^A, \quad \text{for any } A > 0, \]
and
\[ V_k(\xi) \ll \log(2k/\xi), \quad \text{for } \xi \leq k. \]

The purpose of this paper is to show

**Theorem 1.** For any \( \epsilon > 0 \) we have, as \( K \to \infty \),
\[ \sum_{2|K, 2|K+K^{1/2}} \sum_{f \in H_k} |L(1/2, \text{sym}^2(f))|^2 \ll_{\epsilon} K^{3/2 + \epsilon}. \]

**Remark.** It seems to be an open problem (see, for instance, Conjecture 1.2 on p. 5 of [5]) to obtain the sharp bound for the square mean without average over the weight \( k \). Previously the work [3] established a sharp bound for the second moment for \( L(1/2, \text{sym}^2(f)) \), when \( f \) has fixed weight and large level, but its method unfortunately does not extend to the case of large weight. Our work is inspired by [6], which makes use of the Zagier kernel function \( \Phi_s \) for \( L(s, \text{sym}^2(f)) \) to derive a sharp bound for the first moment.

2. **Zagier’s Kernel Function**

Let \( \Delta \) be a discriminant; i.e., \( \Delta \) is an integer such that \( \Delta \equiv 0, 1 (\mod 4) \). Define
\[ L(s, \Delta) = \begin{cases} 
\zeta(2s - 1), & \text{if } \Delta = 0; \\
L_D(s) \sum_{d|f} \mu(d) \left( \frac{D}{f} \right) d^{-s} \sigma_{1-2s} \left( \frac{f}{\tau} \right), & \text{if } \Delta \neq 0,
\end{cases} \]
where if \( \Delta \neq 0 \) we write \( \Delta = Df^2 \) with positive integer \( f \) and \( D \) the fundamental discriminant, \( L_D(s) \) is the Dirichlet \( L \)-function associated to the Kronecker symbol \( \left( \frac{D}{\tau} \right) \), \( \mu \) is the Möbius function, and \( \sigma_\nu(m) = \sum_{d|m} d^\nu \) for a positive integer \( m \) and any complex number \( \nu \).
Moreover for \( t \) an integer with \( \Delta < t^2 \) and \( s \in \mathbb{C} \) with \( 1/2 < \Re(s) < k \) we define (see \cite{8})

\[
I_k(\Delta, t; s) = \int_0^\infty \int_0^\infty \frac{y^{k+s-2}}{x^2 + y^2 + i ty - \frac{1}{4}\Delta} \, dx \, dy
= \frac{\Gamma(k-1/2)\Gamma(1/2)}{\Gamma(k)} \int_0^\infty y^{k+s-2} (y^2 + ity - \frac{1}{4}\Delta)^{k-1/2} \, dy,
\]

where the second integral converges absolutely for \( 1 - k < \sigma < k \) if \( \Delta \neq 0 \). For \( \Delta = 0, \pm t > 0 \), one has

\[
I_k(0, t; s) = c^{\pm i\pi (s-k)/2} \frac{\Gamma(1/2)\Gamma(s-1/2)\Gamma(k-s)}{\Gamma(k)} |t|^{-k+s}.
\]

We also have

\[
2I_k(\Delta, 0, 1/2) = \frac{1}{mk^{-1/2}} \frac{\Gamma\left(\frac{k-1/2}{2}\right)^2}{\Gamma(k)}.
\]

Define

\[
I_{k,s}(z) = \frac{2^{1-k}\Gamma(1/2)}{\Gamma(k-1/2)} \Gamma(k-1+s)\Gamma(k-s)(z^2 - 1)^{-(k-1)/2} P_{-s}^{1-k}(z),
\]

for \( 1 - k < \Re(s) < k, z \in \mathbb{C} \setminus (-\infty, 1] \), where \( P_{-s}^{1-k}(z) \) is the associated Legendre function of the first kind. Then for \( \Delta < 0 \), it follows that

\[
I_k(\Delta, t; s) = (|\Delta|/4)^{s-k/2} \frac{\Gamma(k-1/2)\Gamma(1/2)}{\Gamma(k)} I_{k,s}\left(\frac{it}{\sqrt{|\Delta|}}\right),
\]

while for \( \Delta > 0 \), the following formulas are valid:

\[
I_k(\Delta, t; s) = (\Delta/4)^{(s-k)/2} e^{i\pi (s-k)/2} \frac{\Gamma(k-1/2)\Gamma(1/2)}{\Gamma(k)} I_{k,s}\left(\frac{t}{\sqrt{\Delta}}\right) \quad (\Delta > 0, t > 0);
\]

\[
I_k(\Delta, t; s) = (\Delta/4)^{(s-k)/2} e^{-i\pi (s-k)/2} \frac{\Gamma(k-1/2)\Gamma(1/2)}{\Gamma(k)} I_{k,s}\left(\frac{|t|}{\sqrt{\Delta}}\right) \quad (\Delta > 0, t < 0).
\]

We refer the reader to \cite{8} for the proofs of these formulas for \( I_k(\Delta, t; s) \). Now we are in a position to state the following important theorem due to Zagier, which is the basis for our study on the square mean of \( L(1/2, \text{sym}^2(f)) \) via the kernel function \( \Phi_s \).

**Theorem 2** (\cite{8}). Let \( k \geq 4 \) be an even integer. For a positive integer \( m \) and \( s \in \mathbb{C} \) set

\[
c_m(s) = m^{k-1} \sum_{t = -\infty}^{\infty} \left( I_k(t^2 - 4m, t; s) + I_k(t^2 - 4m, -t; s) \right) L(s, t^2 - 4m)
+ \begin{cases}
-(-1)^{k/2} \frac{\Gamma(k+s-1)\xi(2s)}{2^{k-1-s}\pi^s\Gamma(k)} u^{k-s-1}, & \text{if } m = u^2, u > 0; \\
0, & \text{if } m \text{ is not a perfect square}.
\end{cases}
\]

Then:

(i) The series converges absolutely and uniformly for \( 2 - k < \Re(s) < k - 1 \).

(ii) The function

\[
\Phi_s(z) = \sum_{m=1}^{\infty} c_m(s)e(mz) \quad (z \in \mathbb{H}, 2 - k < \Re(s) < k - 1)
\]

is in \( S_k(\Gamma_0(1)) \).
(iii) Let \( f \in H_k \). Then the Petersson product of \( \Phi_s \) and \( f \) is given by

\[
\langle \Phi_s, f \rangle = c_k \frac{\Gamma(s + k - 1)}{(4\pi)^{s+k-1}} L(s, \text{sym}^2(f)),
\]

where

\[
c_k = \frac{(-1)^{k/2}}{2^{k-1}(k-1)}.
\]

In the above theorem, if \( m \) is a square, then the single term outside the sum has a simple pole at \( s = 1/2 \), while in this case the terms \( t = \pm 2 \sqrt{m} \) in the first sum involve the function

\[
I_k(0, t, s) = 2\pi(-1)^{k/2} \cos \frac{\pi s}{2} \frac{\Gamma(s - 1/2)\Gamma(k - s)}{\Gamma(k)\Gamma(1/2)} |t|^{s-k},
\]

which also has a simple pole at \( s = 1/2 \), and the two poles cancel, so that \( c_m(s) \) is actually holomorphic in the region \( 2 - k < \Re(s) < k - 1 \).

We infer that

\[
\Phi_s(z) = c_k \frac{\Gamma(s + k - 1)}{(4\pi)^{s+k-1}} \sum_{f \in H_k} L(s, \text{sym}^2(f)) \langle f, f \rangle f(z).
\]

Taking the \( m \)-th Fourier coefficient of both sides, we have

\[
c_m(s) = c_k \frac{\Gamma(s + k - 1)}{(4\pi)^{s+k-1}} \sum_{f \in H_k} L(s, \text{sym}^2(f)) |m(k-1)/2a_f(m)|^{1/2}.
\]

Specializing at \( s = 1/2 \) and replacing \( m \) by \( m^2 \), it follows from Lemma 1 that

\[
2 \sum_{m \geq 1} \frac{c_m(1/2)}{m^{k-1/2}} V_k(m) = c_k \frac{\Gamma(k - 1/2)}{(4\pi)^{k-1/2}} \sum_{f \in H_k} \frac{|L(1/2, \text{sym}^2(f))|^2}{\langle f, f \rangle}.
\]

Recall that

\[
\langle f, f \rangle = \frac{\Gamma(k)L(1, \text{sym}^2(f))}{2^{2k-1}m^{k+1}}.
\]

Hence

\[
(2) \quad \sum_{f \in H_k} \frac{|L(1/2, \text{sym}^2(f))|^2}{L(1, \text{sym}^2(f))} = (-1)^{k/2}2^{-2} \pi^{-5/2} \left( \frac{\Gamma(k - 1/2)}{2^{k(k-1/2)}\Gamma(k)} \right)^{-1} \sum_{m \geq 1} \frac{c_m(1/2)}{m^{k-1/2}} V_k(m).
\]

3. The case \( \Delta > 0 \)

We need the following integral representation of \( P_{\nu}^{-\mu}(z) \) ([2, p. 1001, 8.713 (2))):

\[
P_{\nu}^{-\mu}(z) = \frac{\left(z^2 - 1\right)^{\mu/2}}{2\pi \Gamma(\mu - \nu)\Gamma(\nu + 1)} \int_0^\infty \frac{\sinh^{2\nu+1} r (z + \cosh r)^{\nu + \mu + 1}}{(z + \cosh r)^{\nu + \mu + 1}} dr,
\]

for \( \Re(z) > -1, |\arg(z + 1)| < \pi, \Re(\nu + 1) > 0, \Re(\mu - \nu) > 0 \).

Consider first the case where \( \Delta = t^2 - 4m^2 > 0 \). With \( z = \frac{|t|}{\sqrt{t^2 - 4m^2}} \) we deduce that

\[
|P_{-1/2}^1(z)| \ll \frac{t-1}{1 \Gamma(k-1/2)} \int_0^\infty \frac{1}{(z + \cosh r)^{k-1/2}} dr.
\]
Now for any $\delta > 0$, 

$$
\int_0^\infty \frac{1}{(z + \cosh r)^{k-1/2}} dr \\
\ll \frac{1}{(z + 1)^{k-1/2-\delta}} \int_0^\infty \frac{1}{(\cosh r)^{\delta}} dr \\
\ll \frac{1}{(z + 1)^{k-1/2-\delta}} \\
\ll \frac{1}{(z + 1)^{k-1/2-\delta}} \left( t^2 - 4m^2 \right)^{(k-1/2-\delta)/2} \\
\ll \delta \left( \frac{1}{(z + 1)^{k-1/2-\delta}} \right) \frac{1}{|t|^{(k-1/2-\delta)/2}} + \frac{1}{|t|^{(k-1/2-\delta)/2}} \\
\ll \exp(1/3) \frac{(t^2 - 4m^2)^{(k-1/2-\delta)/2}}{|t|^{(k-1/2-\delta)/2}},
$$

since in view of $|t| \geq 2m + 1$, we have 

$$(1 + \sqrt{1 - 2m/|t|})^{(k-1/2-\delta)} \geq \left( 1 + \frac{1}{\sqrt{2m + 1}} \right)^k \geq \left( 1 + \frac{1}{\sqrt{k+\epsilon}} \right)^k \gg \exp(k^{1/2-\epsilon}).$$

Note for $\Delta = Df^2$ as before, the convexity bound yields 

$$|L(1/2, \Delta)| \ll f' |LD(1/2)| \ll \Delta^{1/4+\epsilon}.$$ 

Furthermore, 

$$|I_{k,1/2}(z)| \ll 2^{-k} \Gamma(k - 1/2)|z^2 - 1|^{-(k-1)/2} |P_{1/2}^{1-k}(z)| \\
\ll 2^{-k} \int_0^\infty \frac{1}{(z + \cosh r)^{k-1/2}} dr, \\
|I_k(\Delta, 1/2)| \ll \left( \frac{\Delta}{4} \right)^{(1/2-k)/2} \frac{\Gamma(k - 1/2)}{\Gamma(k)} |I_{k,1/2}(|t|/\sqrt{\Delta})| \\
\ll \Delta^{-(k-1/2)/2} \frac{\Gamma(k - 1/2)}{\Gamma(k)} \int_0^\infty \frac{1}{(|t|/\sqrt{\Delta + \cosh r})^{k-1/2}} dr, \\
\ll \frac{\Gamma(k - 1/2)}{\exp(1/3)|t|(k-1/2-\delta)\Gamma(k)}.$$ 

Hence the contribution to the right-hand side of (2) from the terms with $\Delta = t^2 - 4m^2 > 0$ is 

$$\ll k^{-10} + \left( \frac{\Gamma(k - 1/2)}{2^k k \Gamma(k)} \right)^{-1} \sum_{m \ll k^{1+\epsilon}} \frac{m^{2k-2}}{m^{k-1/2}} \sum_{|t| > 2m} \frac{1}{|t|^{k-1/2}} \\
\times (I_k(t^2 - 4m^2, t; 1/2) + I_k(t^2 - 4m^2, -t; 1/2)) L(1/2, t^2 - 4m^2) \\
(3) \ll 1.$$
4. The case $\Delta \leq 0$

Next we consider the terms in the right-hand side of (2) with $\Delta = t^2 - 4m^2 < 0$ and $z = \frac{it}{\sqrt{|\Delta|}}$. We have

$$I_k(\Delta, t; s) = (|\Delta|/4)^{(1/2-k)/2} \frac{\Gamma(k - 1/2)\Gamma(1/2)}{\Gamma(k)} I_{k,1/2} \left( \frac{it}{\sqrt{|\Delta|}} \right)$$

$$= \left( \frac{|\Delta|/4}{1/2-k} \frac{\Gamma(k - 1/2)\Gamma(1/2)}{\Gamma(k)} \right) \times 2^{-k} \Gamma(1/2) \Gamma(k - 1/2) (z^2 - 1)^{-(k-1)/2} P_{-1/2}^{1-k}(z)$$

$$= 2^{-k} (|\Delta|/4)^{(1/2-k)/2} \frac{\Gamma(k - 1/2)\Gamma(1/2)}{\Gamma(k)} (z^2 - 1)^{-(k-1)/2}$$

$$\times \frac{1}{2^{1/2-k} \Gamma(1/2) \Gamma(k)} \int_0^\infty \frac{1}{(z + \cosh r)^{k-1/2}} dr$$

$$= 2^{3/2-k-1/2} \Gamma(1/2) (|\Delta|/4)^{(1/2-k)/2} \frac{\Gamma(k - 1/2)}{\Gamma(k)} \int_0^\infty \frac{1}{(z + \cosh r)^{k-1/2}} dr$$

$$= 2 \Gamma(1/2) |\Delta|^{(1/2-k)/2} \frac{\Gamma(k - 1/2)}{\Gamma(k)} \int_0^\infty \frac{1}{(z + \cosh r)^{k-1/2}} dr.$$

Making the change of variable $u = \cosh t$, we have

$$\int_0^\infty \frac{1}{(z + \cosh r)^{k-1/2}} dr = \int_1^\infty \frac{du}{\sqrt{u^2 - 1} (z + u)^{k-1/2}}$$

$$= \int_0^\infty \frac{du}{\sqrt{u^2 + 2u(z + 1) + 1)^{k-1/2}}.$$

Applying Cauchy’s theorem we can replace the integral over the positive real line by the line $\{(1+z)v, v \geq 0\}$, and the above integral equals (we may assume $m \ll k^{1+\epsilon}$)

$$\frac{1}{(1+z)^{k-1}} \int_0^\infty \frac{dv}{\sqrt{v(1+v)^{k-1/2} + (1+z)v}}$$

$$= \frac{1}{(1+z)^{k-1}} \left( \int_0^{k^{-1}} \frac{dv}{\sqrt{v(1+v)^{k-1/2} + (1+z)v}} + O(k^{-1}) \right)$$

$$= \frac{1}{\sqrt{2}(1+z)^{k-1}} \left( \int_0^{k^{-1}} \frac{dv}{\sqrt{v(1+v)^{k-1/2} + (1+z)v}} + O(k^{-1+\epsilon}) \right)$$

$$= \frac{\sqrt{2} |\Delta|^{(k-1)/2}}{2^k m^{k-1}} \left( \frac{1 + z}{1 + z} \right)^{-(k-1)} \left( \int_0^{k^{-1}} \frac{dv}{\sqrt{v(1+v)^{k-1/2} + (1+z)v}} + O(k^{-1+\epsilon}) \right).$$
in view of
\[ |1 + z| = \left( \frac{2m}{\sqrt{|\Delta|}} \right) \]
and
\[ |\Delta| = 4m^2 - t^2 = (2m - t)(2m + t) > m. \]

Hence
\[
I_k(\Delta, t; s) = 2\Gamma(1/2)|\Delta|^{(1/2-k)/2} \Gamma(1/2) \Gamma(k) \int_0^\infty \frac{1}{(z + \cosh r)^{k-1/2}} dr
\]
\[ = 2\sqrt{2\Gamma(1/2)}|\Delta|^{-1/4} \Gamma(1/2) \frac{1}{2^k \Gamma(k)} \frac{1 + z}{|1 + z|}^{-(k-1)} \]
\[ \times \left( \int_0^{k^{-1}} \frac{dv}{\sqrt{v(1+v)^{k-1/2}}} + O(k^{-1+\epsilon}) \right). \]

Thus the contribution to the right-hand side of (2) from the terms with \( \Delta = t^2 - 4m^2 < 0 \) is

\[
(4)
O(k^{-10}) + (-1)^{k/2}2^{-2} \pi^{-5/2} \left( \frac{\Gamma(k - 1/2)}{2^k \Gamma(k-1)} \right)^{-1} \sum_{m \ll k^{1+\epsilon}} \frac{m^{2k-2}}{mk^{-1/2}} V_k(m)
\]
\[ \times \sum_{|t| < 2m} (I_k(t^2 - 4m^2, t; 1/2) + I_k(t^2 - 4m^2, -t; 1/2)) L(1/2, t^2 - 4m^2)
\]
\[ = k^{-10} + (-1)^{k/2}2^{-1/2} \pi^{-2} (k - 1) \sum_{m \ll k^{1+\epsilon}} m^{-1/2} V_k(m) \sum_{|t| < 2m} (4m^2 - t^2)^{-1/4}
\]
\[ \times \left( \frac{\sqrt{4m^2 - t^2 + ti}}{|4m^2 - t^2 + ti|} \right)^{-1} + \left( \frac{\sqrt{4m^2 - t^2 - ti}}{|4m^2 - t^2 - ti|} \right)^{-1} \right) L(1/2, t^2 - 4m^2)
\]
\[ \times \left( \int_0^{k^{-1}} \frac{dv}{\sqrt{v(1+v)^{k-1/2}}} + O(k^{-1+\epsilon}) \right)
\]
\[ = O(k^{1+\epsilon}) + (-1)^{k/2}2^{-1/2} \pi^{-2} (k - 1) \left( \int_0^{k^{-1}} \frac{dv}{\sqrt{v(1+v)^{k-1/2}}} \right)
\]
\[ \times \sum_{m \ll k^{1+\epsilon}} m^{-1/2} V_k(m) \sum_{|t| < 2m} (4m^2 - t^2)^{-1/4}
\]
\[ \times \left( \frac{\sqrt{4m^2 - t^2 + ti}}{|4m^2 - t^2 + ti|} \right)^{-1} + \left( \frac{\sqrt{4m^2 - t^2 - ti}}{|4m^2 - t^2 - ti|} \right)^{-1} \right) L(1/2, t^2 - 4m^2). \]
In the last step, we have inferred as follows:

$$
\sum_{m \ll k^{1+\epsilon}} m^{-1/2} \sum_{|t| < 2m} (4m^2 - t^2)^{-1/4} L(1/2, t^2 - 4m^2) \\
\ll k^\epsilon \sum_{2^e \leq k^{1+\epsilon}} \frac{1}{2^{1/2}} \sum_{f \leq 2^{e+1}} \frac{1}{f^{1/2}} \sum_{|D| \leq 2^{2e+2}/f^2} \frac{|L_D(1/2)|}{|D|^{1/4}} \\
\ll k^\epsilon \sum_{2^e \leq k^{1+\epsilon}} \sum_{f \leq 2^{e+1}} \frac{1}{f^{1/2}} \sum_{|D| \leq 2^{2e+2}/f^2} \left( \sum_{|D| \leq 2^{2e+2}/f^2} |L_D(1/2)|^2 \right)^{1/2} \left( \sum_{|D| \leq 2^{2e+2}/f^2} |D|^{-1/2} \right)^{1/2} \\
\ll k^{1+3\epsilon},
$$

where we have used Jutila’s bound [4],

$$
\sum_{|D| \leq X} |L_D(1/2)|^2 \ll X \log^3 X.
$$

Moreover the contribution to the right-hand side of (2) from the terms with \( \Delta = t^2 - 4m^2 = 0 \), i.e. \( t = \pm 2m \), is

$$
\ll k^{-10} + \left( \frac{\Gamma(k - 1/2)}{2^k k \Gamma(k)} \right)^{-1} \sum_{m \ll k^{1+\epsilon}} m^{2k-2} \frac{\Gamma(k - 1/2)}{m^{k-1/2} 2^k m^{k-1/2} \Gamma(k)} \log^2 k
$$

(5) \( \ll k \log^3 k, \)

in view of the bound

$$
\frac{\Gamma'(s)}{\Gamma(s)} = \log s + O(1/s), \text{ for } \Re(s) \geq 1.
$$

5. The mean value in the short interval

The contribution from (4) is bounded trivially by \( k^{3/2+\epsilon} \), but if we average over \( k \) with \( K \leq k \leq K + K^{1/2} \), then the total contribution will be shown below to be \( O(K^{3/2+\epsilon}) \).

When \( K \leq k \leq K + K^{1/2} \), the equality (4) can be replaced by (in light of the formula (2.7) in [5] for \( V_k(\xi) \))

(6)

$$
= O(K^{1+\epsilon}) + (-1)^{k/2} 2^{-1/2} \pi^{-2} K \left( \int_0^{K^{-1}} \frac{dv}{\sqrt{v(1+v)} K^{-1/2}} \right) \\
\times \sum_{m \ll K^{1+\epsilon}} m^{-1/2} V_k(m) \sum_{|t| \leq 2m} (4m^2 - t^2)^{-1/4} \\
\times \left( \frac{\sqrt{4m^2 - t^2 + ti}}{\sqrt{4m^2 - t^2 + ti}} \right)^{-(k-1)} + \left( \frac{\sqrt{4m^2 - t^2 - ti}}{\sqrt{4m^2 - t^2 - ti}} \right)^{-(k-1)} \right) L(1/2, t^2 - 4m^2).
$$
Now we are summing over all even \( k \) in (6) with \( K \leq k \leq K + K^{1/2} \). Denote
\[
e^{i\theta_{t,m}} = \frac{\sqrt{4m^2 - t^2} + ti}{|\sqrt{4m^2 - t^2} + ti|},
\]
and since
\[
\sum_{2|k, K \leq k \leq K + K^{1/2}} (-e^{\pm 2i\theta_{t,m}})^{k/2} \ll \frac{1}{|1 + e^{2i\theta_{t,m}}|} \ll \frac{1}{\cos \theta_{t,m}} \ll \frac{m}{\sqrt{4m^2 - t^2}},
\]
we have, with \( t^2 - 4m^2 = Df^2 \),
\[
\sum_{m \ll K^{1+\varepsilon}} m^{-1/2} V_K(m) \sum_{|t| < 2m} (4m^2 - t^2)^{-1/4} \times \left( \frac{\sqrt{4m^2 - t^2} + ti}{|\sqrt{4m^2 - t^2} + ti|} \right)^{(k-1)} + \left( \frac{\sqrt{4m^2 - t^2} - ti}{|\sqrt{4m^2 - t^2} - ti|} \right)^{(k-1)} L\left( \frac{1}{2}, t^2 - 4m^2 \right)
\ll K^{\varepsilon} \sum_{m \ll K^{1+\varepsilon}} m^{1/2} \sum_{|t| < 2m} (4m^2 - t^2)^{-3/4} |L(1/2, t^2 - 4m^2)|
\ll K^{1/2+\varepsilon} \sum_{m \ll K^{1+\varepsilon}} \sum_{|t| < 2m} (4m^2 - t^2)^{-3/4} |L_D(1/2)|
\ll K^{1/2+\varepsilon} \sum_{f \ll K^{1+\varepsilon}} f^{-3/2} \sum_{|D| \ll K^{2+\varepsilon}} |D|^{-3/4} |L_D(1/2)| \ll K^{1+\varepsilon}.
\]
Hence we conclude that
\[
\sum_{2|k, K \leq k \leq K + K^{1/2}} \sum_{f \ll H_k} |L(1/2, \text{sym}^2(f))|^2 \ll K^{3/2+\varepsilon}.
\]
This completes the proof of Theorem 1.

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