PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 140, Number 10, October 2012, Pages 3349–3363
S 0002-9939(2012)11184-0
Article electronically published on February 17, 2012

ON THE CONSISTENCY OF TWISTED GENERALIZED WEYL ALGEBRAS

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(Communicated by Kailash C. Misra)

Abstract. A twisted generalized Weyl algebra A of degree n depends on a base algebra R, n commuting automorphisms σ_i of R, n central elements t_i of R and on some additional scalar parameters.

In a paper by Mazorchuk and Turowska, it is claimed that certain consistency conditions for σ_i and t_i are sufficient for the algebra to be nontrivial. However, in this paper we give an example which shows that this is false. We also correct the statement by finding a new set of consistency conditions and prove that the old and new conditions together are necessary and sufficient for the base algebra R to map injectively into A. In particular they are sufficient for the algebra A to be nontrivial.

We speculate that these consistency relations may play a role in other areas of mathematics, analogous to the role played by the Yang-Baxter equation in the theory of integrable systems.

1. Introduction

Let R be an algebra over a commutative ring $\Bbbk$, $\sigma_1, \dots, \sigma_n$ commuting $\Bbbk$-algebra automorphisms of R, $t_1, \dots, t_n$ elements from the center of R, and μ_{ij} an $n \times n$ matrix of invertible scalars from $\Bbbk$. To this data one associates a *twisted generalized Weyl algebra* $\mathcal{A}_\mu(R, \sigma, t)$, an associative $\mathbb{Z}^n$-graded algebra (see Section 2 for the definition). These algebras were introduced by Mazorchuk and Turowska [10] and they are generalizations of the much-studied generalized Weyl algebras, defined independently by Bavula [2], Jordan [8], and Rosenberg [13] (there called *hyperbolic rings*).

Simple weight modules over twisted generalized Weyl algebras (TGWA for short) have been studied in [10], [9], [6]. In [11] the authors classified bounded and unbounded $*$-representations over twisted generalized Weyl algebras. Interesting examples of twisted generalized Weyl algebras were given in [9]. In [7] new examples of twisted generalized Weyl algebras were constructed from symmetric Cartan matrices.

It was claimed in [10, Lemma 1] (for the case when $\mu_{ij} = 1\, \forall i, j$) and implicitly in [11, Eq. (1)] (for arbitrary μ_{ij}) that a TGWA $\mathcal{A}_\mu(R, \sigma, t)$ is a nontrivial ring if the following relations are satisfied:

$$t_i t_j = \mu_{ij}\mu_{ji}\sigma_i^{-1}(t_j)\sigma_j^{-1}(t_i), \qquad i \neq j. \tag{1.1}$$

However, in this paper we give an example (Example 2.8) of a TGWA, $A = \mathcal{A}_\mu(R, \sigma, t)$, such that even though the datum (R, σ, t, μ) satisfies (1.1), the algebra

Received by the editors March 22, 2011 and, in revised form, April 7, 2011.
2010 *Mathematics Subject Classification.* Primary 16D30, 16S35; Secondary 16S85.

A is the trivial ring $\{0\}$. This shows that, in fact, (1.1) is not sufficient for a TGWA to be a nontrivial ring.

Our main result in this paper is the discovery of a new consistency relation, which together with (1.1) gives a sufficient condition for a TGWA to be a nontrivial ring. The precise statement is the following.

Theorem A. *Let $\Bbbk$ be a commutative unital ring, R be an associative $\Bbbk$-algebra, n a positive integer, $t = (t_1, \dots, t_n)$ be an n-tuple of regular central elements of R, $\sigma : \mathbb{Z}^n \to \mathrm{Aut}_{\Bbbk}(R)$ a group homomorphism, μ_{ij} $(i, j = 1, \dots, n, i \neq j)$ invertible elements from $\Bbbk$, and $\mathcal{A}_\mu(R, \sigma, t)$ the corresponding twisted generalized Weyl algebra, equipped with the canonical homomorphism of R-rings $\rho : R \to \mathcal{A}_\mu(R, \sigma, t)$. Then the following two statements are equivalent:*

(a) *ρ is injective;*
(b) *the following two sets of relations are satisfied in R:*

$$\sigma_i\sigma_j(t_it_j) = \mu_{ij}\mu_{ji}\sigma_i(t_i)\sigma_j(t_j), \qquad \forall i, j = 1, \dots, n,\ i \neq j, \tag{1.2}$$

$$t_j\sigma_i\sigma_k(t_j) = \sigma_i(t_j)\sigma_k(t_j), \qquad \forall i, j, k = 1, \dots, n,\ i \neq j \neq k \neq i. \tag{1.3}$$

In particular, if (1.2) and (1.3) are satisfied, then $\mathcal{A}_\mu(R, \sigma, t)$ is nontrivial iff R is nontrivial. Moreover, neither of the two conditions (1.2) nor (1.3) implies the other.

One may note that (1.2) and (1.3) may be expressed as the following identities in the localization $T^{-1}R$, where T is the multiplicative submonoid of R generated by all $\sigma_g(t_i)$ for all $g \in \mathbb{Z}^n, i = 1, \dots, n$:

$$\tau_{ij}\tau_{ji} = 1, \qquad \forall i, j = 1, \dots, n,\ i \neq j, \tag{1.4}$$

$$\sigma_k(\tau_{ij}) = \tau_{ij}, \qquad \forall i, j, k = 1, \dots, n,\ i \neq j \neq k \neq i, \tag{1.5}$$

where

$$\tau_{ij} = \frac{\mu_{ji}\sigma_j(t_j)}{\sigma_i\sigma_j(t_j)}. \tag{1.6}$$

We note that the consistency relations (1.2),(1.3) play an analogous role for twisted generalized Weyl algebras as the quantum Yang-Baxter equation plays for Zamolodchikov algebras in factorized S-matrix models [5, Section 1.1.1], and for Faddeev-Reshetikhin-Takhtajan algebras. Therefore we pose the following questions:

(a) Is there a direct relation between the consistency relations (1.2),(1.3) and the quantum Yang-Baxter equation?
(b) Can one construct some physical model for which relations (1.2),(1.3) appear as a condition for the model to be integrable (exactly solvable)?

We leave these open questions to be addressed in future publications.

The structure of the paper is as follows. We first give some constructions of twisted generalized Weyl algebras in full generality. After recalling the definition in Section 2, we show in Section 3 that the construction of a twisted generalized Weyl algebra is functorial in the initial data. In Section 4 and Section 5 we study two natural operations: taking quotients and localizing. In Section 6 we prove Theorem A. In Section 7 (Theorem 7.3(a)), we show that consistent TGWA with all t_i invertible are in fact $\mathbb{Z}^n$-crossed product algebras over R. Thus these TGWA may be thought of as producing solutions to equations (7.1) required for the $\mathbb{Z}^n$-action and twisted 2-cocycle map to produce nontrivial crossed product algebras.

We also define a notion of weak consistency and prove in Theorem 7.3(b) that there is an embedding of any regular, weakly consistent TGWA into an associated $\mathbb{Z}^n$-crossed product algebra.

Notation and conventions. By "ring" ("algebra") we mean a unital associative ring (algebra). All ring and algebra morphisms are required to be unital. By "ideal" we mean a two-sided ideal unless otherwise stated. An element x of a ring R is said to be *regular in* R if for all nonzero $y \in R$ we have $xy \neq 0$ and $yx \neq 0$. The set of invertible elements in a ring R will be denoted by $R^\times$.

Let R be a ring. Recall that an R-ring is a ring A together with a ring morphism $R \to A$. Let X be a set. Let RXR be the free R-bimodule on X. The free R-ring $F_R(X)$ on X is defined as the tensor algebra of the free R-bimodule on X: $F_R(X) = \bigoplus_{n\geq 0}(RXR)^{\otimes_R n}$, where $(RXR)^{\otimes_R 0} = R$ by convention and the ring morphism $R \to F_R(X)$ is the inclusion into the degree zero component. Throughout this paper we fix a commutative ring $\Bbbk$.

2. Definition of TGWA

We recall the definition of twisted generalized Weyl algebras [10, 9]. Here we emphasize the initial data more than usual, which will be useful in the next section to express the functoriality of the construction.

Definition 2.1 (TGW datum)**.** Let n be a positive integer. A *twisted generalized Weyl datum (over $\Bbbk$ of degree n)* is a triple (R, σ, t) where

- R is a unital associative $\Bbbk$-algebra,
- σ is a group homomorphism $\sigma : \mathbb{Z}^n \to \mathrm{Aut}_\Bbbk(R)$, $g \mapsto \sigma_g$,
- t is a function $t : \{1, \dots, n\} \to Z(R)$, $i \mapsto t_i$.

A *morphism* between TGW data over $\Bbbk$ of degree n,

$$\varphi : (R, \sigma, t) \to (R', \sigma', t'),$$

is a $\Bbbk$-algebra morphism $\varphi : R \to R'$ such that $\varphi\sigma_i = \sigma'_i\varphi$ and $\varphi(t_i) = t'_i$ for all $i \in \{1, \dots, n\}$. We let $\mathsf{TGW}_n(\Bbbk)$ denote the category whose objects are the TGW data over $\Bbbk$ of degree n and morphisms are as above.

For $i \in \{1, \dots, n\}$ we put $\sigma_i = \sigma_{e_i}$, where $\{e_i\}_{i=1}^n$ is the standard $\mathbb{Z}$-basis for $\mathbb{Z}^n$. A *parameter matrix (over $\Bbbk^\times$ of size n)* is an $n \times n$ matrix $\mu = (\mu_{ij})_{i\neq j}$ without diagonal where $\mu_{ij} \in \Bbbk^\times\ \forall i \neq j$. The set of all parameter matrices over $\Bbbk^\times$ of size n will be denoted by $\mathrm{PM}_n(\Bbbk)$.

Definition 2.2 (TGW construction)**.** Let $n \in \mathbb{Z}_{>0}$, (R, σ, t) be an object in $\mathsf{TGW}_n(\Bbbk)$, and $\mu \in \mathrm{PM}_n(\Bbbk)$. The *twisted generalized Weyl construction with parameter matrix μ associated to the TGW datum (R, σ, t)* is denoted by $\mathcal{C}_\mu(R, \sigma, t)$ and is defined as the free R-ring on the set $\{x_i, y_i \mid i = 1, \dots, n\}$ modulo the two-sided ideal generated by the following set of elements:

$$x_i r - \sigma_i(r)x_i, \qquad y_i r - \sigma_i^{-1}(r)y_i, \qquad \forall r \in R,\ i \in \{1, \dots, n\}, \tag{2.1a}$$

$$y_i x_i - t_i, \qquad x_i y_i - \sigma_i(t_i), \qquad \forall i \in \{1, \dots, n\}, \tag{2.1b}$$

$$x_i y_j - \mu_{ij} y_j x_i, \qquad \forall i, j \in \{1, \dots, n\},\ i \neq j. \tag{2.1c}$$

The images in $\mathcal{C}_\mu(R,\sigma,t)$ of the elements x_i, y_i will be denoted by $\widehat{X}_i, \widehat{Y}_i$ respectively. The ring $\mathcal{C}_\mu(R,\sigma,t)$ has a $\mathbb{Z}^n$-gradation given by requiring $\deg \widehat{X}_i = e_i, \deg \widehat{Y}_i = -e_i, \deg r = 0 \, \forall r \in R$. Let $\mathcal{I}_\mu(R,\sigma,t) \subseteq \mathcal{C}_\mu(R,\sigma,t)$ be the sum of all graded ideals $J \subseteq \mathcal{C}_\mu(R,\sigma,t)$ having zero intersection with the degree zero component, i.e. such that $\mathcal{C}_\mu(R,\sigma,t)_0 \cap J = \{0\}$. It is easy to see that $\mathcal{I}_\mu(R,\sigma,t)$ is the unique maximal graded ideal having zero intersection with the degree zero component.

Definition 2.3 (TGW algebra)**.** The *twisted generalized Weyl algebra with parameter matrix μ associated to the TGW datum* (R,σ,t) is denoted by $\mathcal{A}_\mu(R,\sigma,t)$ and is defined as the quotient $\mathcal{A}_\mu(R,\sigma,t) := \mathcal{C}_\mu(R,\sigma,t)/\mathcal{I}_\mu(R,\sigma,t)$.

Since $\mathcal{I}_\mu(R,\sigma,t)$ is graded, $\mathcal{A}_\mu(R,\sigma,t)$ inherits a $\mathbb{Z}^n$-gradation from $\mathcal{C}_\mu(R,\sigma,t)$. The images in $\mathcal{A}_\mu(R,\sigma,t)$ of the elements $\widehat{X}_i, \widehat{Y}_i$ will be denoted by X_i, Y_i. By a *monic monomial* in a TGW construction $\mathcal{C}_\mu(R,\sigma,t)$ (respectively TGW algebra $\mathcal{A}_\mu(R,\sigma,t)$) we will mean a product of elements from $\{\widehat{X}_i, \widehat{Y}_i \mid i = 1,\dots,n\}$ (respectively $\{X_i, Y_i \mid i = 1,\dots,n\}$).

The following statements are easy to check.

Lemma 2.4.

(a) *$\mathcal{A}_\mu(R,\sigma,t)$ (respectively $\mathcal{C}_\mu(R,\sigma,t)$) is generated as a left and as a right R-module by the monic monomials in X_i, Y_i $(i = 1,\dots,n)$ (respectively $\widehat{X}_i, \widehat{Y}_i$ $(i = 1,\dots,n)$).*

(b) *The degree zero component of $\mathcal{A}_\mu(R,\sigma,t)$ is equal to the image of R under the natural map $\rho : R \to \mathcal{A}_\mu(R,\sigma,t)$.*

(c) *Any nonzero graded ideal of $\mathcal{A}_\mu(R,\sigma,t)$ has nonzero intersection with the degree zero component.*

Definition 2.5 (μ-Consistency)**.** Let (R,σ,t) be a TGW datum over $\Bbbk$ of degree n and μ be a parameter matrix over $\Bbbk^\times$ of size n. We say that (R,σ,t) is *μ-consistent* if the canonical map $\rho : R \to \mathcal{A}_\mu(R,\sigma,t)$ is injective.

Since $\mathcal{I}_\mu(R,\sigma,t)$ has zero intersection with the zero-component, (R,σ,t) is μ-consistent iff the canonical map $R \to \mathcal{C}_\mu(R,\sigma,t)$ is injective. Even in the cases when ρ is not injective, we will often view $\mathcal{A}_\mu(R,\sigma,t)$ as a left R-module and write for example rX_i instead of $\rho(r)X_i$.

Definition 2.6 (Regularity)**.** A TGW datum (R,σ,t) is called *regular* if t_i is regular in R for all i.

Lemma 2.7. *If $t_i \in R^\times$ for all i, then the canonical projection $\mathcal{C}_\mu(R,\sigma,t) \to \mathcal{A}_\mu(R,\sigma,t)$ is an isomorphism.*

Proof. The algebra $\mathcal{C}_\mu(R,\sigma,t)$ is a $\mathbb{Z}^n$-crossed product algebra over its degree zero subalgebra, since each homogeneous component contains an invertible element. Indeed since $t_i \in R^\times$, each X_i is invertible and thus $X_1^{g_1} \cdots X_n^{g_n}$ has degree g and is invertible. Therefore any nonzero graded ideal in $\mathcal{C}_\mu(R,\sigma,t)$ has nonzero intersection with the degree zero component, a property which holds for any strongly graded ring, in particular for crossed product algebras. Thus $\mathcal{I}_\mu(R,\sigma,t) = 0$, which proves the claim. □

We give an example of a TGWA which is the trivial ring. This shows the need for finding sufficient conditions on the TGW datum which will ensure the algebra is nontrivial. Finding such conditions is the goal of this paper.

Example 2.8. In [11, Equation (2)] it was observed that the relation

$$\sigma_i\sigma_j(t_j)X_jX_i = \mu_{ij}\sigma_j(t_j)X_iX_j \tag{2.2}$$

is satisfied in any TGW algebra $A = \mathcal{A}_\mu(R, \sigma, t)$ for every $i \neq j$. There are two ways to commute a multiple of $X_kX_jX_i$ to a multiple of $X_iX_jX_k$ (starting by commuting X_kX_j or X_jX_i respectively), using relation (2.2). Namely,

$$\sigma_j\sigma_k(t_k)\sigma_i\sigma_j\sigma_k(t_k)\sigma_i\sigma_j(t_j)X_kX_jX_i = \mu_{jk}\sigma_k(t_k)\mu_{ik}\sigma_j\sigma_k(t_k)\mu_{ij}\sigma_j(t_j)X_iX_jX_k \tag{2.3}$$

and

$$\sigma_i\sigma_j\sigma_k(t_j)\sigma_i\sigma_k(t_k)\sigma_i\sigma_j\sigma_k(t_k)X_kX_jX_i = \mu_{ij}\sigma_k\sigma_j(t_j)\mu_{ik}\sigma_k(t_k)\mu_{jk}\sigma_i\sigma_k(t_k)X_iX_jX_k. \tag{2.4}$$

Combining (2.3) and (2.4) we get

$$\begin{aligned}&\big(\sigma_j\sigma_k(t_k)\sigma_i\sigma_j\sigma_k(t_k)\sigma_i\sigma_j(t_j)\mu_{ij}\sigma_k\sigma_j(t_j)\mu_{ik}\sigma_k(t_k)\mu_{jk}\sigma_i\sigma_k(t_k)\\ &\quad - \mu_{jk}\sigma_k(t_k)\mu_{ik}\sigma_j\sigma_k(t_k)\mu_{ij}\sigma_j(t_j)\sigma_i\sigma_j\sigma_k(t_j)\sigma_i\sigma_k(t_k)\sigma_i\sigma_j\sigma_k(t_k)\big)X_iX_jX_k = 0.\end{aligned} \tag{2.5}$$

Assume that t_i is invertible in R for every i. Thus, since $t_i = Y_iX_i$ in A, X_i is invertible in A for every i. Then (2.5) implies that, in the algebra A,

$$\sigma_i\sigma_j(t_j)\sigma_k\sigma_j(t_j) - \sigma_j(t_j)\sigma_i\sigma_j\sigma_k(t_j) = 0. \tag{2.6}$$

(Note that we cannot conclude that (2.6) must also hold in R, unless we know that the canonical map $\rho : R \to A$ is injective.) The following is an example of a TGWA where (1.1) holds but the left-hand side of (2.6) in fact is an invertible element of A, which forces A to be the trivial ring. Namely, let

- $n = 3$, $\Bbbk = \mathbb{C}$, $\mu_{ij} = 1$ for all $i \neq j$;
- $R = \mathbb{C}[\alpha_{12}^{\pm1}, \alpha_{23}^{\pm1}, \alpha_{13}^{\pm1}, t_1^{\pm1}, t_2^{\pm1}, t_3^{\pm1}]$, a Laurent polynomial algebra in 6 variables;
- $\sigma_1, \sigma_2, \sigma_3 \in \mathrm{Aut}_{\mathbb{C}}(R)$, given by

$$\sigma_i(\alpha_{jk}) := \begin{cases}\alpha_{jk} & \text{if } \{i,j,k\} \neq \{1,2,3\},\\ -\alpha_{jk} & \text{if } \{i,j,k\} = \{1,2,3\},\end{cases} \tag{2.7}$$

  for all $i, j, k \in \{1, 2, 3\}$ with $j < k$, and

$$\sigma_i(t_j) := \alpha_{ij}t_j, \tag{2.8}$$

  for all $i, j \in \{1, 2, 3\}$, where $\alpha_{ji} := \alpha_{ij}^{-1}$ for $i < j$ and $\alpha_{ii} := 1$ for $i = 1, 2, 3$.

Then the automorphisms $\sigma_1, \sigma_2, \sigma_3$ commute with each other. Indeed, it is immediate that $\sigma_i\sigma_j(\alpha_{kl}) = \sigma_j\sigma_i(\alpha_{kl})$ holds for all $i, j, k, l \in \{1, 2, 3\}$. On the t_i we have

$$\sigma_i\sigma_j(t_k) = \sigma_i(\alpha_{jk}t_k) = \begin{cases}-\alpha_{jk}\alpha_{ik}t_k, & \text{if } \{i,j,k\} = \{1,2,3\},\\ \alpha_{jk}\alpha_{ik}t_k, & \text{otherwise.}\end{cases}$$

Since the right-hand side is symmetric in i, j, it follows that σ_i and σ_j commute on any t_k. Relation (1.1) is easily checked. However, for any i, j, k such that $\{i, j, k\} = \{1, 2, 3\}$, the result of σ_j^{-1} applied to the left-hand side of (2.6) equals

$$\sigma_i(t_j)\sigma_k(t_j) - \sigma_i\sigma_k(t_j)t_j = \alpha_{ij}\alpha_{kj}t_j^2 - \sigma_i(\alpha_{kj}t_j)t_j = 2\alpha_{ij}\alpha_{kj}t_j^2, \tag{2.9}$$

which is invertible in R; hence the image is invertible in A. Thus (2.6) implies that A is the trivial ring $\{0\}$ despite the fact that (1.1) holds.

Put in different terms, the situation for the TGW datum above is that the ideal in the free R-ring $F = F_R\big(\{x_i, y_i \mid i = 1, \dots, n\}\big)$ generated by the elements (2.1) contains an invertible element of the algebra R and consequently the quotient of F by that ideal becomes trivial.

3. Functoriality of the construction

We prove in this section that the construction of twisted generalized Weyl algebras is functorial in the initial data in a certain sense. This will be applied in the next two sections.

For a group G we let G-$\mathsf{GrAlg}_{\Bbbk}$ denote the category of G-graded $\Bbbk$-algebras where morphisms $\varphi : A \to B$ are graded $\Bbbk$-algebra homomorphisms: $\varphi(A_g) \subseteq B_g$ for all $g \in G$. We have the forgetful functor $\mathcal{F} : \mathsf{TGW}_n(\Bbbk) \to \mathbb{Z}^n$-$\mathsf{GrAlg}_{\Bbbk}$, defined by $\mathcal{F}(R, \sigma, t) = R$ with trivial grading $R_0 = R$.

Theorem 3.1. *Let $n \in \mathbb{Z}_{>0}$ and μ be a parameter matrix over $\Bbbk^\times$ of size n.*

(a) *The construction of a twisted generalized Weyl algebra $\mathcal{A}_\mu(R, \sigma, t)$ from a TGW datum (R, σ, t) defines a functor $\mathcal{A}_\mu : \mathsf{TGW}_n(\Bbbk) \to \mathbb{Z}^n$-$\mathsf{GrAlg}_{\Bbbk}$. That is, for any morphism $\varphi : (R, \sigma, t) \to (R', \sigma', t')$ in $\mathsf{TGW}_n(\Bbbk)$ there is a morphism $\mathcal{A}_\mu(\varphi) : \mathcal{A}_\mu(R, \sigma, t) \to \mathcal{A}_\mu(R', \sigma', t')$ in $\mathbb{Z}^n$-$\mathsf{GrAlg}_{\Bbbk}$ such that $\mathcal{A}_\mu$ preserves compositions and identity morphisms.*

(b) *The family $\rho_\mu = \{\rho_{\mu,(R,\sigma,t)}\}$, where (R, σ, t) ranges over the objects in $\mathsf{TGW}_n(\Bbbk)$, of canonical maps $\rho_{\mu,(R,\sigma,t)} : R \to \mathcal{A}_\mu(R, \sigma, t)$ defines a natural transformation from the forgetful functor $\mathcal{F}$ to the functor $\mathcal{A}_\mu$ from part (a). That is, given any morphism $\varphi : (R, \sigma, t) \to (R', \sigma', t')$ in $\mathsf{TGW}_n(\Bbbk)$, we have the following commutative diagram in $\mathbb{Z}^n$-$\mathsf{GrAlg}_{\Bbbk}$:*

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "\rho_{\mu,(R,\sigma,t)}"] & R' \arrow[d, "\rho_{\mu,(R',\sigma',t')}"] \\ \mathcal{A}_\mu(R, \sigma, t) \arrow[r, "\mathcal{A}_\mu(\varphi)"] & \mathcal{A}_\mu(R', \sigma', t') \end{tikzcd} \tag{3.1}$$

(c) *For any morphism $\varphi : (R, \sigma, t) \to (R', \sigma', t')$ in $\mathsf{TGW}_n(\Bbbk)$, the algebra $\mathcal{A}_\mu(R', \sigma', t')$ is generated as a left and as a right R'-module by the image of $\mathcal{A}_\mu(\varphi)$.*

Proof. Let $\varphi : (R, \sigma, t) \to (R', \sigma', t')$ be a morphism of TGW data. Pulling back the canonical map $R' \to \mathcal{C}_\mu(R', \sigma', t')$ through φ we get a map $R \to \mathcal{C}_\mu(R', \sigma', t')$. The universal property of free R-rings implies that this map extends uniquely to a morphism of R-rings from the free R-ring on the set $\{x_i, y_i \mid i = 1, \dots, n\}$ by requiring that $x_i \mapsto \widehat{X}'_i$, $y_i \mapsto \widehat{Y}'_i$ for all i, where $\widehat{X}'_i, \widehat{Y}'_i$ denote the generators in $\mathcal{C}_\mu(R', \sigma', t')$. Since φ is a morphism of TGW data, one verifies that the ideal with generators (2.1) lies in the kernel, thus inducing a map $\widehat{\varphi}$ such that the following diagram commutes:

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d] & R' \arrow[d] \\ \mathcal{C}_\mu(R, \sigma, t) \arrow[r, "\widehat{\varphi}"] & \mathcal{C}_\mu(R', \sigma', t') \end{tikzcd}$$

Let $\mathcal{I} = \mathcal{I}_\mu(R,\sigma,t)$ (respectively $\mathcal{I}' = \mathcal{I}_\mu(R',\sigma',t')$) denote the sum of all graded ideals in $\mathcal{C}_\mu(R,\sigma,t)$ (respectively $\mathcal{C}_\mu(R',\sigma',t')$) having zero intersection with the degree zero component. We need to verify that $\widehat{\varphi}(\mathcal{I}) \subseteq \mathcal{I}'$. Let $a \in \mathcal{I}$. Since $\mathcal{I}$ is graded we can assume that a is homogeneous. Let $d \in \mathbb{Z}^n$ be its degree. To prove that $\widehat{\varphi}(a) \in \mathcal{I}'$ we show that the two-sided ideal generated by $\widehat{\varphi}(a)$ has zero intersection with the zero component. Then, since $\mathcal{I}'$ is the sum of all such ideals, it must contain $\widehat{\varphi}(a)$. Let $m > 0$ and b'_j, c'_j $(j = 1,\dots,m)$ be nonzero homogeneous elements of $\mathcal{C}_\mu(R',\sigma',t')$ such that $d + \deg b'_j + \deg c'_j = 0$ for all j. We must show that $\sum_j b'_j\varphi(a)c'_j = 0$. Without loss of generality, the elements b_j and c_j are monomials. Since $\widehat{\varphi}(\widehat{X}_i) = \widehat{X}'_i$ and $\widehat{\varphi}(\widehat{Y}_i) = \widehat{Y}'_i$ it follows that we have $b'_j = r'_j\widehat{\varphi}(b_j)$, $c'_j = s'_j\widehat{\varphi}(c_j)$, where $r'_j, s'_j \in R'$ and $b_j, c_j \in \mathcal{C}_\mu(R,\sigma,t)$. Thus

$$\sum_j b'_j\widehat{\varphi}(a)c'_j = \sum_j r'_j\widehat{\varphi}(b_j)\widehat{\varphi}(a)s'_j\widehat{\varphi}(c_j) = \sum_j r'_j\tau_j(s'_j)\varphi(b_jac_j),$$

where $\tau_j = \sigma_{d+\deg b_j}$. But $a \in \mathcal{I}$, so $b_jac_j \in \mathcal{I}$; and hence, since $\deg(b_jac_j) = 0$, it follows that $b_jac_j = 0$ for each j. This proves that $\widehat{\varphi}(a)$ generates a two-sided ideal having zero intersection with the degree zero component, and thus $\widehat{\varphi}(a) \in \mathcal{I}'$. Hence $\widehat{\varphi}(\mathcal{I}) \subseteq \mathcal{I}'$ and therefore $\widehat{\varphi}$ induces a $\Bbbk$-algebra morphism $\mathcal{A}_\mu(\varphi) : \mathcal{A}_\mu(R,\sigma,t) \to \mathcal{A}_\mu(R',\sigma',t')$ such that the diagram (3.1) commutes. That $\mathcal{A}_\mu$ defines a functor is easy to check. This proves part (a). Claim (b) follows from the construction of $\mathcal{A}_\mu(\varphi)$.

(c) By construction, the image of $\mathcal{A}_\mu(\varphi)$ contains all the elements X'_i, Y'_i $(i = 1,\dots,n)$, hence all monic monomials, which generate $\mathcal{A}_\mu(R',\sigma',t')$ as a left and as a right R'-module, by Lemma 2.4(a). □

Corollary 3.2. *If $\varphi : (R,\sigma,t) \to (R',\sigma',t')$ is a surjective morphism in $\mathsf{TGW}_n(\Bbbk)$, then $\mathcal{A}_\mu(\varphi) : \mathcal{A}_\mu(R,\sigma,t) \to \mathcal{A}_\mu(R',\sigma',t')$ is surjective for any $\mu \in \mathrm{PM}_n(\Bbbk)$.*

Proof. This follows from Theorem 3.1(b) and (c). □

Lemma 3.3. *Let $\varphi : (R,\sigma,t) \to (R',\sigma',t')$ be a surjective morphism in $\mathsf{TGW}_n(\Bbbk)$. For $\mu \in \mathrm{PM}_n(\Bbbk)$ put $A = \mathcal{A}_\mu(R,\sigma,t)$ and let $K \subseteq A_0$ be the kernel of the restriction of $\mathcal{A}_\mu(\varphi)$ to A_0. Then $\ker\big(\mathcal{A}_\mu(\varphi)\big)$ equals the sum of all graded ideals J in A such that $J \cap A_0 \subseteq K$.*

Proof. Put $\bar{A} = \mathcal{A}_\mu(R',\sigma',t')$. Let $I(K)$ denote the sum of all graded ideals in A whose intersection with A_0 is contained in K. For brevity we put $\Phi = \mathcal{A}_\mu(\varphi)$. Since Φ is surjective by Corollary 3.2, the image $\Phi\big(I(K)\big)$ is a graded ideal of $\bar{A}$ whose intersection with $\bar{A}_0$ is zero. Lemma 2.4(c) applied to $\bar{A}$ gives $\Phi\big(I(K)\big) = \{0\}$. Thus $I(K) \subseteq \ker\Phi$. Conversely, suppose $a \in \ker(\Phi)$. We can assume a is homogeneous. Clearly $(AaA) \cap A_0 \subseteq K$, which means that $AaA \subseteq I(K)$. So $\ker(\Phi) \subseteq I(K)$. □

Corollary 3.4. *Let $\mu \in \mathrm{PM}_n(\Bbbk)$. If $\varphi : (R,\sigma,t) \to (R',\sigma',t')$ is a morphism in $\mathsf{TGW}_n(\Bbbk)$ which is injective as a $\Bbbk$-algebra morphism and if (R',σ',t') is μ-consistent, then (R,σ,t) is μ-consistent and $\mathcal{A}_\mu(\varphi)$ is injective.*

Proof. That (R,σ,t) is μ-consistent is immediate by the commutativity of the diagram (3.1). Put $A=\mathcal{A}_\mu(R,\sigma,t)$ and $K=\ker\big(\mathcal{A}_\mu(\varphi)|_{A_0}\big)$. Then by Lemma 2.4(b) and commutativity of (3.1), $K=\{0\}$. Thus by Lemma 3.3, $\mathcal{A}_\mu(\varphi)$ is injective. □

4. Quotients of TGWA

The following result generalizes [3, Proposition 2.12] from generalized Weyl algebras to TGW algebras. If $(R,\sigma,t)\in\mathsf{TGW}_n(\Bbbk)$, the group $\mathbb{Z}^n$ acts on R via σ. An ideal J in R is called $\mathbb{Z}^n$-*invariant* if $\sigma_g(r)\in J\,\forall r\in J, g\in\mathbb{Z}^n$.

Theorem 4.1. *Let $A=\mathcal{A}_\mu(R,\sigma,t)$ be a twisted generalized Weyl algebra over $\Bbbk$ of degree n, and J a $\mathbb{Z}^n$-invariant ideal of R. Let $\bar A=\mathcal{A}_\mu(R/J,\bar\sigma,\bar t)$, where $\bar\sigma_g(r+J)=\sigma_g(r)+J$ for all $g\in\mathbb{Z}^n, r\in R$ and $\bar t_i=t_i+J$ for all i. Let $\rho:R\to A$ and $\bar\rho:R/J\to\bar A$ be the natural maps. Put $K=\rho\big(\ker(\bar\rho\circ\pi_J)\big)$, where $\pi_J:R\to R/J$ is the canonical projection. Suppose that*

(i) *$\forall g\in\mathbb{Z}^n$ there exists a monic monomial $u_g\in A_g$ such that $A_g=A_0u_g$,*
(ii) *$\rho(\sigma_g(t_i))+K$ is a regular element in A_0/K for all $i\in\{1,\dots,n\}$, $g\in\mathbb{Z}^n$.*

Let $\langle K\rangle=AKA$ be the ideal in $\mathcal{A}_\mu(R,\sigma,t)$ generated by K. Then $AK=\langle K\rangle=KA$, and there is a graded isomorphism

$$\mathcal{A}_\mu(R,\sigma,t)/\langle K\rangle\simeq\mathcal{A}_\mu(R/J,\bar\sigma,\bar t). \tag{4.1}$$

Proof. The map π_J intertwines the $\mathbb{Z}^n$-actions and maps t_i to $\bar t_i$; hence it is a morphism $\pi_J:(R,\sigma,t)\to(R/J,\bar\sigma,\bar t)$ in $\mathsf{TGW}_n(\Bbbk)$. Applying the functor $\mathcal{A}_\mu$ from Theorem 3.1 we get a $\Bbbk$-algebra morphism $\mathcal{A}_\mu(\pi_J):\mathcal{A}_\mu(R,\sigma,t)\to\mathcal{A}_\mu(R/J,\bar\sigma,\bar t)$ which is surjective by Corollary 3.2. Thus it remains to prove that the kernel of $\mathcal{A}_\mu(\pi_J)$ equals $\langle K\rangle$.

Let $r\in\ker(\bar\rho\circ\pi_J)$. We claim that $\rho(\sigma_g(r))\in K$ for all $g\in\mathbb{Z}$. Let $u_g\in\bar A$ be any monic monomial of degree g and $u_{-g}\in\bar A$ a monic monomial of degree $-g$. Then u_gu_{-g} can be written as $\bar\rho(a+J)$, where a is a product of elements of the form $\sigma_h(t_i)$ ($h\in\mathbb{Z}^n$, $i=1,\dots,n$), possibly multiplied by an element of $\Bbbk^\times$. By assumption (ii), $\rho(a)+K$ is regular in A_0/K. In $\bar A$, we have $0=u_g\bar\rho(r+J)u_{-g}=\bar\rho(\sigma_g(r)+J)\bar\rho(a+J)=\bar\rho(\sigma_g(r)a+J)$. So $\rho(\sigma_g(r))\rho(a)\in K$. Since $\rho(a)+K$ is regular in A_0/K by assumption (ii), we get $\rho(\sigma_g(r))\in K$.

Using this fact and Lemma 2.4(a), we get $AK=\langle K\rangle=KA$. Since $\langle K\rangle\cap A_0=AK\cap A_0=\big(\sum_{g\in\mathbb{Z}^n}A_gK\big)\cap A_0=A_0K=K$, Lemma 3.3 implies that $\langle K\rangle\subseteq\ker\big(\mathcal{A}_\mu(\pi_J)\big)$. For the converse, let $g\in\mathbb{Z}^n$ and let $a\in A_g\cap\ker\big(\mathcal{A}_\mu(\pi_J)\big)$. By assumption (i), $a=rZ_{i_1}\cdots Z_{i_m}$ for some $Z_{i_j}\in\{X_{i_j},Y_{i_j}\}$ and some $r\in A_0$. Put $X_i^*=Y_i$, $Y_i^*=X_i$. Then $b:=aZ_{i_m}^*\cdots Z_{i_1}^*$ has degree zero; hence $b\in\ker\big(\mathcal{A}_\mu(\pi_j)|_{A_0}\big)$, which by Lemma 2.4(b) and naturality (3.1) equals K. On the other hand, using relations (2.1) we get $b=r\cdot\xi u$, where $\xi\in\Bbbk^\times$ and u is a product of elements of the form $\rho(\sigma_h(t_i))$ ($h\in\mathbb{Z}^n$, $i=1,\dots,n$). Since by assumption (ii) all $\rho(\sigma_h(t_i))+K$ are regular in A_0/K, we get $r\in K$ and thus $a\in KA$. □

Remark 4.2. If (R,σ,t) and $(R/J,\bar\sigma,\bar t)$ are μ-consistent, we can identify R and R/J with their images in the corresponding TGW algebras. Under such identifications, the ideal K in Theorem 4.1 is just equal to J. This follows from the commutativity of (3.1).

5. Localizations of TGWA

The following trick was first observed in [9] in the case of R being a commutative domain.

Lemma 5.1. *Let $A = \mathcal{A}_\mu(R, \sigma, t)$ be a twisted generalized Weyl algebra and $\rho : R \to A$ the natural map. Suppose all $\rho(t_i)$ are regular in A_0. Let $g \in \mathbb{Z}^n$. Then*

$$ab = \sigma_g(ba) \tag{5.1}$$

for any monic monomial $a \in A_g$ and any $b \in A_{-g}$.

Proof. Observe that if $Y_i c \in A_0$, then $Y_i c Y_i X_i = \sigma_i^{-1}(cY_i) Y_i X_i$, so since $Y_i X_i = \rho(t_i)$ is regular in A_0 we get $Y_i c = \sigma_i^{-1}(cY_i)$. Similarly $X_i c = \sigma_i(cX_i)$ if $X_i c \in A_0$. Applying this for each of the factors in a gives the claim. □

Theorem 5.2. *Let $A = \mathcal{A}_\mu(R, \sigma, t)$ be a twisted generalized Weyl algebra over $\Bbbk$ of degree n such that $t_1, \dots, t_n$ are regular in R. Suppose $S \subseteq Z(R)$ is a subset such that*

(i) *S is multiplicative: $0 \notin S$, $1 \in S$ and $r, s \in S \Rightarrow rs \in S$,*
(ii) *all elements of S are regular in R,*
(iii) *$\sigma_g(S) \subseteq S$ for all $g \in \mathbb{Z}^n$,*
(iv) *$(S^{-1}R, \widetilde{\sigma}, \widetilde{t})$ is μ-consistent,*

where $\widetilde{\sigma}_g \in \mathrm{Aut}_\Bbbk(S^{-1}R)$ is the unique extension of σ_g for all $g \in \mathbb{Z}^n$, and $\widetilde{t}_i = t_i/1$ for all $i = 1, \dots, n$. By Corollary 3.4, (R, σ, t) is also μ-consistent and we can identify R with its image, A_0, under the natural map $\rho : R \to \mathcal{A}_\mu(R, \sigma, t)$. Then the following statements hold:

(a) *The elements of S are regular in $\mathcal{A}_\mu(R, \sigma, t)$.*
(b) *S is a left and right Ore set in $\mathcal{A}_\mu(R, \sigma, t)$.*
(c) *We have an isomorphism*

$$S^{-1}\mathcal{A}_\mu(R, \sigma, t) \simeq \mathcal{A}_\mu(S^{-1}R, \widetilde{\sigma}, \widetilde{t}). \tag{5.2}$$

Proof. (a) Put $A = \mathcal{A}_\mu(R, \sigma, t)$. Suppose $sa = 0$ for some $s \in S$, $a \in A \backslash \{0\}$. Without loss of generality we can assume a is homogeneous, say $\deg a = g \in \mathbb{Z}^n$. By Lemma 2.4(c) and Lemma 2.4(a) there exist monic monomials $b, c \in A$ such that $bac \in A_0 \setminus \{0\} = R \setminus \{0\}$. By Lemma 5.1, $bac = \sigma_h(acb)$, where $h = \deg(b)$. Thus $acb \in R \setminus \{0\}$ also. But we have $sacb = 0$, which contradicts that s is regular in R.
(b) Easy to check using relations (2.1).
(c) The canonical map $\varphi : R \to S^{-1}R$ intertwines the $\mathbb{Z}^n$-actions and maps t_i to $t_i/1$; hence it is a morphism, $\varphi : (R, \sigma, t) \to (S^{-1}R, \widetilde{\sigma}, \widetilde{t})$, in the category $\mathsf{TGW}_n(\Bbbk)$. Applying the functor $\mathcal{A}_\mu$ from Theorem 3.1 gives a morphism of graded $\Bbbk$-algebras

$$\mathcal{A}_\mu(\varphi) : \mathcal{A}_\mu(R, \sigma, t) \to \mathcal{A}_\mu(S^{-1}R, \sigma, t). \tag{5.3}$$

Since φ is injective, $\mathcal{A}_\mu(\varphi)$ is injective by Corollary 3.4. Since we identify R and $S^{-1}R$ with their images in the respective TGW algebras, commutativity of (3.1) just says that the restriction of $\mathcal{A}_\mu(\varphi)$ to R coincides with φ. In particular $\mathcal{A}_\mu(\varphi)$ maps each element of S to an invertible element. Hence, by the universal property of localization, there is an induced map

$$\psi : S^{-1}\mathcal{A}_\mu(R, \sigma, t) \to \mathcal{A}_\mu(S^{-1}R, \sigma, t). \tag{5.4}$$

Since the image of ψ contains $S^{-1}R$ as well as all the generators X_i, Y_i, ψ is surjective. Suppose $a \in S^{-1}\mathcal{A}_\mu(R,\sigma,t)$, $\psi(a)=0$. By part (a), S consists of regular elements in $\mathcal{A}_\mu(R,\sigma,t)$ and thus the canonical map $\mathcal{A}_\mu(R,\sigma,t) \to S^{-1}\mathcal{A}_\mu(R,\sigma,t)$ is injective and can be regarded as an inclusion. Then $sa \in \mathcal{A}_\mu(R,\sigma,t)$ for some $s \in S$. The restriction of ψ to $\mathcal{A}_\mu(R,\sigma,t)$ coincides with $\mathcal{A}_\mu(\varphi)$. So $\mathcal{A}_\mu(\varphi)(sa) = \psi(sa) = \psi(s)\psi(a) = 0$. Since $\mathcal{A}_\mu(\varphi)$ is injective we get $sa = 0$; hence $a = 0$. This proves that ψ is injective, hence an isomorphism. □

6. μ-Consistency

For any $D = (R,\sigma,t) \in \mathsf{TGW}_n(\Bbbk)$ and any parameter matrix μ over $\Bbbk^\times$ of size n, we let $L_{\mu,D}$ denote the intersection of all $\mathbb{Z}^n$-invariant ideals in R containing the set

$$\begin{aligned} N_{\mu,D} :=& \big\{\sigma_i\sigma_j(t_it_j) - \mu_{ij}\mu_{ji}\sigma_i(t_i)\sigma_j(t_j) \mid i,j = 1,\dots,n, i \neq j\big\} \\ &\cup \big\{t_j\sigma_i\sigma_k(t_j) - \sigma_i(t_j)\sigma_k(t_j) \mid i,j,k = 1,\dots,n, i \neq j \neq k \neq i\big\}. \end{aligned} \tag{6.1}$$

Let T be the intersection of all $\mathbb{Z}^n$-invariant multiplicative subsets of R containing $\{t_1,\dots,t_n\}$. Concretely, T is the set of all products of elements from the set $\big\{\sigma_g(t_i) \mid g \in \mathbb{Z}^n,\ i \in \{1,\dots,n\}\big\}$. For an ideal I of R we let I^{e} be the extension of I in $T^{-1}R$, that is, the ideal in $T^{-1}R$ generated by I. For an ideal J of $T^{-1}R$ we let $J^{\mathrm{c}} = J \cap R$, called the contraction of J. The extension and contraction operations are order-preserving and we have $J^{\mathrm{ce}} = J$ for any ideal $J \subseteq T^{-1}R$ (see for example [1]).

Theorem 6.1. *Let $\mu \in \mathrm{PM}_n(\Bbbk)$ and let $D = (R,\sigma,t) \in \mathsf{TGW}_n(\Bbbk)$ be a regular TGW datum. Put $\bar{R} = R/\big((L_{\mu,D})^{\mathrm{ec}}\big)$. Then*

(a) $(L_{\mu,D})^{\mathrm{ec}} = (\ker\rho_{\mu,D})^{\mathrm{ec}}$,
(b) $(\bar{R},\bar{\sigma},\bar{t})$ *is μ-consistent,*
(c) $\mathcal{A}_\mu(R,\sigma,t)/\mathcal{J}_{\mu,D} \simeq \mathcal{A}_\mu(\bar{R},\bar{\sigma},\bar{t})$,

where $\mathcal{J}_{\mu,D}$ is the sum of all graded ideals J in $A = \mathcal{A}_\mu(R,\sigma,t)$ such that $J \cap A_0 \subseteq \rho\big(\ker(\rho_{\mu,D})^{\mathrm{ec}}\big)$.

Proof. We proceed in steps.

(1) $(\ker\rho_{\mu,D})^{\mathrm{ec}}$ is a $\mathbb{Z}^n$-invariant ideal of R. Indeed, suppose $r \in \ker(\rho_{\mu,D})^{\mathrm{ec}}$. Then $sr \in \ker\rho_{\mu,D}$ for some $s \in T$. Let $i \in \{1,\dots,n\}$. Then, in $\mathcal{A}_\mu(R,\sigma,t)$, we have $0 = X_i\rho(sr)Y_i = \rho(\sigma_i(sr))X_iY_i = \rho(\sigma_i(r)\sigma_i(s)\sigma_i(t_i))$ since s belongs to the center of R, which means that $\sigma_i(r)\sigma_i(st_i) \in \ker\rho_{\mu,D}$. Hence, since $\sigma_i(st_i) \in T$, we have $\sigma_i(r) \in (\ker\rho_{\mu,D})^{\mathrm{ec}}$. Similarly $\sigma_i^{-1}(r) \in (\ker\rho_{\mu,D})^{\mathrm{ec}}$.

(2) $(L_{\mu,D})^{\mathrm{ec}} \subseteq (\ker\rho_{\mu,D})^{\mathrm{ec}}$. By the properties of extension and contraction of ideals, the claim is equivalent to showing $L_{\mu,D} \subseteq (\ker\rho_{\mu,D})^{\mathrm{ec}}$. By step (1), $(\ker\rho_{\mu,D})^{\mathrm{ec}}$ is $\mathbb{Z}^n$-invariant. Thus it is enough to show that $N_{\mu,D} \subseteq (\ker\rho_{\mu,D})^{\mathrm{ec}}$. For any $i \neq j$ we have in $\mathcal{A}_\mu(R,\sigma,t)$,

$$\sigma_i\sigma_j(t_j)X_jX_i = X_jX_it_j = X_jX_iY_jX_j = \mu_{ij}X_jY_jX_iX_j = \mu_{ij}\sigma_j(t_j)X_iX_j \tag{6.2}$$

so that

$$\sigma_i\sigma_j(t_j)\sigma_i\sigma_j(t_i)X_jX_i = \mu_{ij}\mu_{ji}\sigma_j(t_j)\sigma_i(t_i)X_jX_i. \tag{6.3}$$

Multiplying (6.3) from the right by Y_iY_j and using $X_jX_iY_iY_j = \sigma_j\sigma_i(t_i)\sigma_j(t_j) \in T$ we conclude that

$$\sigma_i\sigma_j(t_it_j) - \mu_{ij}\mu_{ji}\sigma_j(t_j)\sigma_i(t_i) \in (\ker\rho_{\mu,D})^{\mathrm{ec}}. \tag{6.4}$$

Let $i, j, k \in \{1, \dots, n\}$ be three different indices. Using that, as before, $r \in R, rX_iX_jX_k = 0 \Rightarrow r \in (\ker \rho_{\mu,D})^{\mathrm{ec}}$, relation (2.5) implies

(6.5)
$$\begin{aligned}&\sigma_j\sigma_k(t_k)\sigma_i\sigma_j\sigma_k(t_k)\sigma_i\sigma_j(t_j)\mu_{ij}\sigma_k\sigma_j(t_j)\mu_{ik}\sigma_k(t_k)\mu_{jk}\sigma_i\sigma_k(t_k)\\ &\quad - \mu_{jk}\sigma_k(t_k)\mu_{ik}\sigma_j\sigma_k(t_k)\mu_{ij}\sigma_j(t_j)\sigma_i\sigma_j\sigma_k(t_j)\sigma_i\sigma_k(t_k)\sigma_i\sigma_j\sigma_k(t_k) \in (\ker \rho_{\mu,D})^{\mathrm{ec}}.\end{aligned}$$

Dividing by $\mu_{ij}\mu_{jk}\mu_{ik}$ and factoring we get

$$\begin{aligned}&\big(\sigma_i\sigma_j(t_j)\sigma_k\sigma_j(t_j) - \sigma_j(t_j)\sigma_i\sigma_j\sigma_k(t_j)\big)\cdot\sigma_k(t_k)\sigma_i\sigma_k(t_k)\sigma_j\sigma_k(t_k)\sigma_i\sigma_j\sigma_k(t_k)\\ &\qquad\qquad \in (\ker \rho_{\mu,D})^{\mathrm{ec}}.\end{aligned} \tag{6.6}$$

Since $\sigma_k(t_k)\sigma_i\sigma_k(t_k)\sigma_j\sigma_k(t_k)\sigma_i\sigma_j\sigma_k(t_k) \in T$ this implies that

$$\sigma_i\sigma_j(t_j)\sigma_k\sigma_j(t_j) - \sigma_j(t_j)\sigma_i\sigma_j\sigma_k(t_j) \in (\ker \rho_{\mu,D})^{\mathrm{ecec}} = (\ker \rho_{\mu,D})^{\mathrm{ec}}. \tag{6.7}$$

Applying σ_j^{-1} to (6.7), using that $(\ker \rho_{\mu,D})^{\mathrm{ec}}$ is $\mathbb{Z}^n$-invariant by step (1), we get together with (6.4) that $N_{\mu,D} \subseteq (\ker \rho_{\mu,D})^{\mathrm{ec}}$.

We will now show that we have the following commutative cube:

$$\begin{tikzcd}
 & \bar{R} \arrow[rr, hook, "\lambda_{\bar{T}}"] \arrow[dd, hook, "\bar{\rho}"] & & \bar{T}^{-1}\bar{R} = \overline{T^{-1}R} \arrow[dd, hook, "\widehat{\rho}"] \\
R \arrow[ur, "\pi_{(L_{\mu,D})^{\mathrm{ec}}}"] \arrow[rr, hook, "\lambda_T"] \arrow[dd, "\rho=\rho_{\mu,D}"] & & T^{-1}R \arrow[ur, "\pi_{L_{\mu,\widetilde{D}}}"] \arrow[dd, "\widetilde{\rho}"] & \\
 & \mathcal{A}_\mu(\bar{R}, \bar{\sigma}, \bar{t}) \arrow[rr, hook] & & \mathcal{A}_\mu(\bar{T}^{-1}\bar{R}, \widehat{\sigma}, \widehat{t}) \\
\mathcal{A}_\mu(R, \sigma, t) \arrow[ur] \arrow[rr] & & \mathcal{A}_\mu(T^{-1}R, \widetilde{\sigma}, \widetilde{t}) \arrow[ur, "\simeq"] &
\end{tikzcd} \tag{6.8}$$

Here $\widetilde{D} = (T^{-1}R, \widetilde{\sigma}, \widetilde{t})$, $\widetilde{\sigma}_g \in \mathrm{Aut}_{\Bbbk}(T^{-1}R)$ is the unique extension of σ_g, $\widetilde{t}_i = t_i/1 \in T^{-1}R$, $\bar{T}$ is the image in $\bar{R}$ of T, and $\widehat{\sigma}_g$ is the extension of $\bar{\sigma}_g$ from $\bar{R}$ to $\bar{T}^{-1}\bar{R}$, $\widehat{t}_i = \bar{t}_i/1 \in \bar{T}^{-1}\bar{R}$, and $\bar{t}_i$ is the image of t_i in $\bar{R}$. Note that localizing at T in $T^{-1}R$ has no effect; thus $(L_{\mu,\widetilde{D}})^{\mathrm{ec}} = L_{\mu,\widetilde{D}}$. The vertical arrows are all instances of the natural transformation ρ_μ. The maps in the bottom square are the result of applying the functor $\mathcal{A}_\mu$ to the respective maps above. The top square is commutative by the exactness of the localization functor. Thus the bottom square is commutative by functoriality of $\mathcal{A}_\mu$. Each vertical square is commutative due to the naturality (3.1). Surjectivity of $\mathcal{A}_\mu(\pi_{(L_{\mu,D})^{\mathrm{ec}}})$ and $\mathcal{A}_\mu(\pi_{L_{\mu,\widetilde{D}}})$ follows by Corollary 3.2. The localization map λ_T ($\lambda_{\bar{T}}$) is injective since elements of T ($\bar{T}$) are regular in R ($\bar{R}$). It remains to prove injectivity of $\widehat{\rho}, \bar{\rho}$, $\mathcal{A}_\mu(\lambda_{\bar{T}})$ and $\mathcal{A}_\mu\big(\pi_{L_{\mu,\widetilde{D}}}\big)$.

(3) $\widehat{\rho}$ is injective; i.e. $(\bar{T}^{-1}\bar{R}, \widehat{\sigma}, \widehat{t})$ is μ-consistent, To prove this we will use the diamond lemma in the form of [4, Theorem 6.1] and will freely use terminology from that paper. Put $k = \bar{T}^{-1}\bar{R}$. Let $X = \{X_1^{\pm 1}, \dots, X_n^{\pm 1}\}$. Let $M_{X_i^{\pm 1}}$ be the k-bimodules which are free of rank 1 as left k-modules: $M_{X_i^{\pm 1}} = kX_i^{\pm 1}$ and

with right k-module structure given by $X_i^{\pm 1} r = \widehat{\sigma}_i^{\pm 1}(r) X_i^{\pm 1}$. Let $k\langle X\rangle$ be the tensor ring on the k-bimodule $M = \bigoplus_{x\in X} M_x$. Let I_1 be the two-sided ideal in $k\langle X\rangle$ generated by the set

$$\{X_i^{\pm 1} X_i^{\mp 1} - 1 \mid i = 1,\dots,n\} \cup \tag{6.9}$$
$$\{X_{i_2}^{\varepsilon_2} X_{i_1}^{\varepsilon_1} - \widehat{\sigma}_{i_2}^{(\varepsilon_2-1)/2} \widehat{\sigma}_{i_1}^{(\varepsilon_1-1)/2} (\tau_{i_2 i_1})^{\varepsilon_1\varepsilon_2} X_{i_1}^{\varepsilon_1} X_{i_2}^{\varepsilon_2} \mid \varepsilon_1,\varepsilon_2 \in \{1,-1\},\ 1 \le i_1 < i_2 \le n\},$$

where

$$\tau_{ji} = \frac{\mu_{ij}\widehat{\sigma}_j(\widehat{t}_j)}{\widehat{\sigma}_i\widehat{\sigma}_j(\widehat{t}_j)} \in \bar{T}^{-1}\bar{R}. \tag{6.10}$$

One verifies that there are well-defined homomorphisms of $\mathbb{Z}^n$-graded algebras

$$\begin{aligned} \mathcal{C}_\mu(\bar{T}^{-1}\bar{R},\widehat{\sigma},\widehat{t}) &\to k\langle X\rangle/I_1, & k\langle X\rangle/I_1 &\to \mathcal{C}_\mu(\bar{T}^{-1}\bar{R},\widehat{\sigma},\widehat{t}),\\ X_i &\mapsto X_i, & X_i &\mapsto X_i,\\ Y_i &\mapsto \widehat{t}_i X_i^{-1}, & X_i^{-1} &\mapsto \widehat{t}_i^{-1} Y_i,\\ r &\mapsto r, & r &\mapsto r, \end{aligned} \tag{6.11}$$

which obviously are inverse to each other. Moreover they are also morphisms of k-bimodules. So, by Lemma 2.7, $\mathcal{A}_\mu(\bar{T}^{-1}\bar{R},\widehat{\sigma},\widehat{t}) \simeq k\langle X\rangle/I_1$ as $\mathbb{Z}^n$-graded $\Bbbk$-algebras and as k-bimodules. Consider the following reduction system:

$$X_{i_2}^{\varepsilon_2} X_{i_1}^{\varepsilon_1} \longrightarrow \sigma_{i_2}^{(\varepsilon_2-1)/2}\sigma_{i_1}^{(\varepsilon_1-1)/2}(\tau_{ji}^{\varepsilon_1\varepsilon_2}) X_{i_1}^{\varepsilon_1} X_{i_2}^{\varepsilon_2}, \quad \varepsilon_1,\varepsilon_2 \in \{1,-1\},\ 1 \le i_1 < i_2 \le n,$$
$$X_i^{\varepsilon} X_i^{-\varepsilon} \longrightarrow 1, \quad \varepsilon \in \{1,-1\},\ i \in \{1,\dots,n\}.$$

These reductions extend uniquely to k-bimodule homomorphisms from the appropriate submodules of $k\langle X\rangle$. Clearly $\langle X\rangle_{\mathrm{irr}} = \{X_1^{g_1}\cdots X_n^{g_n} \mid g \in \mathbb{Z}^n\}$. There are no inclusion ambiguities, but four types of overlap ambiguities:

$$X_{i_3}^{\varepsilon_3} X_{i_2}^{\varepsilon_2} X_{i_1}^{\varepsilon_1}, \quad X_{i_2}^{\varepsilon_2} X_{i_1}^{\varepsilon_1} X_{i_1}^{-\varepsilon_1}, \quad X_{i_2}^{\varepsilon_2} X_{i_2}^{-\varepsilon_2} X_{i_1}^{\varepsilon_1}, \quad X_i^{\varepsilon} X_i^{-\varepsilon} X_i^{\varepsilon},$$

the last of which is trivially resolvable. The two middle ones are easily checked to be resolvable, while the first one is resolvable due to the identity $\widehat{\sigma}_k(\tau_{ij}) = \tau_{ij}$ if $i,j,k \in \{1,\dots,n\}$ are different, which hold in $\bar{T}^{-1}\bar{R}$ since $t_j\sigma_i\sigma_k(t_j) - \sigma_i(t_j)\sigma_k(t_j) \in (L_{\mu,D})^{\mathrm{ec}}$. Thus, by [4, Theorem 6.1], the natural map

$$\bigoplus_{g\in\mathbb{Z}^n} kX_1^{g_1}\cdots X_n^{g_n} \to k\langle X\rangle/I_1$$

is an isomorphism of k-bimodules, in fact of $\mathbb{Z}^n$-graded k-bimodules, since the reductions are homogeneous. Since $\mathcal{A}_\mu(\bar{T}^{-1}R,\widehat{\sigma},\widehat{t})$ is isomorphic to $k\langle X\rangle/I_1$ as graded algebras and as k-bimodules, we in particular have for the degree zero component that the k-bimodule map $k \to k\langle X\rangle/I_1 \to \mathcal{A}_\mu(\bar{T}^{-1}\bar{R},\widehat{\sigma},\widehat{t})_0$ sending 1 to 1 is an isomorphism, which is what we wanted to prove.

(4) $\bar{\rho}$ and $\mathcal{A}_\mu(\lambda_{\bar{T}})$ are injective. This follows from injectivity of $\widehat{\rho}$ and $\lambda_{\bar{T}}$ and Corollary 3.4. This proves statement (b).

(5) $\mathcal{A}_\mu(\pi_{L_{\mu,\widetilde{D}}}) : \mathcal{A}_\mu(T^{-1}R,\sigma,t) \to \mathcal{A}_\mu(\bar{T}^{-1}\bar{R},\sigma,t)$ is an isomorphism. Observe that for $\widetilde{D}$ the contraction and extension are identity operations, since the $\widetilde{t}_i$ are already invertible. Consider $\pi_{L_{\mu,\widetilde{D}}} : (T^{-1}R,\sigma,t) \to (\overline{T^{-1}R},\sigma,t)$. As already noted, Corollary 3.2 implies that $\mathcal{A}_\mu(\pi_{L_{\mu,\widetilde{D}}})$ is surjective. The kernel of $\mathcal{A}_\mu(\pi_{L_{\mu,\widetilde{D}}}) \circ \widetilde{\rho}$ is by the commutativity of (6.8) equal to $\ker(\widehat{\rho}\circ\pi_{L_{\mu,\widetilde{D}}}) = L_{\mu,\widetilde{D}}$ since $(\bar{T}^{-1}\bar{R},\widehat{\sigma},\widehat{t})$ is μ-consistent by step (4). But by step (2), $L_{\mu,\widetilde{D}}$ is contained

in $\ker(\widetilde{\rho})$. Thus the restriction of $\mathcal{A}_\mu(\pi_{L_{\mu,\widetilde{D}}})$ to $\widetilde{\rho}(T^{-1}R)$ is injective; hence $\mathcal{A}_\mu(\pi_{L_{\mu,\widetilde{D}}})$ is injective by Lemma 3.3.

(6) $\ker(\rho_{\mu,D})^{\mathrm{ec}} \subseteq L^{\mathrm{ec}}_{\mu,D}$. By (6.8) and the injectivity of $\bar{\rho}$ we have $\ker(\rho_{\mu,D}) \subseteq \ker\big(\mathcal{A}_\mu(\pi_{(L_{\mu,D})^{\mathrm{ec}}})\circ\rho_{\mu,D}\big) = \ker(\bar{\rho}\circ\pi_{(L_{\mu,D})^{\mathrm{ec}}}) = (L_{\mu,D})^{\mathrm{ec}}$. This proves statement (a).

(7) $\ker\big(\mathcal{A}_\mu(\pi_{(L_{\mu,D})^{\mathrm{ec}}})\big) = \mathcal{J}_{\mu,D}$. This follows from Lemma 3.3 and the fact that the kernel of $\mathcal{A}_\mu(\pi_{(L_{\mu,D})^{\mathrm{ec}}}) \circ \rho_{\mu,D}$ is equal to the kernel of $\pi_{(L_{\mu,D})^{\mathrm{ec}}}$, which is $(L_{\mu,D})^{\mathrm{ec}} = (\ker\rho_{\mu,D})^{\mathrm{ec}}$, by step (2) and step (6). This proves statement (c).

□

We can now deduce the main result of the paper.

Proof of Theorem A. Let D be the TGW datum (R,σ,t) and $\rho_{\mu,D} : R \to \mathcal{A}_\mu(R,\sigma,t)$ be the canonical map of R-rings. By Theorem 6.1(a), $\rho_{\mu,D}$ is injective if and only if $L_{\mu,D} = \{0\}$, which by definition of $L_{\mu,D}$ is equivalent to the set $N_{\mu,D}$, defined in (6.1), being equal to $\{0\}$. But $N_{\mu,D} = \{0\}$ just means that relations (1.2) and (1.3) hold. This proves the required equivalence.

It only remains to prove the independence of the two conditions (1.2) and (1.3). It is clear that (1.2) does not follow from (1.3) because the latter does not depend on μ_{ij}. In Example 2.8, condition (1.2) is satisfied but $\rho_{\mu,D} : R \to \mathcal{A}_\mu(R,\sigma,t)$ is the zero map, and hence the TGW datum (R,σ,t) is not μ-consistent. Thus, by Theorem A, condition (1.3) cannot hold (this can also be verified directly; relation (2.9) shows that $N_{\mu,D}$ contains a nonzero, even invertible, element). This proves that condition (1.3) does not follow from (1.2) either. □

We also obtain the following corollaries.

Corollary 6.2. *Assume $\varphi : (R,\sigma,t) \to (R',\sigma',t')$ is a morphism between regular TGW data. Let $\mu \in \mathrm{PM}_n(\Bbbk)$ and suppose (R,σ,t) is μ-consistent. Then (R',σ',t') is also μ-consistent.*

Proof. This follows from $\varphi(N_{\mu,(R,\sigma,t)}) = N_{\mu,(R',\sigma',t')}$. □

Corollary 6.3. *Let $\mu \in \mathrm{PM}_n(\Bbbk)$. If $(R,\sigma,t) \in \mathsf{TGW}_n(\Bbbk)$ is regular and μ-consistent, then $\mathcal{A}_\mu(\lambda_S) : \mathcal{A}_\mu(R,\sigma,t) \to \mathcal{A}_\mu(S^{-1}R,\widetilde{\sigma},\widetilde{t})$ is injective for any regular $\mathbb{Z}^n$-invariant multiplicative set $S \subseteq R$.*

Proof. Use Corollary 6.2 and Corollary 3.4. □

7. Weak μ-consistency and crossed product algebra embeddings

In this section we show that sometimes a TGW datum (R,σ,t) which is not μ-consistent may nevertheless be replaced by another TGW datum (R',σ',t') which is μ-consistent and such that the corresponding TGW algebras are isomorphic. Such TGW data will be called weakly μ-consistent (see definition below).

Theorem 7.1. *Let $\mu \in \mathrm{PM}_n(\Bbbk)$ and let $D = (R,\sigma,t) \in \mathsf{TGW}_n(\Bbbk)$ be regular. Then the following statements are equivalent.*

(i) *$\rho_{\mu,D}(t_i)$ is regular in $\rho_{\mu,D}(R)$ for all i.*
(ii) *$\big(\ker\rho_{\mu,D}\big)^{\mathrm{ec}} = \ker\rho_{\mu,D}$.*
(iii) *$\mathcal{A}_\mu(\pi_{L^{\mathrm{ec}}_{\mu,D}}) : \mathcal{A}_\mu(R,\sigma,t) \to \mathcal{A}_\mu(\bar{R},\bar{\sigma},\bar{t})$ is an isomorphism.*
(iv) *$\mathcal{A}_\mu(\lambda_T) : \mathcal{A}_\mu(R,\sigma,t) \to \mathcal{A}_\mu(T^{-1}R,\widetilde{\sigma},\widetilde{t})$ is injective.*

(v) *There exists a morphism* $\psi : (R, \sigma, t) \to (R', \sigma', t')$ *from* (R, σ, t) *to a* μ-*consistent regular TGW datum* $(R', \sigma', t') \in \mathsf{TGW}_n(\Bbbk)$ *such that* $\mathcal{A}_\mu(\psi) : \mathcal{A}_\mu(R, \sigma, t) \to \mathcal{A}_\mu(R', \sigma', t')$ *is an isomorphism.*

Proof. That (i) $\Leftrightarrow$ (ii) is immediate by definition. The equivalence (ii) $\Leftrightarrow$ (iii) follows from Theorem 6.1(c). The equivalence (iii) $\Leftrightarrow$ (iv) follows from the cube (6.8). Trivially (iii) implies (v) because $(\bar{R}, \bar{\sigma}, \bar{t})$ is μ-consistent by Theorem 6.1(b). We prove (v)$\Rightarrow$(i). Let $r \in R$, $i \in \{1, \dots, n\}$ and assume that $\rho_{\mu,D}(rt_i) = 0$. Thus $\mathcal{A}_\mu(\psi)(\rho_{\mu,D}(rt_i)) = 0$. By naturality (3.1) we get $\rho_{\mu,D'}(\psi(rt_i)) = 0$, where $D' = (R', \sigma', t')$. But $\psi(t_i) = t'_i$ and $\rho_{\mu,D'}$ is injective, so we get $\psi(r)t'_i = 0$. Since D' is regular we have $\psi(r) = 0$. Hence, by naturality, $\mathcal{A}_\mu(\psi)(\rho_{\mu,D}(r)) = 0$; thus $\rho_{\mu,D}(r) = 0$; since $\mathcal{A}_\mu(\psi)$ is bijective. This proves that $\rho_{\mu,D}(t_i)$ is regular in $\rho_{\mu,D}(R)$. $\square$

Theorem 7.1(v) motivates the following definition.

Definition 7.2. A regular TGW datum satisfying the equivalent conditions in Theorem 7.1 is called *weakly* μ-*consistent*.

As an application, we prove that (weakly) μ-consistent TGWA can be embedded into crossed product algebras.

Recall that a group graded algebra $S = \bigoplus_{g \in G} S_g$ such that each S_g $(g \in G)$ contains an invertible element is called a G-*crossed product over* S_e. One can show (see for example [12]) that any G-crossed product S is isomorphic as a left S_e-module to the group algebra $S_e[G] = \bigoplus_{g \in G} S_e u_g$ with product given by

$$s_1 u_g s_2 u_h = s_1 \zeta_g(s_2) \alpha(g, h) u_{gh}$$

for some unique maps $\zeta : G \to \mathrm{Aut}(S)$ and $\alpha : G \times G \to (S_e)^\times$ satisfying

$$\zeta_g(\zeta_h(a)) = \alpha(g, h) \zeta_{gh}(a) \alpha(g, h)^{-1}, \tag{7.1a}$$

$$\alpha(g, h)\alpha(gh, k) = \zeta_g(\alpha(h, k))\alpha(g, hk), \tag{7.1b}$$

$$\alpha(g, e) = \alpha(e, g) = 1 \tag{7.1c}$$

for all $g, h, k \in G$, $a \in S_e$, where $e \in G$ is the identity element. Then S is denoted by $S_e \rtimes_\alpha^\zeta G$.

Theorem 7.3.

(a) *Let* $\mu \in \mathrm{PM}_n(\Bbbk)$ *and* $D = (R, \sigma, t) \in \mathsf{TGW}_n(\Bbbk)$. *Assume that* $t_1, \dots, t_n \in R^\times$. *Then, if* D *is* μ-*consistent (equivalently, if* (1.2) *and* (1.3) *hold), there is a unique* σ-*twisted* 2-*cocycle* $\alpha : \mathbb{Z}^n \times \mathbb{Z}^n \to R^\times$ *for which there exists a graded* $\Bbbk$-*algebra isomorphism*

$$\xi_{\mu,D} : R \rtimes_\alpha^\sigma \mathbb{Z}^n \to \mathcal{A}_\mu(R, \sigma, t)$$

satisfying

$$\xi(ru_g) = rX_1^{g_1} \cdots X_n^{g_n}.$$

(b) *If* $\mu \in \mathrm{PM}_n(\Bbbk)$ *and* $(R, \sigma, t) \in \mathsf{TGW}_n(\Bbbk)$ *is regular and weakly* μ-*consistent, then* $\mathcal{A}_\mu(R, \sigma, t)$ *can be embedded into a* $\mathbb{Z}^n$-*crossed product algebra:*

$$\mathcal{A}_\mu(R, \sigma, t) \xrightarrow[\simeq]{\mathcal{A}_\mu(\pi_{L_{\mu,D}})} \mathcal{A}_\mu(\bar{R}, \bar{\sigma}, \bar{t}) \xhookrightarrow{\mathcal{A}_\mu(\lambda_{\bar{T}})} \mathcal{A}_\mu(\bar{T}^{-1}\bar{R}, \widehat{\sigma}, \widehat{t}) \xrightarrow[\simeq]{(\xi_{\mu,\widehat{D}})^{-1}} \bar{T}^{-1}\bar{R} \rtimes_\alpha^{\widehat{\sigma}} \mathbb{Z}^n,$$

where $\bar{R} = R/(L_{\mu,D})^{\mathrm{ec}}$, $\bar{\sigma}_g(r + (L_{\mu,D})^{\mathrm{ec}}) = \sigma_g(r) + (L_{\mu,D})^{\mathrm{ec}}$, $\bar{t}_i = t_i + (L_{\mu,D})^{\mathrm{ec}}$, $\bar{T}$ *is the image of* T *in* $\bar{R}$, $\lambda_{\bar{T}} : \bar{R} \to \bar{T}^{-1}\bar{R}$ *is the localization map,* $\widehat{D} = (\bar{T}^{-1}\bar{R}, \widehat{\sigma}, \widehat{t})$, $\widehat{\sigma}_g(s^{-1}r) = \bar{\sigma}_g(s)^{-1}\bar{\sigma}_g(r)$ *for* $s \in \bar{T}$, $r \in \bar{R}$. $\widehat{t}_i = \bar{t}_i/1$ *and the ideal* $(L_{\mu,D})^{\mathrm{ec}}$ *and the set* T *were defined in the beginning of Section* 6.

Proof. (a) This follows by Lemma 2.7 and the above comments on crossed products.

(b) By definition of weakly μ-consistency, $\mathcal{A}_\mu(\pi_{L_{\mu,D}})$ is an isomorphism. By Theorem 6.1(b), $\mathcal{A}_\mu(\bar{R}, \bar{\sigma}, \bar{t})$ is μ-consistent. Corollary 6.3 implies that $\mathcal{A}_\mu(\lambda_{\bar{T}})$ is injective. By Corollary 6.2, $\widehat{D}$ is μ-consistent, so the last isomorphism follows from part (a). □

Acknowledgements

This work was carried out during the second author's postdoc at IME-USP, funded by FAPESP, processo 2008/10688-1. The first author is supported in part by the CNPq grant No. 301743/2007-0 and by the Fapesp grant No. 2010/50347-9.

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