

TRANSITIVITY AND ROTATION SETS
WITH NONEMPTY INTERIOR
FOR HOMEOMORPHISMS OF THE 2-TORUS

FÁBIO ARMANDO TAL

(Communicated by Bryna Kra)

ABSTRACT. We show that if f is a homeomorphism of the 2-torus isotopic to the identity and its lift \tilde{f} is transitive, or even if it is transitive outside the lift of the elliptic islands, then $(0, 0)$ is in the interior of the rotation set of \tilde{f} . This proves a particular case of Boyland's conjecture.

1. INTRODUCTION

In this work we study homeomorphisms of the 2-torus isotopic to the identity. Let f be such a homeomorphism, and let \tilde{f} be a fixed lift of f to the plane, a homeomorphism of \mathbb{R}^2 that satisfies

$$\tilde{f}(x + a, y + b) = \tilde{f}(x, y) + (a, b), \forall (a, b) \in \mathbb{Z}^2,$$

and $\pi(\tilde{f}(x)) = f(\pi(x))$, where $\pi : \mathbb{R}^2 \rightarrow T^2$ is the covering map.

Given f and \tilde{f} we define, following [10] and [7], the rotation set of \tilde{f} as the set of accumulation points of the subset of \mathbb{R}^2 ,

$$\left\{ \frac{\tilde{f}^n(x) - x}{n} \mid x \in \mathbb{R}^2, n \in \mathbb{Z}^+ \right\},$$

which is a compact convex set (see [10]).

One important question is to characterize for which homeomorphisms of the torus the rotation set of its lifts has nonempty interior. There are several consequences associated with this property. In [9], it is shown that if a lift of f has a rotation set with nonempty interior, then f has positive topological entropy. In [10] it is shown that if a point $(\frac{p_1}{q}, \frac{p_2}{q}) \in Q^2$ is in the interior of $\rho(\tilde{f})$, then there exists a point x in the plane such that $\tilde{f}^q(x) = x + (p_1, p_2)$, while the results from [8] and [11] show that, if α is a point in the interior of $\rho(\tilde{f})$, with both coordinates irrational, then there exists a minimal set K_α such that the restriction of f to K_α is regularly semi-conjugate to the translation by α .

While the dynamics tied to the interior of the rotation set is somewhat well understood, a description of the dynamics implied by the extremal points of the rotation set when it has interior is somewhat lacking, even for the cases where f

Received by the editors December 4, 2010 and, in revised form, April 16, 2011.

2010 *Mathematics Subject Classification*. Primary 37E45.

Key words and phrases. Torus homeomorphisms, rotation set, transitivity, omega limits.

Supported by CNPq grant 304360/05-8.

©2012 American Mathematical Society
Reverts to public domain 28 years from publication

is area-preserving. One standing conjecture in this direction was introduced by Phillip Boyland, and is explained below:

Given an area-preserving homeomorphism of the torus f and a fixed lift \tilde{f} , one can consider the rotation vector of the Lebesgue measure λ for \tilde{f} ,

$$\rho_\lambda(\tilde{f}) = \int_{[0,1] \times [0,1]} \tilde{f}(x) - x d\lambda.$$

A homeomorphism f isotopic to the identity such that $\rho_\lambda(\tilde{f}) = (0, 0)$ is usually called in the literature an irrotational homeomorphism. Whenever we speak of an irrotational homeomorphisms, a specific lift for f is fixed.

Boyland's conjecture for the torus claims that, if f is an irrotational homeomorphism and the interior of $\rho(f)$ is nonempty, then $(0, 0)$ is in the interior of the rotation set.

Another important dynamical property of homeomorphisms which has received attention recently is transitivity. In [4] it was shown that \mathcal{C}^1 generically, an area-preserving diffeomorphism of the torus is transitive. In [3], the authors showed that, \mathcal{C}^0 generically, if f is irrotational, then \tilde{f} is transitive in the whole plane. The work [4] suggests this is also true \mathcal{C}^1 generically.

In [1] it is shown that, if g is a homeomorphism of the closed annulus whose lift \tilde{g} to the strip $\mathbb{R} \times [0, 1]$ is transitive, and if there are no fixed points in the boundary of the annulus, then 0 is in the interior of the rotation interval of \tilde{g} . Some of the ideas presented in this paper follow from that paper.

In [5] the authors study the relationship between transitivity, entropy and rotation set, showing that if f is $\mathcal{C}^{1+\varepsilon}$, has topological entropy, and the lift of f to every finite covering of the torus is transitive, then the rotation set of \tilde{f} has nonempty interior.

Here we study homeomorphisms of the 2-torus isotopic to the identity which have the following property:

Definition 1. A lift \tilde{f} of a homeomorphism of the torus isotopic to the identity is said to have the Property T if there exists a point x_T in \mathbb{R}^2 such that the omega limit of the orbit of x_T by \tilde{f} , $\omega(x_T)$, contains all \mathbb{Z}^2 translates of x_T , that is, $\pi^{-1}(\pi(x_T)) \subset \omega(x_T)$.

The above hypothesis is clearly weaker than requesting transitivity of \tilde{f} . In fact, it covers the following case, which is of relevance for measure-preserving homeomorphisms, a hypothesis we do not require. A common and stable trait of sufficiently differentiable area-preserving diffeomorphisms of the torus is the existence of elliptic islands. Of course such diffeomorphisms cannot be transitive, nor can their lift be, but it is still possible for such mappings to be transitive in the complement of the interior of these islands, which in many relevant cases is a torus with infinitely many holes. If a diffeomorphism f has a lift \tilde{f} which is transitive in the complement of the lift of the elliptic islands, then \tilde{f} obeys Property T.

Our main result is:

Theorem 1. *Let f be a homeomorphism of the 2-torus isotopic to the identity such that its lift \tilde{f} obeys Property T. Then $(0, 0)$ is in the interior of $\rho(f)$.*

In section 2 we introduce the sets B_θ and $\omega(B_\theta)$, and we study their structure. In section 3 we study the projection of the sets B_θ and $\omega(B_\theta)$ to the torus and

derive some properties of their closure. The final section is devoted to proving the main theorem.

In what follows f will always be a homeomorphism of the 2-torus isotopic to the identity, and \tilde{f} a lift that satisfies Property T; x_T will denote a point in \mathbb{R}^2 such that $\pi^{-1}(\pi(x_T)) = x_T + \mathbb{Z}^2 \subset \omega(x_T)$.

2. THE SETS $B_\theta, C_\theta, \omega(B_\theta)$, AND $\omega(C_\theta)$

We begin by considering the one-point compactification of \mathbb{R}^2 , the set $\mathbb{R}^2 \cup \{\infty\}$ which is homeomorphic to S^2 . We associate to the homeomorphism f of the torus and its lift \tilde{f} to the plane a third homeomorphism \hat{f} , a homeomorphism of S^2 induced by \tilde{f} that keeps the infinity fixed.

Given any θ in $[0, 2\pi)$, we define the sets

$$V_\theta = \{x \in \mathbb{R}^2 : \langle x; e_\theta \rangle = 0\}$$

and

$$V_\theta^+ = \{x \in \mathbb{R}^2 : \langle x; e_\theta \rangle \geq 0\},$$

where $e_\theta = (\cos(\theta), \sin(\theta))$.

We consider also the corresponding sets in S^2 , which we denote by \widehat{V}_θ and \widehat{V}_θ^+ .

Let \widehat{B}_θ be the connected component of

$$\bigcap_{n \leq 0} \hat{f}^n(\widehat{V}_\theta^+)$$

that contains the infinity, and denote by B_θ the corresponding set in \mathbb{R}^2 .

Let \widehat{C}_θ be the connected component of

$$\bigcap_{n \geq 0} \hat{f}^n(\widehat{V}_\theta^+)$$

that contains the infinity, and denote by C_θ the corresponding set in \mathbb{R}^2 .

It follows from these definitions that B_θ is the union of all closed unbounded connected sets whose forward orbit remains in V_θ^+ for all times and that C_θ is the union of the connected components whose backward orbit remains in V_θ^+ .

Proposition 1. *For every θ in $[0, 2\pi)$, we have:*

- (1) B_θ and C_θ are closed and $\tilde{f}(B_\theta) \subset B_\theta, \tilde{f}^{-1}(C_\theta) \subset C_\theta$.
- (2) Every connected component of B_θ or C_θ is unbounded.
- (3) If (p, q) is a point in $V_\theta^+ \cap \mathbb{Z}^2$, then $B_\theta + (p, q) \subset B_\theta$, and the same holds for C_θ .
- (4) B_θ^C has a single connected component, as does C_θ^C .

Proof. We prove these properties only for B_θ ; the reasoning is the same for C_θ . Properties (1) and (2) are direct consequences of the definition of B_θ . If x is a point of B_θ , let Γ be the connected component of B_θ that contains x . Then, for every y in Γ and every positive integer j , $\tilde{f}^j(y)$ is in V_θ^+ . But this implies that if z is a point of the set $\Gamma + (p, q)$, then also for every positive integer j we have $\tilde{f}^j(z) \in V_\theta^+$. Therefore $\Gamma + (p, q)$ is a connected closed set whose future orbit does not leave V_θ^+ , and so $\Gamma + (p, q) \subset B_\theta$ and $x + (p, q) \in B_\theta$. This proves property (3).

Now let Ω be the connected component of B_θ^C that contains $\mathbb{R}^2 \setminus V_\theta^+$. Our goal is to show that $\Omega^C = B_\theta$, which implies property (4). Since $\tilde{f}(B_\theta) \subset B_\theta$, we have that

$B_\theta^C \subset \tilde{f}(B_\theta^C)$, and so every connected component of $\tilde{f}(B_\theta^C)$ contains a connected component of B_θ^C . As $\tilde{f}(\Omega) \cap \Omega \neq \emptyset$, this implies $\Omega \subset \tilde{f}(\Omega)$. In particular, Ω^C is forward-invariant.

Let $z \in \Omega^C$, and let Γ be the closed connected component of Ω^C that contains z . Since Ω^C is forward-invariant, for every positive integer j we have $\tilde{f}^j(\Gamma) \cap \Omega = \emptyset$ but since $(V_\theta^+)^C \subset \Omega$, Γ is a closed connected set whose forward orbit remains in V_θ^+ . Furthermore, since $\partial(\Omega^C) \subset B_\theta$, then $\Gamma \cap B_\theta \neq \emptyset$. Let $z \in \Gamma \cap B_\theta$ and Θ be the connected component of B_θ that contains z . Θ is unbounded and since $\Theta \subset \Omega^C$ and $\Theta \cap \Gamma \neq \emptyset$, then $\Theta \subset \Gamma$. So Γ is unbounded, closed and its forward orbit remains in V_θ^+ ; therefore $\Gamma \subset B_\theta$. This shows that $\Omega^C = B_\theta$. □

Lemma 1. *For every θ in $[0, 2\pi)$, we have $B_\theta \cap [-1, 1] \times [-1, 1] \neq \emptyset$, and $C_\theta \cap [-1, 1] \times [-1, 1] \neq \emptyset$.*

Proof. We assume, without loss of generality, that $0 \leq \theta < \frac{\pi}{2}$.

Since \tilde{f} satisfies Property T, there is a translate y of x_T which lies in $V_{\theta+\pi}^+$ such that, for every positive integer m , there exists a positive integer $n_0(m)$ such that $\tilde{f}^{n_0(m)}(y) \in V_\theta^+ + (m, 0)$. Let $n = n(m)$ be the first time that $\tilde{f}^n(V_{\theta+\pi}^+)$ intersects $V_\theta^+ + (m, 0)$, that is, $\tilde{f}^n(V_{\theta+\pi}^+) \cap V_\theta^+ + (m, 0) \neq \emptyset$ and such that, for all $0 \leq j < n$, $\tilde{f}^j(V_{\theta+\pi}^+) \cap V_\theta^+ + (m, 0)$ is empty.

As $\tilde{f}^{n(m)}(V_{\theta+\pi}^+) \cap V_{\theta+\pi}^+$ is not empty, this implies that

$$\left(\tilde{f}^{n(m)}(\partial V_{\theta+\pi}^+) \cap V_\theta^+ + (m, 0)\right) = \left(\tilde{f}^{n(m)}(V_\theta) \cap V_\theta^+ + (m, 0)\right) \neq \emptyset.$$

Let m_k be a sequence of integers such that $\lim_{k \rightarrow \infty} n(m_k) = \infty$. For each k there is an unbounded closed curve γ_k that starts at $\tilde{f}^{n(m_k)}(V_\theta)$ and such that $\gamma_k \subset \tilde{f}^{n(m_k)}(V_\theta^+) \cap V_\theta^+ + (m_k, 0)$. It follows that the sets $\beta_k = \tilde{f}^{-n(m_k)}(\gamma_k)$ all intersect V_θ and that for $0 \leq i \leq n(m_k)$, $\tilde{f}^i(\beta_k) \subset V_\theta^+$.

Also, since for every point z in V_θ there are integers p and q such that $\langle (p, q); e_\theta \rangle \geq 0$ and such that $z + (p, q)$ is in $[0, 1] \times [0, 1]$, we have that there is a curve $\alpha_k = \beta_k + (p, q)$ that starts in $[0, 1] \times [0, 1]$ and such that, for $0 \leq i \leq n(m_k)$, $\tilde{f}^i(\alpha_k) \subset V_\theta^+$.

Now consider the closed sets $\widehat{\alpha}_k$ in S^2 corresponding to the sets α_k . $\widehat{\alpha}_k$ is a sequence of compact connected sets, and as such there is a subsequence $\widehat{\alpha}_{k_j}$ that is convergent in the Hausdorff topology to a closed connected set $\widehat{\Gamma}$. It is clear that $\widehat{\alpha}_k$ is in \widehat{B}_θ and that $\widehat{\Gamma}$ intersects $[0, 1] \times [0, 1]$. The subset Γ of \mathbb{R}^2 corresponding to $\widehat{\Gamma}$ shows that the lemma is true. □

Another interesting property of the set B_θ is the following:

Lemma 2. *There exists $(p, q) \in \mathbb{Z}^2$ such that $\tilde{f}^{-1}(B_\theta) \subset B_\theta - (p, q)$. In particular, $f(\pi(B_\theta)) = \pi(B_\theta)$.*

Proof. Since \tilde{f} is biperiodic, there exists a positive real K such that, for all $x \in \mathbb{R}^2$, $\|\tilde{f}(x) - x\| \leq K$.

Let $y \in B_\theta$, and let $(p, q) \in \mathbb{Z}^2$ be such that $\langle (p, q); e_\theta \rangle \geq 2K$. Let Γ be the connected component of B_θ containing y . Then $\tilde{f}^{-1}(\Gamma) + (p, q) \subset V_\theta^+$. As $\tilde{f}(\tilde{f}^{-1}(\Gamma) + (p, q))$ is a subset of $B_\theta + (p, q) \subset B_\theta \subset V_\theta^+$ and as $\tilde{f}^{-1}(\Gamma) + (p, q)$ is connected and unbounded, we have that $\tilde{f}^{-1}(\Gamma) + (p, q) \subset B_\theta$, which shows the result. □

Also,

Lemma 3. *There exists $(p, q) \in \mathbb{Z}^2$ such that $\tilde{f}(C_\theta) \subset C_\theta - (p, q)$. In particular, $f(\pi(C_\theta)) = \pi(C_\theta)$.*

The proof is the same as the one of the previous lemma.

We will need to also consider the following sets:

$$\omega(B_\theta) = \bigcap_{i=1}^\infty \overline{\bigcup_{j=i}^\infty \tilde{f}^j(B_\theta)} = \bigcap_{i=1}^\infty \widehat{\tilde{f}^i(B_\theta)},$$

where the last equality comes from the positive invariance of B_θ and the respective set in S^2 ,

$$\widehat{\omega(B_\theta)} = \bigcap_{i=1}^\infty \widehat{\tilde{f}^i(B_\theta)}.$$

We also define the sets

$$\omega(C_\theta) = \bigcap_{i=1}^\infty \tilde{f}^{-i}(C_\theta)$$

and $\widehat{\omega(C_\theta)}$.

The following properties of $\omega(B_\theta)$ and $\widehat{\omega(B_\theta)}$ follow directly from the definition and Proposition 1.

Proposition 2. *For every θ in $[0, 2\pi)$, we have:*

- (1) $\{\infty\} \subset \widehat{\omega(B_\theta)}$, and $\widehat{\omega(B_\theta)}$ is connected.
- (2) $\omega(B_\theta)$ is closed and $\tilde{f}(\omega(B_\theta)) = \omega(B_\theta)$.
- (3) Every connected component of $\omega(B_\theta)$ is unbounded.
- (4) If (p, q) is a point in $V_\theta^+ \cap \mathbb{Z}^2$, then $\omega(B_\theta) + (p, q) \subset \omega(B_\theta)$.
- (5) $\omega(B_\theta)^C$ has a single connected component.

Proposition 3. $\omega(B_\theta) = \omega(C_\theta)$.

Proof. We show only that $\omega(B_\theta) \subset \omega(C_\theta)$. Let Γ be a connected component of $\omega(B_\theta)$. Since $\omega(B_\theta)$ is invariant, we have $\tilde{f}^{-k}(\Gamma) \subset \omega(B_\theta) \subset V_\theta^+$, for all positive integers k , which shows that $\Gamma \subset C_\theta$. Since the choice of Γ was arbitrary, we have $\omega(B_\theta) \subset C_\theta$. Again, we use the invariance of $\omega(B_\theta)$ and get

$$\tilde{f}^i(\omega(B_\theta)) = \omega(B_\theta) \subset C_\theta,$$

so that $\omega(B_\theta) \subset \tilde{f}^{-i}(C_\theta)$ for all integers i , proving the proposition. □

The following lemma will be our main tool in establishing the existence of vectors in the rotation set of \tilde{f} .

Lemma 4. *If $\omega(B_\theta) = \emptyset$, then there exists a positive real number a such that:*

- For all x in B_θ ,

$$\liminf_{n \rightarrow \infty} \frac{\langle \tilde{f}^n(x); e_\theta \rangle}{n} \geq a.$$

- For all x in C_θ ,

$$\liminf_{n \rightarrow \infty} \frac{\langle \tilde{f}^{-n}(x); e_\theta \rangle}{n} \geq a.$$

Proof. We again assume, without loss of generality, that θ is in $[0, \pi/2)$, and we prove just the first item. We claim that there is a positive n such that $\tilde{f}^i(B_\theta)$ is contained in $V_\theta^+ + (1, 0)$, for all $i > n$. If this were not the case, then there would be a sequence $i_j \rightarrow \infty$ such that, for all j , there is a point $x_j \in B_\theta$ such that $\tilde{f}^{i_j}(x_j) \in V_\theta^+ \setminus V_\theta^+ + (1, 0)$.

But this implies that there is a sequence of pairs of integers p_j, q_j such that $\langle (p_j, q_j); e_\theta \rangle \geq 0$ and such that $\tilde{f}^{i_j}(x_j) + (p_j, q_j)$ is in $[1, 2] \times [0, 1]$. But since $B_\theta + (p_j, q_j) \subset B_\theta$, we would have that if $z_j = x_j + (p_j, q_j)$, then the sequence $(\tilde{f}(z_j))_{j \in \mathbb{N}}$ would have a convergent subsequence, which contradicts $\omega(B_\theta) = \emptyset$, since $z_j \in B_\theta$ for all j .

Let Γ be a connected component of B_θ . By the choice of n , for all $j \geq 0$ we have $\tilde{f}^{j+n}(\Gamma) \subset V_\theta^+ + (1, 0)$, and so $\tilde{f}^{n+j}(\Gamma) - (1, 0) \subset V_\theta^+$. Since $\tilde{f}^n(\Gamma)$ is closed, connected and unbounded, it follows from the definition of B_θ that $\tilde{f}^n(\Gamma) - (1, 0) \subset B_\theta$, and so $\tilde{f}^n(\Gamma) \subset B_\theta + (1, 0)$. As the choice of Γ was arbitrary, we have

$$\tilde{f}^n(B_\theta) \subset B_\theta + (1, 0).$$

A simple induction argument now shows that for every positive integer k , $\tilde{f}^{nk}(B_\theta) \subset B_\theta + (k, 0) \subset V_\theta^+ + (k, 0)$, and so, for every x in B_θ ,

$$\liminf_{i \rightarrow \infty} \frac{\langle \tilde{f}^i(x); e_\theta \rangle}{i} \geq \frac{\cos(\theta)}{n},$$

which ends the proof. □

The following is an immediate consequence of the previous lemma:

Corollary 1. *If $\omega(B_\theta) = \emptyset$, then there exist $r_1, r_2 \in \rho(\tilde{f})$ such that $\langle r_1; e_\theta \rangle > 0$ and $\langle r_2; e_\theta \rangle < 0$.*

We can now state the main lemma of this work, which will be proved in the last section:

Lemma 5. *For every $\theta \in [0, \pi)$, at least one of $\omega(B_\theta)$ and $\omega(B_{\theta+\pi})$ is empty.*

We will postpone the proof of this result to the last section, but let us now show how this lemma yields Theorem 1.

Proof of Theorem 1. We first note that since \tilde{f} satisfies Property T, it has a recurrent point, and so $(0, 0) \in \rho(\tilde{f})$. Since $\rho(\tilde{f})$ is convex, either $(0, 0)$ is in the interior of $\rho(\tilde{f})$ and we are done or there exists a straight line passing through $(0, 0)$ such that $\rho(\tilde{f})$ is contained on one side of this line. Assume that the latter happens. This means that there is θ_1 such that for all $r \in \rho(\tilde{f})$, $\langle r; e_{\theta_1} \rangle \geq 0$.

But from Lemma 5 either $\omega(B_{\theta_1}) = \emptyset$ or $\omega(B_{\theta_1+\pi}) = \emptyset$, and in both cases this implies, by Corollary 1, that there exists $r \in \rho(\tilde{f})$ with $\langle r; e_{\theta_1} \rangle < 0$. □

3. THE SETS $\pi(B_\theta)$ AND $\pi(\omega(B_\theta))$

Let us state this general result:

Lemma 6. *Let g be a homeomorphism of the torus isotopic to the identity and \tilde{g} be its lift to the plane. Assume there exists points \tilde{x}, \tilde{y} in \mathbb{R}^2 such that*

$$\tilde{z} + \mathbb{Z}(1, 0) \subset \omega(\tilde{x}) = \bigcap_{i=1}^{\infty} \overline{\bigcup_{j=i}^{\infty} \tilde{g}^j(\tilde{x})}.$$

If $z = \pi(\tilde{z})$, then for every $\varepsilon > 0$, there exists a curve $\gamma \subset \mathbb{R}^2$ which is unbounded in the e_0 direction but bounded in the $e_{\pi/2}$ direction, and such that, for every point $y \in \gamma$, there exists a positive $n(y)$ such that $\pi(\tilde{g}^{n(y)}(y)) \in B_\varepsilon(z)$.

Proof. Since $\tilde{z} + \mathbb{Z}(1, 0) \subset \omega(\tilde{x})$, there are points x_1 and x_2 , both in $B_\varepsilon(\tilde{z})$, and integers k_1 and k_2 , both strictly negative, such that $\tilde{g}^{k_1}(x_1) = y_1 \in B_\varepsilon(\tilde{z})$ and $\tilde{g}^{k_2}(x_2) = y_2 \in B_\varepsilon(\tilde{z} + (1, 0))$. Let α_0 be the segment connecting x_1 and x_2 , α_1 be the segment connecting x_1 and y_1 , α_2 be the segment connecting x_2 and $y_2 - (1, 0)$, α_3 be the segment connecting y_1 and \tilde{z} , and α_4 be the segment connecting $y_2 - (1, 0)$ and \tilde{z} . We note that $\pi(\alpha_i) \subset B_\varepsilon(z)$, for $i \in \{0, 1, 2, 3, 4\}$.

Also, since x_1 and y_1 belong to α_1 and since $y_1 = \tilde{g}^{k_1}(x_1)$ belongs to $\tilde{g}^{k_1}(\alpha_1)$, we have that $y_1 \in \tilde{g}^{k_1}(\alpha_1) \cap \alpha_1$, and so $\tilde{g}^{k_1}(\alpha_1) \cup \alpha_1$ is a connected set. A simple induction shows that

$$\beta_1 = \left(\bigcup_{i=0}^{-k_2} \tilde{g}^{ik_1}(\alpha_1) \right)$$

is a connected arc joining $\tilde{g}^{-k_1 k_2}(x_1)$ and y_1 .

Analogously, since x_2 and $y_2 - (1, 0)$ belong to α_2 and $\tilde{g}^{k_2}(x_2) = y_2$, it follows that the set $\tilde{g}^{k_2}(\alpha_2) \cup \alpha_2 + (1, 0)$ is an arc connecting y_2 and $y_2 - (1, 0)$. Therefore the set

$$\beta_2 = \left(\bigcup_{i=0}^{-k_1} \tilde{g}^{ik_2}(\alpha_2 + (i - k_1, 0)) \right)$$

is a connected arc joining $\tilde{g}^{-k_1 k_2}(x_2)$ and $y_2 - (k_1 - 1, 0)$. This implies that

$$\tilde{g}^{-k_1 k_2}(\alpha_0) \cup \beta_1 \cup \beta_2$$

is a connected arc joining y_1 and $y_2 - (k_1 - 1, 0)$.

Consider the set

$$\gamma_0 = \tilde{g}^{-k_1 k_2}(\alpha_0) \cup \beta_1 \cup \beta_2 \cup \alpha_3 \cup \alpha_4 + (-k_1, 0),$$

which is also connected and bounded in the $e_{\pi/2}$ direction and which contains both \tilde{z} and $\tilde{z} - (k_1, 0)$. If $\gamma = \bigcup_{i=-\infty}^{\infty} (\gamma_0 + (ik_1, 0))$, then γ is connected, unbounded in the e_0 direction and bounded in the $e_{\pi/2}$ direction, and satisfies that if $y \in \gamma$, then there is a $n(y) > 0$ such that $\pi(\tilde{g}^{n(y)}(y)) \in \pi(\alpha_i)$ for some $0 \leq i \leq 4$. Since $\pi(\alpha_i) \subset B_\varepsilon(z)$, $0 \leq i \leq 4$, we have the result. \square

The previous lemma yields these three simple corollaries, the first one of which will be used in the remainder of the paper; the second is interesting in itself.

Corollary 2. *Let g be a homeomorphism of the torus and \tilde{g} be its lift to the plane. Assume there exists \tilde{x}, \tilde{y} in \mathbb{R}^2 such that $\tilde{y} + \mathbb{Z}^2 \subset \omega(\tilde{x})$.*

Let $B \subset \mathbb{R}^2$ be a set such that $\tilde{g}(B) \subset B$ and such that B has an unbounded connected component. Then $\pi(\tilde{y}) \in \overline{\pi(B)}$.

Proof. Let $\tilde{z} \in B$ be a point in an unbounded connected component of B . Suppose, by contradiction, that there exists a point $\tilde{y} \in \tilde{A}$ and a positive real ε such that $\pi(B) \cap B_\varepsilon(\pi(\tilde{y})) = \emptyset$.

The previous lemma implies that there are two connected sets, α and β , with the following properties:

- For all \tilde{x} in $\alpha \cup \beta$, there exists a positive integer $n(\tilde{x})$ such that $\pi(\tilde{g}^n(\tilde{x})) \in B_\varepsilon(\pi(\tilde{y}))$.
- α is unbounded in the direction of $e_{\pi/2}$, bounded in the direction of e_0 and disconnects the plane.
- β is bounded in the direction of $e_{\pi/2}$, unbounded in the direction of e_0 and disconnects the plane.

Now, by the contradiction hypothesis and the first item,

$$(1) \quad \pi(B) \cap \pi(\alpha \cup \beta) = \emptyset.$$

By the second item, there is an integer k_1 such that \tilde{z} is to the left of $\alpha + (k_1, 0)$, to the right of $\alpha - (k_1, 0)$, below $\beta + (0, k_1)$ and above $\beta - (0, k_1)$. This shows that \tilde{z} is in a bounded connected component of the complement

$$D = \alpha + (k_1, 0) \cup \alpha - (k_1, 0) \cup \beta + (0, k_1) \cup \beta - (0, k_1).$$

But, from (1), $B \cap D = \emptyset$, which is absurd since the connected component of B that contains \tilde{z} must be unbounded by hypothesis and remain in the same connected component of D^C that contains \tilde{z} . □

We finish this section with two direct applications of this result:

Corollary 3. *Let g be a homeomorphism of the torus isotopic to the identity and \tilde{g} be its lift. Assume \tilde{g} is transitive in \mathbb{R}^2 .*

Let $B \subset \mathbb{R}^2$ be a set such that $\tilde{g}(B) \subset B$ and such that B has an unbounded connected component. Then $\overline{\pi(B)} = T^2$.

Corollary 4. *Let \tilde{f} be a homeomorphism of the torus satisfying property T. Then, for all $0 \leq \theta < 2\pi$, if $\omega(B_\theta)$ is nonempty, then $\pi(x_T) \subset \pi(\omega(B_\theta))$.*

4. PROOF OF LEMMA 5

It suffices to show that for every $\theta \in [0, \pi)$, either $\omega(B_\theta) = \emptyset$ or $\omega(B_{\theta+\pi}) = \emptyset$.

Assume, by contradiction, that there exists θ for which this does not hold. Without loss in generality, we will assume that $0 \leq \theta < \pi/2$.

There are two possibilities:

- (1) $\pi(\omega(B_\theta)) \cap \pi(\omega(B_{\theta+\pi})) \neq \emptyset$;
- (2) $\pi(\omega(B_\theta)) \cap \pi(\omega(B_{\theta+\pi})) = \emptyset$.

4.1. Case 1. If $\pi(\omega(B_\theta)) \cap \pi(\omega(B_{\theta+\pi})) \neq \emptyset$, then there exist (p_1, q_1) and (p_2, q_2) such that

$$\omega(B_\theta) + (p_1, q_1) \cap \omega(B_{\theta+\pi}) + (p_2, q_2) \neq \emptyset,$$

and so, if $(p, q) = (p_2 - p_1, q_2 - q_1)$,

$$\omega(B_\theta) \cap \omega(B_{\theta+\pi}) + (p, q) \neq \emptyset.$$

Let

$$O = (\omega(B_\theta) \cup \omega(B_{\theta+\pi}) + (p, q))^C.$$

Consider also the set

$$O_1 = ((\omega(B_\theta) - (1, 1)) \cup (\omega(B_{\theta+\pi}) + (p + 1, q + 1)))^C.$$

Proposition 4. *If \tilde{x} is a point of O_1 and i, j are integers such that $-1 \leq \langle (i, j); e_\theta \rangle \leq 1$, then $\tilde{x} + (i, j) \in O$.*

Proof. We recall that since $(i, j) \in \mathbb{Z}^2$ is such that $\langle (i, j); e_\theta \rangle \leq 1 \leq \langle (1, 1); e_\theta \rangle$, then $\langle (1 - i, 1 - j); e_\theta \rangle \geq 0$, and from Proposition 2 we have $\omega(B_\theta) + (1 - i, 1 - j) \subset \omega(B_\theta)$ and $\omega(B_\theta) \subset \omega(B_\theta) + (i - 1, j - 1)$. Therefore, if $\tilde{x} \notin \omega(B_\theta) - (1, 1)$, then $\tilde{x} + (i, j) \notin \omega(B_\theta) + (i - 1, j - 1)$ and so $\tilde{x} + (i, j) \notin \omega(B_\theta)$. A similar argument shows that since $\langle (i, j); e_\theta \rangle \geq -1 \geq \langle (-1, -1); e_\theta \rangle$, then, if $\tilde{x} \notin \omega(B_{\theta+\pi}) + (p + 1, q + 1)$, it follows that $\tilde{x} + (i, j) \notin \omega(B_{\theta+\pi}) + (p, q)$. \square

Proposition 5. *If $tg(\theta)$ is rational and $v \in \mathbb{Z}^2$ is such that $\langle v; e_\theta \rangle = 0$ and $\tilde{x} \in O_1$, then \tilde{x} and $\tilde{x} + v$ are not in the same connected component of O_1 .*

Proof. Assume, by contradiction, that \tilde{x} and $\tilde{x} + v$ are in the same connected component of O_1 . Let $\alpha \subset O_1$ be a curve joining these two points and consider the set $\gamma = \bigcup_{l=-\infty}^{\infty} \alpha + lv$. Also, let $M = \max_{x \in \alpha} \{|\langle x; e_\theta \rangle|\}$.

Clearly γ is connected, does not intersect $O_1 \subset O$, is unbounded in the $e_{\theta+\pi/2}$ direction, and bounded in the e_θ direction. This shows that the sets $\{x \in \mathbb{R}^2 \mid \langle x, e_\theta \rangle > M\}$ and $\{x \in \mathbb{R}^2 \mid \langle x, e_\theta \rangle < -(M)\}$ are in different connected components of γ^C .

But since $\omega(B_\theta) \cap \omega(B_{\theta+\pi}) + (p, q)$ is not empty, there exists x in the intersection. Let Γ be the connected component of $\omega(B_\theta)$ that contains x , and let Θ be the connected component of $\omega(B_{\theta+\pi}) + (p, q)$ that contains x . Then $\Gamma_1 = \Gamma \cup \Theta$ is a connected component of $\omega(B_\theta) \cup \omega(B_{\theta+\pi}) + (p, q)$ such that $\sup_{w \in \Gamma_1} \langle w; e_\theta \rangle = +\infty$ and such that $\inf_{w \in \Gamma_1} \langle w; e_\theta \rangle = -\infty$. But this contradicts the fact that $\gamma \cap \Gamma_1 = \emptyset$ and that γ is connected. \square

Proposition 6. *Let E_1, E_2 be two connected components of O_1 , and $v_1, v_2, v_3 \in \mathbb{Z}^2$ be three vectors of integer coordinates such that*

$$-\frac{1}{2} \leq \langle v_1; e_\theta \rangle \leq 0 \leq \langle v_2; e_\theta \rangle \leq \frac{1}{2},$$

$|\langle v_3; e_\theta \rangle| \leq \frac{1}{2}$, and such that $\langle v_l; e_{\theta+\pi/2} \rangle > 0, l \in \{1, 2\}$. Then, for every \tilde{x} in the plane, it cannot be that \tilde{x} and $\tilde{x} + v_1$ are both in E_1 and that $\tilde{x} + v_3$ and $\tilde{x} + v_3 + v_2$ are both in E_2 .

Proof. Assume, by contradiction, that for some \tilde{x} the proposition is false. Let α be an arc contained in E_1 connecting \tilde{x} and $\tilde{x} + v_1$ and let β be an arc connecting $\tilde{x} + v_3$ and $\tilde{x} + v_3 + v_2$, contained in E_2 .

From Proposition 4 we see that if $v \in \mathbb{Z}^2$ is such that $-1 \leq \langle v; e_\theta \rangle \leq 1$, then $E_1 + v \subset O$. Also, if $-\frac{1}{2} \leq \langle v; e_\theta \rangle \leq \frac{1}{2}$, then $E_2 - v_3 + v \subset O$.

Now let $\tilde{x}_0 = \tilde{x}, u_0 = (0, 0)$ and define, inductively, $\gamma_i = \alpha + u_i, u_{i+1} = u_i + v_1$ if $\langle u_i; e_\theta \rangle \geq 0$, and $\gamma_i = \beta + u_i - v_3, u_{i+1} = u_i + v_2$ if $\langle u_i; e_\theta \rangle < 0$.

It should be clear that γ_i is an arc connecting $\tilde{x} + u_i$ to $\tilde{x} + u_{i+1}$ and that $-1 \leq \langle u_i; e_\theta \rangle \leq 1$. Therefore, for all integers i , $\gamma_i \cup \gamma_{i+1}$ is connected and, since γ_i is either a subset of $E_1 + u_i$ or of $E_2 - v_3 + u_i$, we have that $\gamma_i \subset O$.

Let $\gamma = \bigcup_{i=-\infty}^{\infty} \gamma_i$. γ be connected, not intersect $\omega(B_\theta) \cup \omega(B_{\theta+\pi}) + (p, q)$, be unbounded in the $e_{\theta+\pi/2}$ direction, and bounded in the e_θ direction. From here, the same reasoning as in the proof of Proposition 5 yields a contradiction. \square

These two previous propositions yield:

Lemma 7. *O_1 has infinitely many connected components. Furthermore, the set $x_T + \mathbb{Z}^2$ intersects infinitely many of these connected components.*

Proof. First, note that the orbit of x_T is not bounded in any direction, and as such $x_T + \mathbb{Z}^2 \subset O \subset O_1$.

If $tq(\theta)$ is rational, then the result follows from Proposition 5 by taking $v \in \mathbb{Z}^2$ perpendicular to e_θ , since each of the points $x_T + iv$, $i \in \mathbb{Z}$ needs to be in a different connected component of O_1 .

If $tq(\theta)$ is not rational, then let N be a fixed integer, and let $w_1, w_2 \in \mathbb{Z}^2$ be such that

$$(2) \quad -\frac{1}{2N} \leq \langle w_1; e_\theta \rangle \leq 0 \leq \langle w_2; e_\theta \rangle \leq \frac{1}{2N},$$

$$(3) \quad \langle w_i; e_{\theta+\pi/2} \rangle \geq 0, \quad i \in \{1, 2\}.$$

We will show that O_1 has more than N connected components by proving that at least N of the points of the form $x_T, x_T + iw_1$, and $x_T + jw_2$, for $1 \leq i \leq N, 1 \leq j \leq N$, are in distinct connected components of O_1 , which clearly proves the lemma.

But, if this is false, then there are i_1, i_2, j_1 , and j_2 in $[0, N]$, $i_1 < i_2, j_1 < j_2$, and E_1 and E_2 connected components of O_1 , such that $y_1 = x_T + i_1w_1$ and $y_1 + (i_2 - i_1)(w_1)$ are in E_1 and such that $y_2 = x_T + j_1w_2$ and $y_2 + (j_2 - j_1)(w_2)$ are in E_2 . This contradicts Proposition 6, with $\tilde{x} = y_1, v_1 = (i_2 - i_1)w_1, v_2 = (j_2 - j_1)w_2$ and $v_3 = y_2 - y_1$. □

Since \tilde{f} satisfies Property T, the forward orbit of x_T is not confined to any half-plane and, as such, $x_T \notin \omega(B_\theta) \cup \omega(B_{\theta+\pi}) + (p, q)$, so that $x_T \in O_1$.

Now x_T is a recurrent point, so there is an n such that $\tilde{f}^n(x_T)$ and x_T are in the same connected component of O_1 . Therefore, if U is the connected component of O_1 that contains x_T , then $\tilde{f}^n(U) \cap U$ is not empty, and since O_1 is an invariant set, this implies that $\tilde{f}^n(U) = U$. Thus, the orbit of x_T only intersects n connected components of O_1 .

But Lemma 7 tells us that the set $x_T + \mathbb{Z}^2$ both intersects infinitely many connected components of O_1 and is contained in the closure of the forward orbit of x_T , which is absurd. Therefore Case 1 leads to a contradiction.

4.2. Case 2. We still need to deal with the case where

$$\pi(\omega(B_\theta)) \cap \pi(\omega(B_{\theta+\pi})) = \emptyset,$$

which we assume throughout the section.

We begin with the following proposition, whose proof can be seen in [12].

Proposition 7. *Let K_1 and K_2 be compact connected sets of \mathbb{R}^2 such that $K_1 \cap K_2$ is connected. If x and y are in the same connected component of K_1^C and are also in the same connected component of K_2^C , then x and y are in the same connected component of $(K_1 \cup K_2)^C$.*

Proposition 8. *For every $(p_1, q_1), (p_2, q_2) \in \mathbb{Z}^2$, the complement of the set*

$$\omega(B_\theta) + (p_1, q_1) \cup \omega(B_{\theta+\pi}) + (p_2, q_2)$$

has a single connected component.

Proof. We already know that $\omega(B_\theta) + (p_1, q_1) \cap \omega(B_{\theta+\pi}) + (p_2, q_2) = \emptyset$. We also know, from Lemma 1, that both the complements of $\omega(B_\theta) + (p_1, q_1)$ and of $\omega(B_{\theta+\pi}) + (p_2, q_2)$ have a single connected component. This means that if we consider the corresponding sets in S^2 , $\widehat{\omega(B_\theta) + (p_1, q_1)}$ and $\widehat{\omega(B_{\theta+\pi}) + (p_2, q_2)}$, their intersection consists of the point ∞ , and so we can apply the previous proposition, which gives us the result. \square

The following is a result that follows from a construction very similar to the one done in the proof of Proposition 6.

Proposition 9. *For every point x in T^2 and every (p_1, q_1) and (p_2, q_2) in \mathbb{Z}^2 , there is a curve $H : \mathbb{R} \rightarrow \mathbb{R}^2$ satisfying:*

- (1)
$$\lim_{t \rightarrow -\infty} \langle H(t); e_{\theta+\pi/2} \rangle = -\infty, \lim_{t \rightarrow \infty} \langle H(t); e_{\theta+\pi/2} \rangle = \infty.$$
- (2)
$$\inf\{t \in \mathbb{R} \mid \pi(H(t)) = x\} = -\infty, \sup\{t \in \mathbb{R} \mid \pi(H(t)) = x\} = +\infty.$$
- (3) *If $\gamma = H(\mathbb{R})$, then $\omega(B_\theta) + (p_1, q_1)$ and $\omega(B_{\theta+\pi}) + (p_2, q_2)$ are in different connected components of γ^C .*

Proof. Since $\pi(\omega(B_\theta)) \cap \pi(\omega(B_{\theta+\pi})) = \emptyset$, we will assume, without loss of generality, that $x \notin \pi(\omega(B_\theta))$. Therefore, there is a point y in $\pi^{-1}(x)$ such that $\langle y; e_\theta \rangle \geq \langle (p_2, q_2); e_\theta \rangle + 3$. This means that the points $y, y + (0, 1)$ and $y - (1, 0)$ are all in the semi-plane $V_\theta^+ + (p_2 + 1, q_2 + 1)$, and therefore none of these three points belongs to $O_1 = \omega(B_\theta) + (p_1 - 1, q_1 - 1) \cup \omega(B_{\theta+\pi}) + (p_2 + 1, q_2 + 1)$.

Now, from Proposition 8, we know that all three points are in the same connected component of O_1^C , and so there are curves $h_1, h_2 : [0, 1] \rightarrow O_1, h_1(0) = y$ and $h_1(1) = y + (0, 1)$ and $h_2(0) = y$ and $h_2(1) = y - (1, 0)$.

We construct H in the following way: Let $H(0) = y, H|_{[0,1]}(t) = h_1(t)$. For $i \in \mathbb{Z}$, we define $H|_{[i, i+1]}(t) = h_1(t + i)$ if $\langle H(i); e_\theta \rangle < \langle y; e_\theta \rangle$, and $H|_{[i, i+1]}(t) = h_2(t + i)$ if $\langle H(i); e_\theta \rangle \geq \langle y; e_\theta \rangle$.

This construction is analogous to the one done in the proof of Proposition 6, and from that proof and the construction we see that H satisfies the first two conditions, and that if $\gamma = H(\mathbb{R})$, then

$$-\infty < m = \inf_{z \in \gamma} \langle z; e_\theta \rangle \leq \sup_{z \in \gamma} \langle z; e_\theta \rangle = M < \infty,$$

and that γ^C has two connected unbounded components, one of which contains all points of $V_\theta^+ + Me_\theta$, and the other contains all points of $\mathbb{R}^2 \setminus (V_\theta^+ + me_\theta)$.

Since $\omega(B_\theta) + (p_1, q_1) \cap V_\theta^+ + Me_\theta \neq \emptyset$, and since

$$\omega(B_{\theta+\pi}) + (p_2, q_2) \cap \mathbb{R}^2 \setminus (V_\theta^+ + me_\theta)$$

is also nonempty, we have the result. \square

Since $x_T + \mathbb{Z}^2 \subset \omega(x_T)$, $\pi(x_T)$ is not a fixed point. Let $\varepsilon > 0$ be such that if $B_\varepsilon[\pi(x_T)]$ is the closed ball with center $\pi(x_T)$ and radius ε , then $f(B_\varepsilon[\pi(x_T)]) \cap B_\varepsilon[\pi(x_T)] = \emptyset$.

Now, from Corollary 4,

$$\pi(x_T) \in \overline{\pi(\omega(B_\theta))} \cap \overline{\pi(\omega(B_{\theta+\pi}))};$$

therefore there exists $y \in x_T + \mathbb{Z}^2$, and two integers p and q , such that the distance from y to B_θ is less than ε , as is the distance from $y - (p, q)$ to $B_{\theta+\pi}$.

Since we assumed that $\pi(\omega(B_\theta)) \cap \pi(\omega(B_{\theta+\pi})) = \emptyset$, the distance between the compact sets $\omega(B_\theta) \cap B_\varepsilon[y]$ and $(\omega(B_{\theta+\pi}) + (p, q)) \cap B_\varepsilon[y]$ is a strictly positive number a , and $a < \varepsilon$. Let v be an open line segment with length a such that its endpoints are u and w , where $u \in \omega(B_\theta)$ and $w \in \omega(B_{\theta+\pi}) + (p, q)$. Note that, as $v \subset B_\varepsilon[y]$, $\tilde{f}(v) \cap v = \emptyset$.

Finally, let Γ be the connected component of $\omega(B_\theta)$ that contains u and let Θ be the connected component of $\omega(B_{\theta+\pi}) + (p, q)$ that contains w .

Lemma 8. *The set $(\Gamma \cup \Theta)^C$ contains a single connected component. The set $(\Gamma \cup \Theta \cup v)^C$ has exactly two connected components, Ω_1 and Ω_2 . Also, for every x in T^2 $\pi^{-1}(x)$ intersects both Ω_1 and Ω_2 .*

Proof. The first claim follows from Proposition 8.

Let p be the midpoint of the segment v , and let $r = \text{dist}(p, \Theta) = \text{dist}(p, \Gamma)$. The closed ball $B_r[p]$ is split in two by v . Of course, since $(\Gamma \cup \Theta)^C$ has a single connected component, $(\Gamma \cup \Theta \cup v)^C$ has, at most, two components. Let q_1 be a point in one of the connected components of $B_r[p] \setminus v$ and let q_2 be a point in the second connected component. Suppose, by contradiction, that q_1 and q_2 are in the same connected component of $(\Gamma \cup \Theta \cup v)^C$. Then, there is an arc α connecting q_1 and q_2 lying entirely in this set. We can add to α the segment joining q_1 and q_2 to obtain a Jordan curve J , which does not intersect $\Gamma \cup \Theta$. But one endpoint of v must be in the $\text{int}(J)$, the bounded component of J^C , and so one of Γ and Θ must be in the $\text{int}(J)$, absurd since both are unbounded. \square

Proposition 10. *For $i = \{1, 2\}$, $\tilde{f}(\Omega_i) \cap \Omega_i$ is not empty.*

Proof. Let y be a fixed point of \tilde{f} (since x_T is a recurrent point of \tilde{f} , \tilde{f} cannot be a Brouwer homeomorphism and so it has fixed points; see [6]).

Let H be the curve given by Proposition 9 for the point y , and let $T \in \mathbb{R}$ be such that, if $|t| > T$, then $H(t) \cap v = \emptyset$. Such a T exists by the first property of H . Let $t_1 < -T, t_2 > T$ be reals such that $H(t_1) = y_1 = y + u_1$, and $H(t_2) = y_2 = y + u_2$, where both u_1 and u_2 belong to \mathbb{Z}^2 . We claim that y_1 and y_2 are in different connected components of $(\Gamma \cup \Theta \cup v)^C$, which shows that both Ω_1 and Ω_2 have fixed points and finishes the proof.

If, by contradiction, both y_1 and y_2 are in Ω_1 , then there is an arc α contained in Ω_1 joining these two points. Let $\gamma = H((-\infty, t_1]) \cup \alpha \cup H([t_2, \infty))$. This arc also separates $\omega(B_\theta)$ and $\omega(B_{\theta+\pi}) + (p, q)$, since it is bounded in the e_θ direction and unbounded in the $e_{\theta+\pi/2}$ direction. Since $\Gamma \cup \Theta \cup v$ is connected, $\gamma \cap v \neq \emptyset$, a contradiction. \square

Lemma 9. *Either $\tilde{f}(\overline{\Omega_1}) \subset \overline{\Omega_1}$ or $\tilde{f}(\overline{\Omega_2}) \subset \overline{\Omega_2}$.*

Proof. By the definition of v , $v \cap \omega(B_\theta) = \emptyset$, as well as $v \cap \omega(B_{\theta+\pi}) + (p, q) = \emptyset$. Since both $\omega(B_\theta)$ and $\omega(B_{\theta+\pi})$ are invariant, and since $\tilde{f}(B_\varepsilon[y]) \cap B_\varepsilon[y] = \emptyset$, we have that $\tilde{f}(v) \cap (\Gamma \cup \Theta \cup v) = \emptyset$.

This means that either $\tilde{f}(v) \subset \Omega_1$ or $\tilde{f}(v) \subset \Omega_2$. We will assume, without loss of generality, that the former is true.

Since Γ is a connected component of $\omega(B_\theta)$ and the latter is invariant, $\tilde{f}(\Gamma)$ is also a connected component of $\omega(B_\theta)$ and so either $\tilde{f}(\Gamma) = \Gamma$ or $\tilde{f}(\Gamma) \cap \Gamma = \emptyset$.

Also, since $v \cap \omega(B_\theta) = \emptyset$ and $\Theta \cap \omega(B_\theta)$ is also empty, and since $\omega(B_\theta)$ is invariant, it follows that $\tilde{f}(\Gamma) \cap (v \cup \Theta) = \emptyset$. If $\tilde{f}(\Gamma) = \Gamma$, then $\tilde{f}(\Gamma) \cap \Omega_2 = \emptyset$. If $\tilde{f}(\Gamma) \neq \Gamma$, since $\Gamma \cup v$ is connected, and since $\tilde{f}(v) \subset \Omega_1$ and $\tilde{f}(\Gamma) \cap \partial\Omega_1 = \emptyset$, we have $\tilde{f}(\Gamma) \subset \Omega_1$, and thus $\tilde{f}(\Gamma) \cap \Omega_2$ is also empty.

By a similar argument, $\tilde{f}(\Theta) \cap \Omega_2 = \emptyset$, and so $\tilde{f}(\partial\Omega_1) \cap \Omega_2$ is empty. Since both Ω_1 and Ω_2 are connected, this implies that either $\Omega_2 \subset \tilde{f}(\Omega_1)$ or $\Omega_2 \cap \tilde{f}(\Omega_1) = \emptyset$. By Proposition 10 the first possibility cannot be, since \tilde{f} is a homeomorphism. The second possibility implies that $\tilde{f}(\overline{\Omega_1}) \subset \overline{\Omega_1}$. \square

Finally since \tilde{f} satisfies property T, the future orbit of x_T must visit any neighborhood of any \mathbb{Z}^2 translate of x_T infinitely many times, and so, from Lemma 8, there must be integers $n_1 < n_2 < n_3$ such that

$$(4) \quad \tilde{f}^{n_1}(x_T) \in \Omega_1, \tilde{f}^{n_2}(x_T) \in \Omega_2 \text{ and } \tilde{f}^{n_3}(x_T) \in \Omega_1.$$

But, from the previous lemma, if $\tilde{f}(\overline{\Omega_1}) \subset \overline{\Omega_1}$, then for all $n \geq n_1$, $\tilde{f}^n(x_T) \notin \Omega_2$, which violates (4), and if $\tilde{f}(\overline{\Omega_2}) \subset \overline{\Omega_2}$, then for all $n \geq n_2$, $\tilde{f}^n(x_T) \notin \Omega_1$, which again violates (4), and we are done.

ACKNOWLEDGMENT

The author would like to thank Prof. Addas-Zanata for the insightful discussions on this work and for presenting ideas that helped to clarify some of the proofs presented here.

REFERENCES

1. Addas-Zanata S. and Tal F. A. (2011) Homeomorphisms of the annulus with a transitive lift. *Math. Zeit.* **267**, 971–980. MR2776069
2. Addas-Zanata S. and Tal F. A. (2010) On generic rotationless diffeomorphisms of the annulus. *Proc. Amer. Math. Soc.* **138**, 1023–1031. MR2566568 (2011e:37094)
3. Alpern S. and Prasad V. (1995) Topical transitivity for lifts of rotationless annulus or torus homeomorphisms. *Bull. London Math. Soc.* **27**, 79–81. MR1331686 (96g:58097)
4. Bonatti C. and Crovisier S. (2004) Récurrence et généricité. *Invent. Math.* **158**, 33–104. MR2090361 (2007b:37036)
5. Enrich H., Guelman N., Larcanche A. and Liousse I. (2009) Diffeomorphisms having rotation sets with non-empty interior. *Nonlinearity* **22**, 1899–1907. MR2525816 (2010h:37089)
6. Franks J. (1988) Generalisations of the Poincaré-Birkhoff theorem. *Ann. of Math. (2)* **128**, 139–151. MR951509 (89m:54052)
7. Franks J. (1989) Realizing rotation vectors for torus homeomorphisms. *Trans. Amer. Math. Soc.* **311**, 107–115. MR958891 (89k:58239)
8. Jaeger T. (2009) Linearization of conservative toral homeomorphisms. *Inventiones Math.* **176**, 601–616. MR2501297 (2011e:37095)
9. Llibre J. and Macay R.S. (1991) Rotation vectors and entropy for homeomorphisms of the torus isotopic to the identity. *Erg. Th. and Dyn. Sys.* **11**, 115–128. MR1101087 (92b:58184)
10. Misiurewicz M. and Ziemian K. (1989) Rotation sets of toral maps. *J. London Math. Soc.* **40**, 490–506. MR1053617 (91f:58052)
11. Misiurewicz M. and Ziemian K. (1991) Rotation sets and ergodic measures for torus homeomorphisms. *Fundam. Math.* **137**(1), 45–52. MR1100607 (92d:58106)
12. Newman M. H. A. (1939) Elements of the Topology of Plane Sets of Points. Cambridge University Press, 216 pp. MR0044820 (13:483a)

INSTITUTO DE MATEMÁTICA E ESTATÍSTICA, UNIVERSIDADE DE SÃO PAULO, RUA DO MATÃO 1010, CIDADE UNIVERSITÁRIA, 05508-090 SÃO PAULO, SP, BRAZIL

E-mail address: fabiot@ime.usp.br