ON THE EXISTENCE OF J-CLASS OPERATORS ON BANACH SPACES

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(Communicated by Thomas Schlumprecht)

Abstract. In this paper we answer in the negative the question raised by G. Costakis and A. Manoussos whether there exists a J-class operator on every non-separable Banach space. In particular we show that there exists a non-separable Banach space constructed by S. Argyros, A. Arvanitakis and A. Tolias such that the J-set of every operator on this space has empty interior for each non-zero vector. On the other hand, on non-separable spaces which are reflexive there always exists a J-class operator.

1. Preliminaries and the main result

Let \( X \) be a real or complex Banach space. If \( X \) is a real Banach space, then \( X_C \) we denote the complexification of \( X \). By \( L(X) \) we mean the space of all bounded linear operators acting on \( X \). If \( T \in L(X) \), the symbol \( \sigma(T) \) stands for the spectrum of \( T \). Consider any subset \( C \) of \( X \). The symbol \( C^\circ \) denotes the interior of \( C \) in the norm topology of \( X \). The symbol \( \text{orb}(T, x) \) denotes the orbit of \( x \) under \( T \), i.e. \( \text{orb}(T, x) := \{ T^n x : n \in \mathbb{N} \cup \{0\} \} \). If \( X \) is separable and \( \text{orb}(T, x) \) is dense, then \( T \) is called hypercyclic, which is equivalent to \( T \) being topologically transitive; i.e. for every pair of non-empty open sets \( U, V \subset X \), there exists a non-negative integer \( n \) such that \( T^n(U) \cap V \neq \emptyset \). Following [7], by \( J_T(x) \) we denote the J-set of \( x \) under \( T \), i.e.

\[
J_T(x) := \{ y \in X : \text{there exists a strictly increasing sequence of natural numbers } (k_n) \text{ and a sequence } (x_n) \text{ in } X, \text{ such that } x_n \to x \text{ and } T^{k_n}x_n \to y \}.
\]

If \( J_T(x) = X \) for some \( x \in X \setminus \{0\} \), then \( T \) is called a J-class operator. By \( A_T \) we denote the set of all \( x \in X \) such that \( J_T(x) = X \). On separable spaces every hypercyclic operator is J-class, but the converse is not true. It is known [4] that on \( l^\infty \), there does not exist a topological transitive operator. On the other hand there exist J-class operators such as the weighted backward shift \( \lambda B : l^\infty \to l^\infty \), \( \lambda B(x_1, x_2, \ldots) := (\lambda x_2, \lambda x_3, \ldots) \) for \( |\lambda| > 1 \). Therefore it is natural to ask whether every non-separable Banach space admits a J-class operator [7]. Our main result is the following:

Received by the editors September 28, 2010 and, in revised form, April 13, 2011 and April 15, 2011.

2000 Mathematics Subject Classification. Primary 47A16; Secondary 37B99, 54H20.

Key words and phrases. J-class operators, hypercyclicity.

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**Theorem 1.1.** There exists a non-separable complex Banach space \( X \) on which the \( J \)-set of every operator has empty interior for every non-zero vector. Consequently there exists no \( J \)-class operator on \( X \).

The statement in Theorem 1.1 gives us a stronger result than the question raised by G. Costakis and A. Manoussos, since in general \((J_T(x))^o \neq \emptyset\) does not imply that \( J_T(x) = X \). As is clear from our proof, the conclusion of Theorem 1.1 is satisfied for every complex non-separable Banach space, for which every \( T \in L(X) \) is of the form \( T = \lambda I + S \) with \( \lambda \in \mathbb{C} \) and \( S \) a strictly singular operator with separable range. A real non-separable HI (Hereditarily Indecomposable) Banach space containing no reflexive subspace for which every \( T \in L(X) \) takes the form \( T = \lambda I + S \) with \( \lambda \in \mathbb{R} \) and \( S \) a weakly compact operator with separable range has been constructed by S. Argyros, A. Arvanitakis and A. Tolias in [3]. The complexification of this space is easily shown to satisfy our requirements, and thus the conclusion of Theorem 1.1. In contrast we show in Theorem 1.18 that every non-separable reflexive Banach space admits a \( J \)-class operator.

**Definition 1.2.** Let \( X, Y \) be infinite dimensional Banach spaces. A linear and bounded operator \( S : X \to Y \) is called strictly singular if for every infinite dimensional subspace \( M \subset X \) the restriction \( S|M : M \to S(M) \) is not an isomorphism (linear homeomorphism).

**Remark 1.3.** If \( X = Y \), then an immediate consequence of the above definition is that \( 0 \in \sigma(S) \). The spectrum of \( S \) is at most countable with 0 as the only possible accumulation point, [1].

The next two theorems can be found in [1].

**Theorem 1.4.** Let \( X \) be a Banach space. The collection of all strictly singular operators is a closed subspace in \( L(X) \), which is also a two-sided ideal.

**Proposition 1.5.** Let \( X \) be a real Banach space and \( X_C \) its complexification. A bounded operator \( T : X \to X \) is strictly singular if and only if \( T_C : X_C \to X_C \) is strictly singular.

We show now that a certain class of operators on a complex Banach space can not be \( J \)-class. For this purpose we need the following proposition.

**Proposition 1.6.** Let \( X \) be a complex Banach space and \( T \in L(X) \). Suppose \((J_T(x))^o \neq \emptyset\) for some \( x \in X \setminus \{0\} \). Then \( \sigma(T) \cap \partial D \neq \emptyset \).

**Proof.** Assume that \( \sigma(T) \cap \partial D = \emptyset \). We decompose \( \sigma \) in \( \sigma_1 := \{ \lambda \in \sigma : |\lambda| > 1 \} \) and \( \sigma_2 := \{ \lambda \in \sigma : |\lambda| < 1 \} \). Then \( \sigma_1 \) and \( \sigma_2 \) are disjoint and closed. By the Riesz decomposition theorem we can decompose \( X = M_1 \oplus M_2 \), where \( M_1 \) and \( M_2 \) are closed and \( T \)-invariant subspaces and \( \sigma_1 = \sigma(T|_{M_1}) \), \( \sigma_2 = \sigma(T|_{M_2}) \). Assume now that there exists a non-zero vector \( x \in X \), such that \( J_T(x) \) has non-empty interior. We can write \( x = x_1 + x_2 \), with \( x_1 \in M_1 \) and \( x_2 \in M_2 \). Since \( x \neq 0 \), either \( x_1 \) or \( x_2 \) is not equal to zero. Consider the projection \( P_1 : X \to M_1 \) along \( M_2 \) onto \( M_1 \). Since \( J_T(x) \subset J_{T|_{M_1}}(x_1) + J_{T|_{M_2}}(x_2) \) it follows that \( P_1(J_T(x)) \subset J_{T|_{M_1}}(x_1) \). By the open mapping theorem we get that \( P_1(J_T(x)) \) has non-empty interior and hence \((J_{T|_{M_1}}(x_1))^o \neq \emptyset \). From the spectral radius formula we obtain that \( \|T^n \tilde{x}\| \leq a^n \|\tilde{x}\| \) for some \( a \in \mathbb{R} \) with \( 0 \leq a < 1 \) and for all \( \tilde{x} \in M_1 \). This implies that \( J_{T|_{M_1}}(x_1) = \{0\} \) and therefore \( M_1 = \{0\} \). So we get \( x_1 = 0 \). Again
from the spectral radius formula we know that \( \| T^n(x) \| \geq A^n \| x \| \) for some \( A > 1 \) and all \( x \in M_2 \). This implies \( x_2 = 0 \), which is a contradiction to our assumption that \( x \neq 0 \). \( \square \)

**Theorem 1.7.** Let \( X \) be a complex infinite dimensional Banach space. Then for every operator of the form \( T = \lambda I + S \), where \( S \) is strictly singular and \( |\lambda| > 1 \), and every \( x \neq 0 \), the set \( J_T(x) \) has empty interior.

**Proof.** By Remark 1.3 it follows that \( \lambda \in \sigma(T) \) and it is the only possible accumulation point. We decompose the spectrum in \( \sigma_1 := \{ \mu \in \sigma(T) : |\mu| \leq 1 \} \) and \( \sigma_2 := \{ \mu \in \sigma(T) : |\mu| > 1 \} \). Clearly then \( \lambda \in \sigma_2 \). The set \( \sigma_1 \) is closed, and since \( \lambda \in \sigma_2 \), and \( \lambda \) is the only possible accumulation point, we conclude that \( \sigma_2 \) is also closed. Furthermore \( \sigma_1 \) and \( \sigma_2 \) are disjoint. By the Riesz decomposition theorem we can decompose \( X = M_1 \oplus M_2 \), where \( M_1 \) and \( M_2 \) are closed and \( T \)-invariant subspaces and \( \sigma_1 = \sigma(T_{|M_1}) \), \( \sigma_2 = \sigma(T_{|M_2}) \). Now assume that there exists a non-zero vector \( x \) with \((J_T(x))^o \neq \emptyset \). Let \( x = x_1 + x_2 \), where \( x_1 \in M_1 \) and \( x_2 \in M_2 \). Then since \( x \) is non-zero, either \( x_1 \) or \( x_2 \) is not equal to zero. We claim that \( M_1 \) is finite dimensional. Otherwise suppose \( M_1 \) is infinite dimensional. Then \( S_{|M_1} \) is strictly singular and it follows from Remark 1.3 that \( 0 \notin \sigma(S_{|M_1}) \) and therefore \( \lambda \in \sigma(T_{|M_1}) = \sigma_1 \), which is not possible. Now consider the projection \( P_1 : X \rightarrow M_1 \) along \( M_2 \) onto \( M_1 \). Since \( J_T(x) \subset J_{T_{M_1}}(x_1) + J_{T_{M_2}}(x_2) \), it follows that
\[
P_1(J_T(x)) \subset J_{T_{M_1}}(x_1).
\]

By the open mapping theorem it follows that \( P_1(J_T(x)) \) has non-empty interior and so \( J_{T_{M_1}}(x_1) \) has non-empty interior. This is possible only if \( x_1 = 0 \), since \( M_1 \) is finite dimensional; see [7].

As above we conclude that \( J_{T_{M_2}}(x_2) \) has non-empty interior. Therefore since \( \sigma(T_{|M_2}) = \sigma_2 \subset \mathbb{C} \setminus \mathbb{D} \), it follows by Proposition 1.6 that \( x_2 = 0 \). Thus \( x = x_1 + x_2 = 0 \), which is a contradiction. \( \square \)

S. Argyros, A. Arvanitakis and A. Tolias constructed a non-separable real Banach space, on which every operator \( T \) has the form \( T = \lambda I + S \), where \( S \) is strictly singular and has separable range; see [3].

**Theorem 1.8** (Argyros, Arvanitakis, Tolias). There exists a real non-separable Banach space \( X_A \), containing no reflexive subspace, on which every operator \( T \) is of the form \( T = \lambda I + S \) with \( \lambda \in \mathbb{R} \) and \( S \) a weakly compact operator (and hence of separable range).

The fact that \( X_A \) contains no reflexive subspace implies that every weakly compact operator on \( X_A \) is strictly singular; thus every operator \( T : X_A \rightarrow X_A \) is of the form \( T = \lambda I + S \) with \( \lambda \in \mathbb{R} \) and \( S \) a strictly singular operator with separable range. The proof of the following corollary is essentially contained in the proof of Lemma 4.3 in [3].

**Corollary 1.9.** Consider \( X := (X_A)_{\mathbb{C}} \). Then every operator \( T \in L(X) \) is of the form \( T = wI + S \) (w \( \in \mathbb{C} \)), where \( S \) is strictly singular and has separable range.

**Proof.** Every operator \( T \in L(X) \) can be written as \( T = T_1 + iT_2 \), where \( T_1, T_2 \in L(X_A) \). By the previous theorem, \( T_1 = \lambda I + S_1 \) and \( T_2 = \mu I + S_2 \) (\( \lambda, \mu \in \mathbb{R} \)), where
$S_1, S_2 \in L(X_A)$ are strictly singular and have separable range. Therefore we get

$$T = T_1 + iT_2 = \lambda I + S_1 + i(\mu I + S_2) = (\lambda + i\mu)I + (S_1)C + i(S_2)C.$$  

By Proposition 1.5, $(S_i)_C$ is strictly singular for $i \in \{1, 2\}$ and by Theorem 1.4, $S := (S_1)C + i(S_2)C$ is strictly singular and has separable range. With $w := \lambda + i\mu$ we get $T = wI + S$. □

The next lemma can be found in [7].

**Lemma 1.10.** Let $X$ be a Banach space and $T \in L(X)$. If $J_T(x)$ has non-empty interior for some $x \neq 0$, then $T - \lambda I$ has dense range for each $|\lambda| \leq 1$.

**Theorem 1.11.** There exists a non-separable complex Banach space $X$ on which the $J$-set of every operator has empty interior for every non-zero vector. In particular there does not exist a $J$-class operator on $X$.

**Proof.** We consider the space $X = (X_A)_C$. Then every operator $T$ is of the form $T = \lambda I + S$ by Corollary 1.9, where $S$ is strictly singular and has separable range. If $|\lambda| > 1$, then it follows by Theorem 1.7 that the interior of $J_T(x)$ is empty for each non-zero vector $x$. Now consider $|\lambda| \leq 1$. Then by Lemma 1.10 the operator $T - \lambda I = S$ has dense range. This is not possible since $S$ has separable range and $X$ is non-separable. □

Our next aim is to show that on the space $Y := X \oplus X$, where $X = (X_A)_C$, the $J$-set of every $T \in L(Y)$ also has empty interior for each non-zero vector in $Y$. The next lemma gives us some information about the form of the operators in $L(Y)$.

**Lemma 1.12.** Consider $Y := X \oplus X$, where $X = (X_A)_C$. Then for every operator $T \in L(Y)$ there exists an isomorphism $J \in L(Y)$, such that $J^{-1}TJ$ has one of the following two matrix representations:

$$J^{-1}TJ = \begin{pmatrix} \lambda I & I \\ 0 & \lambda I \end{pmatrix} + \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}$$

or

$$J^{-1}TJ = \begin{pmatrix} \lambda_1 I & 0 \\ 0 & \lambda_2 I \end{pmatrix} + \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix},$$

where $S_i \in L(X)$ are strictly singular and have separable range for $i \in \{1, 2, 3, 4\}$.

**Proof.** Every operator $T \in L(Y)$ has the following matrix representation:

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix},$$

where $T_i \in L(X)$ for $i \in \{1, 2, 3, 4\}$. From Corollary 1.9 every $T_i = a_i I + \tilde{S}_i$, where $\tilde{S}_i$ is strictly singular with separable range. So we get

$$T = \begin{pmatrix} a_1 I & a_2 I \\ a_3 I & a_4 I \end{pmatrix} + \begin{pmatrix} \tilde{S}_1 \\ \tilde{S}_2 \\ \tilde{S}_3 \\ \tilde{S}_4 \end{pmatrix}.$$
Applying the Jordan decomposition of matrices, there exists an isomorphism such that

\[ J^{-1}AJ = \begin{pmatrix} \lambda I & I \\ 0 & \lambda I \end{pmatrix} \quad \text{or} \quad J^{-1}AJ = \begin{pmatrix} \lambda_1I & 0 \\ 0 & \lambda_2I \end{pmatrix}. \]

By Theorem 1.4, \( S := J^{-1}\tilde{S}J \) is also strictly singular and therefore there exist some \( S_i \in L(X), i \in \{1, 2, 3, 4\} \) strictly singular with separable range (see [1]) such that

\[ S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}. \]

The desired statement now follows. \( \square \)

The next theorem can be found in [1].

**Theorem 1.13.** Assume that \( S \in L(X) \) is strictly singular and that an operator \( T \in L(X) \) has an at most countable spectrum. Then the spectrum of \( S+T \) is at most countable and zero and the points of \( \sigma(T) \) are the only possible accumulation points of \( \sigma(S+T) \).

**Theorem 1.14.** Consider \( Y = X \oplus X \) with \( X = (X_A)_C \) and \( T \in L(Y) \). Then \( (J_T((x,y)))^* = \emptyset \) for \( (x,y) \in Y \setminus \{(0,0)\} \).

**Proof.** We argue by contradiction. So suppose there exists a \( T \in L(Y) \) such that \( (J_T((x,y)))^* \neq \emptyset \) for some \( (x,y) \in Y \setminus \{(0,0)\} \). By Lemma 1.12 there exists an isomorphism \( D \in L(Y) \) such that

\[ D^{-1}TD = \begin{pmatrix} \lambda I & 0 \\ 0 & \lambda I \end{pmatrix} + \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} \quad (*) \]

or

\[ D^{-1}TD = \begin{pmatrix} \lambda_1I & 0 \\ 0 & \lambda_2I \end{pmatrix} + \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} \quad (**). \]

Then for \( \tilde{T} := D^{-1}TD \) the \( J \)-set \( J_{\tilde{T}}(D^{-1}(x,y)) \) also has non-empty interior.

**Case 1.** \( \tilde{T} = (*) \).

If \( |\lambda| \neq 1 \), then we decompose \( \sigma(\tilde{T}) \) in \( \sigma_1 = \{ \mu \in \sigma(\tilde{T}) : |\mu| = 1 \} \) and \( \sigma_2 = \{ \mu \in \sigma(\tilde{T}) : |\mu| \neq 1 \} \). The set \( \sigma_1 \) is closed and by Theorem 1.13, \( \sigma_2 \) is also closed, since \( \lambda \) and 0 are the only possible accumulation points of \( \sigma(\tilde{T}) \) and hence of \( \sigma_2 \). Furthermore the corresponding \( \tilde{T} \)-invariant closed subspace \( M_1 \) for \( \sigma_1 \), resulting from the Riesz decomposition theorem is finite dimensional; otherwise,

\[ \tilde{T}|_{M_1} = \begin{pmatrix} \lambda I & 0 \\ 0 & \lambda I \end{pmatrix} \quad \text{is not invertible and hence} \quad \lambda \in \sigma(\tilde{T}|_{M_1}) = \sigma_1, \quad \text{which is not possible.} \]

The rest of the proof for \( |\lambda| \neq 1 \) is similar to Theorem 1.7.

Now consider \( |\lambda| = 1 \). Then \( \tilde{T} - \begin{pmatrix} \lambda I & 0 \\ 0 & \lambda I \end{pmatrix} \) does not have dense range, which is a contradiction to Lemma 1.10.
Case 2. $\tilde{T} = (**).$

If $\lambda_1 = \lambda_2$ the argumentation is almost similar to case 1. So suppose $\lambda_1 \neq \lambda_2$. Assume $|\lambda_1| = 1$ or $|\lambda_2| = 1$. Without loss of generality $|\lambda_1| = 1$. Then $\tilde{T} - (\lambda_1 I \ 0 \ 0_0)$ does not have dense range, which is a contradiction as in Case 1.

For $|\lambda_1| \neq 1$ and $|\lambda_2| \neq 1$ the argumentation is identical to Case 1. \(\square\)

Remark 1.15. It is also possible with some more technicalities to prove the same result in Theorem 1.14 for $Y = X \oplus \ldots \oplus X$, where $X = (X_A)_{C}$. We will now show that there is a large class of non-separable Banach spaces on which there always exists a $J$-class operator, namely the reflexive non-separable Banach spaces. The next theorem can be found in \cite{2}.

**Theorem 1.16.** Let $X$ be a Banach space and $Y$ a separable Banach space. Consider $S \in L(X)$ with $\sigma(S) \subset \{\lambda : |\lambda| > 1\}$. Also let $T \in L(Y)$ be hypercyclic. Then:

1. $S \times T : X \times Y \to X \times Y$ is a $J$-class operator, but not hypercyclic.
2. $A_{S \times T} = \{0\} \times Y$.

The next theorem by Lindenstrauss (\cite{8}) gives us some information about the decomposition of reflexive non-separable Banach spaces.

**Theorem 1.17 (Lindenstrauss).** Let $X$ be a non-separable reflexive Banach space and $Y \subset X$ a separable and closed subspace. Then there exists a separable closed subspace $W$ of $X$ that contains $Y$ and a linear bounded projection $P_W : X \to W$ with $\|P_W\| = 1$.

**Theorem 1.18.** Let $X$ be a non-separable reflexive Banach space. Then for every infinite dimensional separable and closed subspace $Y$ and for every $\lambda \in (1, \infty)$ there exists a $J$-class operator $T$ with $Y \subset A_T$ and $\|T\| = \lambda$.

**Proof.** By Theorem 1.17 there exists a separable infinite dimensional subspace $W$ that contains $Y$ and a linear bounded projection $P_W : X \to W$ with $\|P_W\| = 1$. There exists a closed subspace $U$ of $X$ such that $X = U \oplus W$. For given $\epsilon > 0$ we can find a hypercyclic operator $T_1 : W \to W$, $T_1 := I_W + K$, with $K$ compact and $\|K\| < \epsilon$; see (\cite{2}, \cite{3}). Then by Theorem 1.16 the operator $T_1 := \lambda I_W \oplus T_1 = \lambda I + (1 - \lambda)P_W + K \circ P_W$ is $J$-class for $\lambda > 1$. Furthermore $Y \subset W = A_{T_1}$. Now define the function $g : (1, \infty) \to \mathbb{R}$ by $g(\delta) := \|T_\delta\|$. Then it is easy to see that $g$ is continuous. For given $\lambda$ we choose $\delta > 1$ and $\epsilon > 0$ such that $2\delta + \epsilon < 1 + \lambda$. Therefore we get

$$g(\delta) = \|T_\delta\| = \|\delta I + (1 - \delta)P_W + K \circ P_W\|,$$

$$\leq \delta + |1 - \delta| \|P_W\| + \|P_W\| \|K\| \leq 2\delta - 1 + \epsilon < \lambda.$$

On the other hand we can find a $\mu > 1$ large enough such that $g(\mu) > \lambda$. By the intermediate value theorem there exists a $\xi \in [\delta, \mu]$ with $g(\xi) = \|T_\xi\| = \lambda$. \(\square\)

**ACKNOWLEDGEMENTS**

The author is grateful to Prof. Dr. Rainer Brück for his scientific support. He would also like to express his gratitude to Professor Dr. W. Kaballo for some helpful comments.
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