

## $C^*$ -ALGEBRAS ASSOCIATED WITH REAL MULTIPLICATION

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ABSTRACT. Noncommutative tori with real multiplication are the irrational rotation algebras that have special equivalence bimodules. Y. Manin proposed the use of noncommutative tori with real multiplication as a geometric framework for the study of abelian class field theory of real quadratic fields. In this paper, we consider the Cuntz-Pimsner algebras constructed by special equivalence bimodules of irrational rotation algebras. We shall show that the associated  $C^*$ -algebras are simple and purely infinite. We compute the  $K$ -groups of the associated  $C^*$ -algebras and show that these algebras are related to the solutions of Pell's equation and the unit groups of real quadratic fields. We consider the Morita equivalent classes of the associated  $C^*$ -algebras.

### 1. INTRODUCTION

Let  $\theta$  be an irrational number. An irrational rotation algebra  $A_\theta$  is the crossed product  $C^*$ -algebra for the action of the integers on the circle by powers of the rotation by angle  $2\pi\theta$ . It is simple and has a unique normalized trace  $\tau_\theta$ . These algebras are also called noncommutative tori and have been classified up to  $C^*$ -isomorphism and Morita equivalence [18],[19]. If  $A_{\theta_1}$  is Morita equivalent to  $A_{\theta_2}$ , then there exist  $a, b, c, d \in \mathbb{Z}$  such that  $\theta_1 = \frac{a\theta_2 + b}{c\theta_2 + d}$  and  $ad - bc = \pm 1$  by Theorem 4 in [19].

Suppose that  $\sigma$  is a free and proper action of a locally compact group  $G$  on a locally compact Hausdorff space  $X$ . Then  $X/\sigma$  is a locally compact Hausdorff space and  $C_0(X) \rtimes_\sigma G$  is Morita equivalent to  $C_0(X/\sigma)$  (see [21]). The action of the integers on the circle by powers of the rotation by angle  $2\pi\theta$  is free, but every orbit is dense and the orbit space  $(\mathbb{R}/\mathbb{Z})/\mathbb{Z}\theta$  behaves badly. By the fact above, we may consider an irrational rotation algebra  $A_\theta$  as a well-behaved quotient space of  $(\mathbb{R}/\mathbb{Z})/\mathbb{Z}\theta$ . This idea is due to A. Connes [4].

An elliptic curve  $E_\omega$  over  $\mathbb{C}$  can be described as the quotient  $E_\omega \cong \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\omega)$  of the complex plane by a 2-dimensional lattice  $\mathbb{Z} + \mathbb{Z}\omega$ , where we can take  $\text{Im}(\omega) > 0$ . Most elliptic curves over  $\mathbb{C}$  have only the multiplication-by- $m$  endomorphisms. An elliptic curve  $E_\omega$  over  $\mathbb{C}$  has extra endomorphisms if and only if  $\omega$  is in an imaginary quadratic field. In this case, an elliptic curve  $E_\omega$  is said to have complex multiplication. Such curves have many special properties. One of the important properties is that the  $j$ -invariant and the torsion points of  $E_\omega$  generate a maximal abelian extension of an imaginary quadratic field  $\mathbb{Q}(\omega)$ . For real quadratic fields,

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a similar description is not known. By the idea above and regarding  $(\mathbb{R}/\mathbb{Z})/\mathbb{Z}\theta$  as  $\mathbb{R}/(\mathbb{Z} + \mathbb{Z}\theta)$ , Y. Manin proposed to use irrational rotation algebras corresponding to real quadratic irrationalities as a replacement of elliptic curves with complex multiplication. We shall explain that there exist analogous properties with complex multiplication in the irrational rotation algebras.

Two  $C^*$ -algebras  $A$  and  $B$  are Morita equivalent if there exists an  $A$ - $B$ -equivalence bimodule. An  $A$ - $B$ -equivalence bimodule is an  $A$ - $B$ -bimodule  $\mathcal{E}$  which is simultaneously a full left Hilbert  $A$ -module under an  $A$ -valued inner product  ${}_A\langle \cdot, \cdot \rangle$  and a full right Hilbert  $B$ -module under a  $B$ -valued inner product  $\langle \cdot, \cdot \rangle_B$ , satisfying  ${}_A\langle \xi, \eta \rangle \zeta = \xi \langle \eta, \zeta \rangle_B$  for any  $\xi, \eta, \zeta \in \mathcal{E}$ . We recall some definitions and elementary facts on the Picard groups of  $C^*$ -algebras introduced by Brown, Green and Rieffel in [2]. For  $A$ - $A$ -equivalence bimodules  $\mathcal{E}_1$  and  $\mathcal{E}_2$ , we say that  $\mathcal{E}_1$  is isomorphic to  $\mathcal{E}_2$  as an equivalence bimodule if there exists a  $\mathbb{C}$ -linear one-to-one map  $\Phi$  from  $\mathcal{E}_1$  onto  $\mathcal{E}_2$  with the properties such that  $\Phi(a\xi b) = a\Phi(\xi)b$ ,  ${}_A\langle \Phi(\xi), \Phi(\eta) \rangle = {}_A\langle \xi, \eta \rangle$  and  $\langle \Phi(\xi), \Phi(\eta) \rangle_A = \langle \xi, \eta \rangle_A$  for  $a, b \in A$ ,  $\xi, \eta \in \mathcal{E}_1$ . The set of isomorphic classes  $[\mathcal{E}]$  of the  $A$ - $A$ -equivalence bimodules  $\mathcal{E}$  forms a group under the product defined by  $[\mathcal{E}_1][\mathcal{E}_2] = [\mathcal{E}_1 \otimes_A \mathcal{E}_2]$ . We call it the *Picard group* of  $A$  and denote it by  $\text{Pic}(A)$ . The identity of  $\text{Pic}(A)$  is given by the  $A$ - $A$ -bimodule  $\mathcal{E} := A$  with  ${}_A\langle a, b \rangle = ab^*$  and  $\langle a, b \rangle_A = a^*b$  for  $a, b \in A$ . The dual module  $\mathcal{E}^*$  of an  $A$ - $A$ -equivalence bimodule  $\mathcal{E}$  is a set  $\{\xi^*; \xi \in \mathcal{E}\}$  with the operations such that  $\xi^* + \eta^* = (\xi + \eta)^*$ ,  $\lambda\xi^* = (\bar{\lambda}\xi)^*$ ,  $b\xi^*a = (a^*\xi b^*)^*$ ,  ${}_A\langle \xi^*, \eta^* \rangle = \langle \eta, \xi \rangle_A$  and  $\langle \xi^*, \eta^* \rangle_A = {}_A\langle \eta, \xi \rangle$ . Then  $[\mathcal{E}^*]$  is the inverse element of  $[\mathcal{E}]$  in the Picard group of  $A$ . Let  $\alpha$  be an automorphism of  $A$ , and let  $\mathcal{E}_\alpha = A$  with the obvious left  $A$ -action and the obvious left  $A$ -valued inner product. We define the right  $A$ -action on  $\mathcal{E}_\alpha$  by  $\xi \cdot a = \xi\alpha(a)$  for any  $\xi \in \mathcal{E}_\alpha$  and  $a \in A$ , and the right  $A$ -valued inner product by  $\langle \xi, \eta \rangle_A = \alpha^{-1}(\xi^*\eta)$  for any  $\xi, \eta \in \mathcal{E}_\alpha$ . Then  $\mathcal{E}_\alpha$  is an  $A$ - $A$ -equivalence bimodule. For  $\alpha, \beta \in \text{Aut}(A)$ ,  $\mathcal{E}_\alpha$  is isomorphic to  $\mathcal{E}_\beta$  if and only if there exists a unitary  $u \in A$  such that  $\alpha = ad u \circ \beta$ . Moreover,  $\mathcal{E}_\alpha \otimes \mathcal{E}_\beta$  is isomorphic to  $\mathcal{E}_{\alpha \circ \beta}$ . Hence we obtain a homomorphism  $\rho_A$  from  $\text{Aut}(A)/\text{Inn}(A)$  to  $\text{Pic}(A)$  and may regard  $A$ - $A$ -equivalence bimodules as a generalization of automorphisms of  $C^*$ -algebras.

The equivalence bimodules of irrational rotation algebras are constructed by M. Rieffel. These bimodules are defined as completions of  $C_c(\mathbb{R} \times \mathbb{Z}/c\mathbb{Z})$  or  $S(\mathbb{R} \times \mathbb{Z}/c\mathbb{Z})$  with the certain actions of irrational rotation algebras [22], [23]. There exist special  $A_\theta$ - $A_\theta$ -equivalence bimodules in the case where  $\theta$  is a quadratic irrational number. In fact, K. Kodaka showed that if  $\theta$  is not a quadratic irrational number, then the Picard group of  $A_\theta$  is isomorphic to  $\text{Aut}(A_\theta)/\text{Inn}(A_\theta)$  and that if  $\theta$  is a quadratic number, then the Picard group of  $A_\theta$  is isomorphic to a semidirect product of  $\text{Aut}(A_\theta)/\text{Inn}(A_\theta)$  with  $\mathbb{Z}$  [11]. We may consider that this is analogous to the complex multiplication of an elliptic curve. Y. Manin called this real multiplication and proposed the use of noncommutative tori with real multiplication as a geometric framework for the study of abelian class field theory of real quadratic fields [12]. But we have no idea of how to solve this problem.

In this paper, we consider the Cuntz-Pimsner algebras constructed by the equivalence bimodules of irrational rotation algebras that are not generated by automorphisms. In the theory of  $C^*$ -algebras, the crossed product algebras help us to understand the automorphisms. Since the Cuntz-Pimsner construction is a generalization of the crossed product construction, the Cuntz-Pimsner algebras constructed by special equivalence bimodules help us to understand special bimodules

of irrational rotation algebras and may give us a hint for studying the abelian class field theory of real quadratic fields by using irrational rotation algebras. In fact, we show that these algebras are related to the solutions of Pell's equation and the unit groups of real quadratic fields. This shows that there exists a deep relation between irrational rotation algebras and real quadratic fields. (See, for example, [3] and [7] for the relation between the algebraic number theory of quadratic fields and Pell's equation.)

In Section 2 we consider the  $A_\theta$ - $A_\theta$ -equivalence bimodules. K. Kodaka considered the  $A_\theta$ - $A_\theta$ -equivalence bimodules as the automorphisms of  $A_\theta \otimes \mathbb{K}$ , where  $\mathbb{K}$  consists of the  $C^*$ -algebras of all compact operators on a countably infinite dimensional Hilbert space, and determined the Picard groups of the irrational rotation algebras in [11]. We consider the equivalence bimodules of irrational rotation algebras neither as the completions of  $C_c(\mathbb{R} \times \mathbb{Z}/c\mathbb{Z})$  ( $S(\mathbb{R} \times \mathbb{Z}/c\mathbb{Z})$ ) nor as the automorphisms of  $A_\theta \otimes \mathbb{K}$  in this paper. If  $A$  is unital, then an  $A$ - $A$ -equivalence bimodule is a finitely generated projective  $A$ -module as a right  $A$ -module. Hence an  $A$ - $A$ -equivalence bimodule is isomorphic to  $qA^n$  as a right Hilbert  $A$ -module, where  $q$  is a projection in  $M_n(A)$ . Note that we need to be careful with a left action. We consider the  $A_\theta$ - $A_\theta$ -equivalence bimodules from this viewpoint. But we analyze the  $A_\theta$ - $A_\theta$ -equivalence bimodules by the similar arguments in [11]. We show that  $\tau_\theta \otimes Tr(q)$  is an invertible element of an order of a real quadratic field. The unit group (invertible elements) of real quadratic fields is an important invariant in algebraic number theory. Hence the  $A_\theta$ - $A_\theta$ -equivalence bimodules are deeply related to the real quadratic fields.

In Section 3 we construct  $C^*$ -algebras associated with the bimodules considered in Section 2. We show that these algebras are purely infinite, simple, nuclear and in the UCT class. Therefore these  $C^*$ -algebras are classifiable by  $K$ -groups [9], [16].

In Section 4 we compute the  $K$ -groups of the associated  $C^*$ -algebras and show that these groups are related to the solutions of Pell's equation, where Pell's equation is  $x^2 - Dy^2 = \pm 4$  for a given integer  $D$ , to be solved in integers  $x$  and  $y$ . Moreover, we consider the Morita equivalent classes of the associated  $C^*$ -algebras. We show that the Morita equivalent classes of the associated  $C^*$ -algebras depend on the discriminants of irrational numbers. Let  $\theta_1$  and  $\theta_2$  be quadratic irrational numbers such that the discriminant of  $\theta_1$  is equal to the discriminant of  $\theta_2$ . In general  $A_{\theta_1}$  is not Morita equivalent to  $A_{\theta_2}$ . We denote by  $D_{\theta_1}$  the discriminant of  $\theta_1$ . If either  $D_{\theta_1} \equiv 1 \pmod{4}$  is square free or  $D_{\theta_1} \equiv 8$  or  $12 \pmod{16}$  and  $D_{\theta_1}/4$  is square free and the class number of  $\mathbb{Q}(\theta_1)$  is equal to 1, then  $A_{\theta_1}$  is Morita equivalent to  $A_{\theta_2}$ . See [3] and [7] for basic facts of the class numbers of real quadratic fields. But there exists no relation between  $A_{\theta_1}$  and  $A_{\theta_2}$  in general. Hence this result is very surprising.

## 2. REAL MULTIPLICATION

In this section, we shall consider the  $A_\theta$ - $A_\theta$ -equivalence bimodules and review the result in [11] from a slightly different perspective. We refer the reader to [1] for basic facts of  $C^*$ -algebras and equivalence bimodules.

Let  $A$  be a unital  $C^*$ -algebra. Then an  $A$ - $A$ -equivalence bimodule  $\mathcal{E}$  is a finitely generated projective  $A$ -module as a right module [20]. Hence  $\mathcal{E}$  is isomorphic to  $qA^n$  as a right Hilbert  $A$ -module, where  $q$  is a projection in  $M_n(A)$ . A right Hilbert  $A$ -module  $qA^n$  has a structure of a  $qM_n(A)q$ - $A$ -equivalence bimodule with

the obvious left  $qM_n(A)q$ -action and the  $qM_n(A)q$ -valued inner product such that  ${}_{qM_n(A)q}\langle q(a_i)_i, q(b_i)_i \rangle = q(a_j b_k^*)_{j,k} q$  for any  $a_i, b_i \in A$ . Since  $\mathcal{E}$  is an  $A$ - $A$ -equivalence bimodule,  $qM_n(A)q$  is isomorphic to  $A$ . Note that the left  $A$ -action on  $qA^n$  and the left  $A$ -valued inner product are dependent on an isomorphism  $\phi$  from  $A$  to  $qM_n(A)q$ . Therefore we need to be careful with an isomorphism from  $A$  to  $qM_n(A)q$ .

We shall consider the condition of  $q$  that  $qA_\theta^n$  is an  $A_\theta$ - $A_\theta$ -equivalence bimodule. We need the following well-known fact [22].

**Lemma 2.1.** *Let  $q_1$  and  $q_2$  be projections in  $A_\theta$ . If  $\tau_\theta(q_1) \geq \tau_\theta(q_2)$  (resp.  $\tau_\theta(q_1) = \tau_\theta(q_2)$ ), then there exists a unitary element  $w$  in  $A_\theta$  such that  $q_1 \geq w^* q_2 w$  (resp.  $q_1 = w^* q_2 w$ ).*

Let  $Tr_\theta := \tau_\theta \otimes Tr$  be the unnormalized trace on  $M_n(A_\theta)$ , where  $Tr$  is the usual trace on  $M_n(\mathbb{C})$ . We have the following lemma (see [10] and [13]).

**Lemma 2.2.** *If  $q$  is a proper projection in  $M_n(A_\theta)$  such that  $Tr_\theta(q) = k(c\theta + d)$ , where  $k$  is a natural number and  $c, d$  are integers such that  $\gcd(c, d) = 1$ , then  $qM_n(A_\theta)q \cong M_k(A_{\frac{c\theta+b}{c\theta+d}})$  for any  $a, b \in \mathbb{Z}$  such that  $ad - bc = \pm 1$ .*

Since  $qA_\theta^n$  is a  $qM_n(A_\theta)q$ - $A_\theta$ -equivalence bimodule, the condition of  $q$  is equivalent to  $Tr_\theta(q) = c\theta + d$ , where there exist integers  $a$  and  $b$  such that  $\frac{a\theta+b}{c\theta+d} = \theta$  by the lemma above. We shall consider the case where  $\theta$  is not a quadratic number. The following proposition is Corollary 9 in [11].

**Proposition 2.3.** *Let  $\theta$  be an irrational number and  $\mathcal{E}$  an  $A_\theta$ - $A_\theta$ -equivalence bimodule. Assume that  $\theta$  is not a quadratic number. Then there exists an automorphism  $\alpha$  such that  $\mathcal{E}$  is isomorphic to  $\mathcal{E}_\alpha$ .*

We shall consider the case where  $\theta$  is a quadratic irrational number. We may assume that  $\theta$  satisfies  $k\theta^2 + l\theta + m = 0$  with a natural number  $k$  and integers  $l, m$  such that  $\gcd(k, l, m) = 1$ . The equation is uniquely determined. Let  $D_\theta = l^2 - 4km$  be the discriminant of  $\theta$ . The following fact is well known (see [6] and [10, Remark 7]).

*Fact 2.4.* Let  $\theta$  be a quadratic number. Then there exists a real number  $\epsilon_0$  such that

$$\{c\theta + d : \frac{a\theta + b}{c\theta + d} = \theta, a, b, c, d \in \mathbb{Z}, ad - bc = \pm 1\} = \{\pm \epsilon_0^n : n \in \mathbb{Z}\}.$$

The number  $\epsilon_0$  is only dependent on  $D_\theta$ .

We may assume  $\epsilon_0 > 1$ . A real number  $\epsilon_0$  is called the fundamental unit. It is known that the set  $\{\pm \epsilon_0^m; m \in \mathbb{Z}\}$  is the group of units (invertible elements) of  $\mathbb{Z}[k\theta]$  which is an order of a real quadratic field  $\mathbb{Q}(\theta)$ . Especially, if either  $D_\theta \equiv 1 \pmod{4}$  is square free or  $D_\theta \equiv 8$  or  $12 \pmod{16}$  and  $D_\theta/4$  is square free, then  $\mathbb{Z}[k\theta]$  is a ring of integers of  $\mathbb{Q}(\theta)$ . We refer the reader to [3] and [7] for basic facts of the quadratic number fields. We obtain the following proposition.

**Proposition 2.5.** *Let  $\theta$  be a quadratic number with  $k\theta^2 + l\theta + m = 0$ , and let  $q$  be a projection in  $M_n(A_\theta)$  for  $n \in \mathbb{N}$ . Then  $qA^n$  is an  $A_\theta$ - $A_\theta$ -equivalence bimodule if and only if  $Tr_\theta(q)$  is a unit (an invertible element) in  $\mathbb{Z}[k\theta]$  which is an order of a real quadratic field  $\mathbb{Q}(\theta)$ .*

Since  $Tr_\theta(q)$  is a positive number, we have

$$\{Tr_\theta(q) : qA_\theta^n \text{ is an } A_\theta\text{-}A_\theta\text{-equivalence bimodule}\} = \{\epsilon_0^m : m \in \mathbb{Z}\}.$$

We shall consider the relation between the Picard groups of the irrational rotation algebras and the unit groups of the real quadratic fields. We shall show that the  $A_\theta$ - $A_\theta$ -equivalence bimodules are of simple form.

**Lemma 2.6.** *Let  $\mathcal{E}$  be an  $A_\theta$ - $A_\theta$ -equivalence bimodule. Then there exist a projection  $q \in A_\theta$  and an isomorphism  $\phi$  of  $A_\theta$  to  $qA_\theta q$  such that  $\mathcal{E}$  is isomorphic to  $qA_\theta$  with the obvious right  $A_\theta$ -action, the obvious right  $A_\theta$ -valued inner product, the left  $A_\theta$ -action such that  $a \cdot qb = \phi(a)b$  and the left  $A_\theta$ -valued inner product such that  ${}_A\langle qa, qb \rangle = \phi^{-1}(qab^*q)$  or  $A_\theta q$  with the obvious left  $A_\theta$ -action, the obvious left  $A_\theta$ -valued inner product, the right  $A_\theta$ -action such that  $bq \cdot a = b\phi(a)$  and the right  $A_\theta$ -valued inner product such that  $\langle aq, bq \rangle_{A_\theta} = \phi^{-1}(qa^*bq)$  for any  $a, b \in A_\theta$ .*

*Proof.* There exist a natural number  $n$  and a projection  $q \in M_n(A_\theta)$  such that  $\mathcal{E}$  is isomorphic to  $qA_\theta^n$ . We denote by  $\rho$  an isomorphism of  $A_\theta$  to  $qM_n(A_\theta)q$  such that the left  $A_\theta$ -action of  $qA_\theta^n$  is defined by  $a \cdot \xi = \rho(a)\xi$  for any  $a \in A_\theta$  and  $\xi \in qA_\theta^n$ .

(1) The case  $Tr_\theta(q) \leq 1$ .

By Lemma 2.1, there exists a unitary element  $w$  in  $M_n(A_\theta)$  such that  $w^*qw = q' \otimes e_{11}$ , where  $e_{11}$  is a rank one projection in  $M_n(\mathbb{C})$  and  $q'$  is a projection in  $A_\theta$ . Since  $w^*qwA_\theta^n$  where the left  $A_\theta$ -action is defined by  $a \cdot \xi = w^*\rho(a)w\xi$  for any  $a \in A_\theta$  and  $\xi \in w^*qwA_\theta^n$  is isomorphic to  $qA_\theta^n$ ,  $\mathcal{E}$  is isomorphic to  $(q' \otimes e_{11})A_\theta^n$ . It is easy to see that  $(q' \otimes e_{11})A_\theta^n$  is isomorphic to  $q'A_\theta$ .

(2) The case  $Tr_\theta(q) \geq 1$ .

By Lemma 2.1, there exists a unitary element  $w$  in  $M_n(A_\theta)$  such that  $w^*qw \geq 1 \otimes e_{11}$ . It is easy to see that  $w^*qwM_n(A_\theta)(1 \otimes e_{11})$  is isomorphic to  $w^*qwA_\theta^n$ . Hence  $\mathcal{E}$  is isomorphic to  $w^*qwM_n(A_\theta)(1 \otimes e_{11})$ . Since  $w^*qw(1 \otimes e_{11}) = 1 \otimes e_{11}$ ,  $\mathcal{E}$  is isomorphic to  $w^*qwM_n(A_\theta)w^*qw(1 \otimes e_{11})$ . Define a map  $\psi$  from  $A_\theta$  to  $w^*qwM_n(A_\theta)w^*qw$  by  $\psi(a) = w^*\rho(a)w$  for any  $a \in A_\theta$ . Then  $\psi$  is an isomorphism of  $A_\theta$  to  $w^*qwM_n(A_\theta)w^*qw$ . Therefore  $\mathcal{E}$  is isomorphic to  $A_\theta\psi^{-1}(1 \otimes e_{11})$ , where the right  $A_\theta$ -action is defined by  $b\psi^{-1}(1 \otimes e_{11}) \cdot a = b\psi^{-1}(a \otimes e_{11})$  for any  $a, b \in A_\theta$ . □

*Remark 2.7.* (1) By the proof above,  $A_\theta q_1$  is isomorphic to  $q_2 A_\theta^n$  such that  $Tr_\theta(q_2) = \tau_\theta(q_1)^{-1}$ .

(2) Lemma 2.6 is important in the next section.

(3) Let  $A$  be a unital simple  $C^*$ -algebra with a unique normalized trace  $\tau$ . Assume that the normalized trace on  $A$  separates equivalence classes of projections. Then  $A$  satisfies the conclusion of Lemma 2.6.

We denote by  $\mathcal{E}_{\theta,\phi,q}$  (resp.  $\mathcal{F}_{\theta,\phi,q}$ ) the  $A_\theta$ - $A_\theta$ -equivalence bimodule  $qA_\theta$  with the left  $A_\theta$ -action such that  $a \cdot qb = \phi(a)b$  (resp.  $A_\theta q$  with the right  $A_\theta$ -action such that  $bq \cdot a = b\phi(a)$ ) for any  $a, b \in A_\theta$ . We call  $\mathcal{E}_{\theta,\phi,q}$  and  $\mathcal{F}_{\theta,\phi,q}$  *real multiplication* in the case  $q \neq 1$ . We shall consider the Picard groups of the irrational rotation algebras.

**Proposition 2.8.** *Let  $q_1$  and  $q_2$  be projections in  $A_\theta$  such that  $A_\theta$  is isomorphic to  $q_1 A_\theta q_1$  and  $q_2 A_\theta q_2$ . Assume that  $\phi_1$  (resp.  $\phi_2$ ) is an isomorphism of  $A_\theta$  to  $q_1 A_\theta q_1$  (resp.  $q_2 A_\theta q_2$ ). Then  $\mathcal{E}_{\theta,\phi_1,q_1} \otimes \mathcal{E}_{\theta,\phi_2,q_2}$  is isomorphic to  $\mathcal{E}_{\theta,\phi_2 \circ \phi_1,q}$ , where  $q$  is a projection in  $A_\theta$  such that  $\tau_\theta(q) = \tau_\theta(q_1)\tau_\theta(q_2)$ . Moreover,  $\mathcal{E}_{\theta,\phi_1,q_1} \otimes \mathcal{F}_{\theta,\phi_1,q_1}$  and  $\mathcal{F}_{\theta,\phi_1,q_1} \otimes \mathcal{E}_{\theta,\phi_1,q_1}$  are isomorphic to  $A_\theta$  with the obvious actions and the obvious inner products.*

*Proof.* Let  $q = \phi_2(q_1)$ . Then  $\tau_\theta(q) = \tau_\theta(q_1)\tau_\theta(q_2)$  and  $q \leq q_2$ . Define a map  $F$  from  $\mathcal{E}_{\theta,\phi_1,q_1} \otimes \mathcal{E}_{\theta,\phi_2,q_2}$  to  $\mathcal{E}_{\theta,\phi_2 \circ \phi_1,q}$  by  $F(q_1a \otimes q_2b) = q\phi_2(a)b$  for any  $a, b \in A_\theta$  and extend it by the universality. Since  $qq_2 = q$ , if  $q_1 \otimes q_2a = q_1 \otimes q_2b$ , then  $qa = qb$ . Hence  $F$  is well-defined. Easy computations show that  $F$  is an isomorphism of equivalence bimodules. Since  $\mathcal{F}_{\theta,\phi_1,q_1}$  is a dual module of  $\mathcal{E}_{\theta,\phi_1,q_1}$ ,  $\mathcal{E}_{\theta,\phi_1,q_1} \otimes \mathcal{F}_{\theta,\phi_1,q_1}$  and  $\mathcal{F}_{\theta,\phi_1,q_1} \otimes \mathcal{E}_{\theta,\phi_1,q_1}$  are isomorphic to  $A_\theta$  with the obvious actions and the obvious inner products.  $\square$

K. Kodaka showed that if  $\theta$  is a quadratic number, then the Picard group of  $A_\theta$  is isomorphic to a semidirect product of  $\text{Aut}(A_\theta)/\text{Inn}(A_\theta)$  with  $\mathbb{Z}$  [11]. By Remark 2.7 and Proposition 2.8, the  $\mathbb{Z}$  part is related to the unit group of  $\mathbb{Z}[k\theta]$  which is an order of a real quadratic field  $\mathbb{Q}(\theta)$ . This is shown by another method in [14].

### 3. ASSOCIATED $C^*$ -ALGEBRAS

In this section, we shall construct  $C^*$ -algebras associated with real multiplication. We show that these algebras are simple and purely infinite. We recall Cuntz-Pimsner algebras [17]. Let  $A$  be a  $C^*$ -algebra and  $\mathcal{E}$  a right Hilbert  $A$ -module. For  $\xi, \eta \in \mathcal{E}$ , the “rank one operator”  $\Theta_{\xi,\eta}$  is defined by  $\Theta_{\xi,\eta}(\zeta) = \xi\langle\eta, \zeta\rangle_A$  for any  $\zeta \in \mathcal{E}$ . We denote by  $B_A(\mathcal{E})$  the algebra of the adjointable operators on  $\mathcal{E}$  and by  $K_A(\mathcal{E})$  the closure of the linear span of rank one operators of  $\mathcal{E}$ . We say that  $\mathcal{E}$  is a *Hilbert bimodule* over  $A$  if  $\mathcal{E}$  is a right Hilbert  $A$ -module with a homomorphism  $\phi : A \rightarrow B_A(\mathcal{E})$ . We assume that  $\mathcal{E}$  is full and  $\phi$  is injective. Define  $I_\mathcal{E} = \phi^{-1}(K_A(\mathcal{E}))$ . The Cuntz-Pimsner algebra  $\mathcal{O}_\mathcal{E}$  is the universal  $C^*$ -algebra generated by  $A$  and  $\{S_\xi : \xi \in \mathcal{E}\}$  with the relation

$$S_{\alpha\xi+\beta\eta} = \alpha S_\xi + \beta S_\eta, \quad aS_\xi b = S_{\phi(a)\xi} b, \quad S_\xi^* S_\eta = \langle \xi, \eta \rangle_A$$

for any  $a, b \in A$ ,  $\xi, \eta \in \mathcal{E}$ ,  $\alpha, \beta \in \mathbb{C}$  and

$$i_K(\phi(a)) = a$$

for  $a \in I_\mathcal{E}$ , where  $i_K : I_\mathcal{E} \rightarrow \mathcal{O}_\mathcal{E}$  is defined by  $i_K(\Theta_{\xi,\eta}) = S_\xi S_\eta^*$ . We shall consider the Cuntz-Pimsner algebras generated by the equivalence bimodules of irrational rotation algebra that are not generated by automorphisms. By Lemma 2.6, an equivalence bimodule of an irrational rotation algebra  $A_\theta$  is isomorphic to  $\mathcal{E}_{\theta,\phi,q}$  or  $\mathcal{F}_{\theta,\phi,q}$ . It is easy to see that the Cuntz-Pimsner algebra generated by  $\mathcal{E}_{\theta,\phi,q}$  is isomorphic to the Cuntz-Pimsner algebra generated by  $\mathcal{F}_{\theta,\phi,q}$ . We denote by  $\mathcal{O}_{qA_\theta,\phi}$  the Cuntz-Pimsner algebra generated by  $\mathcal{E}_{\theta,\phi,q}$ . Since  $\mathcal{E}_{\theta,\phi,q}$  is an equivalence bimodule,  $K_{A_\theta}(\mathcal{E}_{\theta,\phi,q})$  is isomorphic to  $A_\theta$  and  $S_\xi S_\eta^* = {}_{A_\theta}\langle \xi, \eta \rangle$  for  $\xi, \eta \in \mathcal{E}_{\theta,\phi,q}$ . Hence  $\mathcal{O}_{qA_\theta,\phi}$  is the universal  $C^*$ -algebra generated by  $A_\theta$  and  $S_q$  with the following relation: for  $a \in A_\theta$ ,

$$aS_q = S_q\phi(a), \quad S_q^* S_q = \langle q, q \rangle_{A_\theta} = q, \quad S_q S_q^* = {}_{A_\theta}\langle q, q \rangle = \phi^{-1}(q) = 1.$$

Note that  $\phi$  is regarded as an endomorphism such that  $\phi : A_\theta \rightarrow qA_\theta q \subseteq A_\theta$ . An endomorphism  $\rho$  on a unital  $C^*$ -algebra  $A$  is called a corner endomorphism if its image is equal to the corner  $\rho(1)A\rho(1)$  of  $A$ . It is easy to see that  $\mathcal{O}_{qA_\theta,\phi}$  is isomorphic to the corner endomorphism crossed product  $A_\theta \rtimes_\phi \mathbb{N}$ . (See [15] and [24].)

**Theorem 3.1.** *Let  $\mathcal{E}_{\theta,q,\phi}$  be an  $A_\theta$ - $A_\theta$ -equivalence bimodule that is not generated by an automorphism. Then the Cuntz-Pimsner algebra  $\mathcal{O}_{qA_\theta,\phi}$  generated by  $\mathcal{E}_{\theta,q,\phi}$  is purely infinite, simple, nuclear and in the UCT class.*

*Proof.* By [24] (Theorem 3.1),  $\mathcal{O}_{qA_\theta, \phi}$  is simple and purely infinite because  $A_\theta$  is a simple unital  $C^*$ -algebra of real rank zero and with the property of Lemma 2.1. Since  $A_\theta$  is nuclear and in the UCT class,  $\mathcal{O}_{qA_\theta, \phi}$  is nuclear and in the UCT class.  $\square$

The isomorphism class of the  $C^*$ -algebra  $\mathcal{O}_{qA_\theta, \phi}$  is completely determined by the  $K$ -groups together with the class of the unit by the classification theorem by Kirchberg-Phillips [9], [16]. Especially, the Morita equivalence class of  $C^*$ -algebra  $\mathcal{O}_{qA_\theta, \phi}$  is completely determined by the  $K$ -groups.

4.  $K$ -GROUPS

In this section, we shall compute the  $K$ -groups of  $\mathcal{O}_{qA_\theta, \phi}$ . Let  $D_{\mathcal{E}_{\theta, q, \phi}}$  be the linking algebra of  $\mathcal{E}_{\theta, q, \phi}$ . The linking algebra of  $D_{\mathcal{E}_{\theta, q, \phi}}$  has the following form:

$$D_{\mathcal{E}_{\theta, q, \phi}} = \left\{ \begin{pmatrix} x & \xi \\ \eta & y \end{pmatrix} : x \in K_{A_\theta}(\mathcal{E}_{\theta, q, \phi}) = qA_\theta q, y \in A_\theta, \xi \in \mathcal{E}_{\theta, q, \phi}, \eta \in \mathcal{E}_{\theta, q, \phi}^* \right\},$$

where  $\mathcal{E}_{\theta, q, \phi}^*$  is the dual module of  $\mathcal{E}_{\theta, q, \phi}$ . The linking algebra  $D_{\mathcal{E}_{\theta, q, \phi}}$  has the unique normalized trace  $Tr_D$  such that  $Tr_D\left(\begin{pmatrix} x & \xi \\ \eta & y \end{pmatrix}\right) = \frac{\tau_\theta(x) + \tau_\theta(y)}{1 + \tau_\theta(q)}$ . The natural embeddings are denoted by

$$i_{K_{A_\theta}(\mathcal{E}_{\theta, q, \phi})} : K_{A_\theta}(\mathcal{E}_{\theta, q, \phi}) \rightarrow D_{\mathcal{E}_{\theta, q, \phi}}, i_{A_\theta} : A_\theta \rightarrow D_{\mathcal{E}_{\theta, q, \phi}}, i : I_{\mathcal{E}_{\theta, q, \phi}} \rightarrow A_\theta.$$

By [8] (Proposition B.3), the inclusion  $i_{A_\theta} : A_\theta \rightarrow D_{\mathcal{E}_{\theta, q, \phi}}$  induces an isomorphism of the  $K$ -groups. We can define a map  $K_*([\mathcal{E}_{\theta, q, \phi}]) : K_*(I_{\mathcal{E}_{\theta, q, \phi}}) \rightarrow K_*(A_\theta)$  by the composition of the map  $K_*(\phi)$  induced by the restriction of  $\phi$  to  $I_{\mathcal{E}_{\theta, q, \phi}}$ , the map  $K_*(i_{K_{A_\theta}(\mathcal{E}_{\theta, q, \phi})})$  induced by  $i_{K_{A_\theta}(\mathcal{E}_{\theta, q, \phi})}$  and the inverse of the isomorphism  $K_*(i_{A_\theta})$ . We have the following exact sequence [8]:

$$\begin{array}{ccccc} K_0(I_{\mathcal{E}_{\theta, q, \phi}}) & \xrightarrow{K_0(i) - K_0([\mathcal{E}_{\theta, q, \phi}])} & K_0(A_\theta) & \longrightarrow & K_0(\mathcal{O}_{qA_\theta, \phi}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{O}_{qA_\theta, \phi}) & \longleftarrow & K_1(A_\theta) & \xleftarrow{K_1(i) - K_1([\mathcal{E}_{\theta, q, \phi}])} & K_1(I_{\mathcal{E}_{\theta, q, \phi}}). \end{array}$$

Since  $\mathcal{E}_{\theta, q, \phi}$  is an equivalence bimodule,  $K_*(i_{K_{A_\theta}(\mathcal{E}_{\theta, q, \phi})})$  is an isomorphism and  $I_{\mathcal{E}_{\theta, q, \phi}} = A_\theta$ . Therefore  $K_*([\mathcal{E}_{\theta, q, \phi}]) \in GL(2, \mathbb{Z})$  because  $K_0(A_\theta) \cong K_1(A_\theta) \cong \mathbb{Z}^2$  and  $K_*(\phi)$  is an isomorphism. For any  $g \in GL(2, \mathbb{Z})$ , there exists an automorphism  $\alpha$  of  $A_\theta$  such that  $K_1(\alpha) = g$  [5]. Hence  $K_1([\mathcal{E}_{\theta, q, \phi}])$  depends on  $\phi$  and can be any element in  $GL(2, \mathbb{Z})$ . Let  $B_{\theta, q, \phi} = K_1(i) - K_1([\mathcal{E}_{\theta, q, \phi}])$ . We shall consider  $K_0([\mathcal{E}_{\theta, q, \phi}])$ .

**Proposition 4.1.** *Let  $\mathcal{E}_{\theta, q, \phi}$  be an  $A_\theta$ - $A_\theta$ -equivalence bimodule such that  $\tau_\theta(q) = c\theta + d$  for  $c, d \in \mathbb{Z}$  and  $0 < \theta < 1$ . Then there exist integers  $a$  and  $b$  such that  $\frac{a\theta + b}{c\theta + d} = \theta$  and  $K_0([\mathcal{E}_{\theta, q, \phi}]) = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$  for some basis of  $\mathbb{Z}^2$ .*

*Proof.* We can choose  $\langle [1], [p] \rangle$ , where  $\tau_\theta(p) = \theta$  as a basis of  $K_0(A_\theta)$  (see [18] and [19]). Since  $K_0(i_{A_\theta})$  is an isomorphism,  $\langle [i_{A_\theta}(1)], [i_{A_\theta}(p)] \rangle$  is a basis of  $K_0(D_{\mathcal{E}_{\theta, q, \phi}})$ . Easy computations show that  $Tr_D(i_{K_{A_\theta}(\mathcal{E}_{\theta, q, \phi})}(\phi(1))) = \frac{\tau_\theta(q)}{1 + \tau_\theta(q)}$ ,  $Tr_D(i_{K_{A_\theta}(\mathcal{E}_{\theta, q, \phi})}(\phi(p))) = \frac{\theta\tau_\theta(q)}{1 + \tau_\theta(q)}$ ,  $Tr_D(i_{A_\theta}(1)) = \frac{1}{1 + \tau_\theta(q)}$  and  $Tr_D(i_{A_\theta}(p)) = \frac{\theta}{1 + \tau_\theta(q)}$ . We denote by  $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$  the matrix  $K_0([\mathcal{E}_{\theta, q, \phi}])$  with respect to the

basis  $\langle [i_{A_\theta}(1)], [i_{A_\theta}(p)] \rangle$ . Then we have  $\frac{\tau_\theta(q)}{1+\tau_\theta(q)} = \frac{x_{11}+x_{21}\theta}{1+\tau_\theta(q)}$  and  $\frac{\theta\tau_\theta(q)}{1+\tau_\theta(q)} = \frac{x_{12}+x_{22}\theta}{1+\tau_\theta(q)}$ . By the discussion in Section 2, there exist integers  $a$  and  $b$  such that  $\frac{a\theta+b}{c\theta+d} = \theta$ . Therefore we have  $x_{11} = d, x_{12} = b, x_{21} = c$  and  $x_{22} = a$ .  $\square$

By the proposition above, we shall show some examples.

**Example 4.2.** Let  $\theta = \frac{-1+\sqrt{5}}{2}$  and  $q$  be a projection in  $A_\theta$  such that  $\tau_\theta(q) = \frac{-1+\sqrt{5}}{2} = \theta$ . Then  $\frac{-\theta+1}{\theta} = \theta$  and  $K_0([\mathcal{E}_{\theta,q,\phi}]) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$ . By the exact sequence, the  $K$ -groups have the following form:

$$K_0(\mathcal{O}_{qA_\theta,\phi}) \cong \ker(B_{\theta,q,\phi}), \quad K_1(\mathcal{O}_{qA_\theta,\phi}) \cong \mathbb{Z}^2/\text{Im}(B_{\theta,q,\phi}).$$

**Example 4.3.** Let  $\theta = \frac{5+\sqrt{5}}{10}$  and  $q$  be a projection in  $A_\theta$  such that  $\tau_\theta(q) = \frac{-1+\sqrt{5}}{2} = 5\theta - 3$ . Then  $\frac{2\theta-1}{5\theta-3} = \theta$  and  $K_0([\mathcal{E}_{\theta,q,\phi}]) = \begin{pmatrix} -3 & -1 \\ 5 & 2 \end{pmatrix}$ . By the exact sequence, the  $K$ -groups have the following form:

$$K_0(\mathcal{O}_{qA_\theta,\phi}) \cong \ker(B_{\theta,q,\phi}), \quad K_1(\mathcal{O}_{qA_\theta,\phi}) \cong \mathbb{Z}^2/\text{Im}(B_{\theta,q,\phi}).$$

**Example 4.4.** Let  $\theta = \frac{-1+\sqrt{5}}{2}$  and  $q$  be a projection in  $A_\theta$  such that  $\tau_\theta(q) = \sqrt{5} - 2 = 2\theta - 1$ . Then  $\frac{-3\theta+2}{2\theta-1} = \theta$  and  $K_0([\mathcal{E}_{\theta,q,\phi}]) = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$ . By the exact sequence, the  $K$ -groups have the following form:

$$K_0(\mathcal{O}_{qA_\theta,\phi}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \ker(B_{\theta,q,\phi}), \quad K_1(\mathcal{O}_{qA_\theta,\phi}) \cong \mathbb{Z}^2/\text{Im}(B_{\theta,q,\phi}).$$

We shall consider the condition of  $\theta_1, \theta_2, q_1, q_2$  that  $\mathcal{O}_{q_1A_{\theta_1},\phi_1}$  is Morita equivalent to  $\mathcal{O}_{q_2A_{\theta_2},\phi_2}$ . The following proposition is natural.

**Proposition 4.5.** *Let  $\mathcal{E}_{\theta_1,q_1,\phi_1}$  be an  $A_{\theta_1}$ - $A_{\theta_1}$ -equivalence bimodule and  $\mathcal{E}_{\theta_2,q_2,\phi_2}$  an  $A_{\theta_2}$ - $A_{\theta_2}$ -equivalence bimodule such that  $A_{\theta_1}$  is Morita equivalent to  $A_{\theta_2}$  and  $\tau_{\theta_1}(q_1) = \tau_{\theta_2}(q_2)$ . Assume that  $\mathbb{Z}^2/\text{Im}(B_{\theta_1,q_1,\phi_1})$  is isomorphic to  $\mathbb{Z}^2/\text{Im}(B_{\theta_2,q_2,\phi_2})$ . Then  $\mathcal{O}_{q_1A_{\theta_1},\phi_1}$  is Morita equivalent to  $\mathcal{O}_{q_2A_{\theta_2},\phi_2}$ .*

*Proof.* We may assume that  $0 < \theta_1, \theta_2 < 1$ . Since  $A_{\theta_1}$  is Morita equivalent to  $A_{\theta_2}$ , there exists  $g \in GL(2, \mathbb{Z})$  such that  $\theta_1 = g\theta_2$  by Theorem 4 in [19]. There exist  $c_1, c_2, d_1, d_2 \in \mathbb{Z}$  such that  $\tau_{\theta_1}(q_1) = \tau_{\theta_2}(q_2) = c_1\theta_1 + d_1 = c_2\theta_2 + d_2$ . The discussion in Section 2 shows that there exist  $a_1, a_2, b_1, b_2 \in \mathbb{Z}$  such that  $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\theta_1 = \theta_1$  and  $\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\theta_2 = \theta_2$ . By the computation, we have  $g\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}g^{-1} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ . Hence  $\text{Ker}(K_0(i) - K_0([\mathcal{E}_{\theta_1,q_1,\phi_1}]))$  is isomorphic to  $\text{Ker}(K_0(i) - K_0([\mathcal{E}_{\theta_2,q_2,\phi_2}]))$ , and  $\text{Im}(K_0(i) - K_0([\mathcal{E}_{\theta_1,q_1,\phi_1}]))$  is isomorphic to  $\text{Im}(K_0(i) - K_0([\mathcal{E}_{\theta_2,q_2,\phi_2}]))$  by Proposition 4.1. Therefore the  $K$ -groups of  $\mathcal{O}_{q_1A_{\theta_1},\phi_1}$  are isomorphic to the  $K$ -groups of  $\mathcal{O}_{q_2A_{\theta_2},\phi_2}$ . Consequently  $\mathcal{O}_{q_1A_{\theta_1},\phi_1}$  is Morita equivalent to  $\mathcal{O}_{q_2A_{\theta_2},\phi_2}$  by Theorem 3.1 and the classification theorem by Kirchberg-Phillips [9], [16].  $\square$

We shall show that the  $K_0$ -group of  $\mathcal{O}_{qA_\theta,\phi}$  is related to Pell's equation. We shall review some facts of elementary number theory. Let  $\theta$  be a quadratic irrational number. We may assume that  $\theta$  satisfies  $k\theta^2 + l\theta + m = 0$  with a natural number  $k$  and integers  $l, m$  such that  $\text{gcd}(k, l, m) = 1$ . The equation is uniquely determined. Let  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  be an element in  $GL(2, \mathbb{Z})$  such that  $\frac{a\theta+b}{c\theta+d} = \theta$ . Then the  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  can be written in the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{t+lu}{2} & ku \\ -mu & \frac{t-lu}{2} \end{pmatrix},$$

where  $t, u$  are integers such that

$$\begin{aligned} t^2 - D_\theta u^2 &= 4 \quad \text{if } ad - bc = 1, \\ t^2 - D_\theta u^2 &= -4 \quad \text{if } ad - bc = -1. \end{aligned}$$

The equation above is called Pell's equation. We see that  $c\theta + d = \frac{t+u\sqrt{D_\theta}}{2}$ . Especially, if  $t, u > 0$  are minimum integers satisfying one of the equations above, then  $\frac{t+u\sqrt{D_\theta}}{2}$  is the fundamental unit. We shall determine the  $K$ -groups of  $\mathcal{O}_{qA_\theta, \phi}$ .

**Theorem 4.6.** *Let  $\mathcal{E}_{\theta, q, \phi}$  be an  $A_\theta$ - $A_\theta$ -equivalence bimodule such that  $\tau_\theta(q) = \frac{t+u\sqrt{D_\theta}}{2}$ , where  $D_\theta$  is the discriminant of  $\theta$ . Then  $B_{\theta, q, \phi}$  can be any element in  $\{1 - g; g \in GL(2, \mathbb{Z})\}$  by the choice of  $\phi$  and the  $K$ -groups have the following form:*  
 (1) *If  $t^2 - D_\theta u^2 = 4$ , then*

$$K_0(\mathcal{O}_{qA_\theta, \phi}) \cong \mathbb{Z}/s_1\mathbb{Z} \oplus \mathbb{Z}/\frac{2-t}{s_1}\mathbb{Z} \oplus \ker(B_{\theta, q, \phi}), \quad K_1(\mathcal{O}_{qA_\theta, \phi}) \cong \mathbb{Z}^2/\text{Im}(B_{\theta, q, \phi}),$$

where  $s_1$  is a maximal integer such that  $2 - t = es_1^2$  and  $u = fs_1$  for some  $e, f \in \mathbb{Z}$ .

(2) *If  $t^2 - D_\theta u^2 = -4$ , then*

$$K_0(\mathcal{O}_{qA_\theta, \phi}) \cong \mathbb{Z}/s_2\mathbb{Z} \oplus \mathbb{Z}/\frac{t}{s_2}\mathbb{Z} \oplus \ker(B_{\theta, q, \phi}), \quad K_1(\mathcal{O}_{qA_\theta, \phi}) \cong \mathbb{Z}^2/\text{Im}(B_{\theta, q, \phi}),$$

where  $s_2$  is a maximal integer such that  $t = es_2^2$  and  $u = fs_2$  for some  $e, f \in \mathbb{Z}$ .

*Proof.* We may assume  $0 < \theta < 1$ . Let  $\theta' = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \theta$  and  $q'$  be a projection in  $A_{\theta'}$  such that  $\tau_{\theta'}(q') = \tau_\theta(q)$ . Then there exists  $g \in GL(2, \mathbb{Z})$  such that  $K_0(i) - K_0([\mathcal{E}_{\theta, q, \phi}]) = g^{-1}(K_0(i) - K_0([\mathcal{E}_{\theta', q', \phi}]))g$  by Proposition 4.5. It is easy to see that  $m(\theta')^2 + (l + m)\theta' + k + l + m = 0$  and  $0 < \theta' < 1$ .

*Proof of (1).* The equation  $t^2 - D_\theta u^2 = 4$  gives  $e(2 + t) = -f^2 D_\theta$ . It is easy to see that  $es_1$  is an odd number if and only if  $t$  is an odd number, and  $D_\theta$  is an odd number if and only if  $l$  is an odd number. Hence we see that  $\frac{es_1 - lf}{2}$  and  $\frac{es_1 + lf}{2}$  are integers by elementary computations. Therefore there exist integers  $x_{11}, x_{12}, x_{21}, x_{22}$  such that

$$K_0(i) - K_0([\mathcal{E}_{\theta, q, \phi}]) = \begin{pmatrix} 1 - \frac{t-lu}{2} & -ku \\ mu & 1 - \frac{t+lu}{2} \end{pmatrix} = s_1 \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}.$$

The greatest common divisor of the matrix elements of  $K_0(i) - K_0([\mathcal{E}_{\theta, q, \phi}])$  is equal to the greatest common divisor of the matrix elements of  $K_0(i) - K_0([\mathcal{E}_{\theta', q', \phi}])$  because  $K_0(i) - K_0([\mathcal{E}_{\theta, q, \phi}]) = g^{-1}(K_0(i) - K_0([\mathcal{E}_{\theta', q', \phi}]))g$ . Since we have

$$K_0(i) - K_0([\mathcal{E}_{\theta', q', \phi}]) = \begin{pmatrix} 1 - \frac{t-(l+m)u}{2} & -mu \\ (k+l+m)u & 1 - \frac{t+(l+m)u}{2} \end{pmatrix}$$

and  $\gcd(k, m, k + l + m) = 1$ , we see that  $\gcd(x_{11}, x_{12}, x_{21}, x_{22}) = 1$ . An easy computation shows that  $\det(K_0(i) - K_0([\mathcal{E}_{\theta, q, \phi}])) = 2 - t$ . Hence there exist  $g_1, g_2 \in GL(2, \mathbb{Z})$  such that  $\text{Im}(g_1(K_0(i) - K_0([\mathcal{E}_{\theta, q, \phi}]))g_2) = \text{Im}\left(\begin{smallmatrix} s_1 & 0 \\ 0 & \frac{2-t}{s_1} \end{smallmatrix}\right)$ . By the exact sequence, we obtain the conclusion.  $\square$

*Proof of (2).* By an easy computation,  $\det(K_0(i) - K_0([\mathcal{E}_{\theta, q, \phi}])) = -t$ . The rest is proved similarly as above.  $\square$

We shall show some examples.

**Example 4.7.** Let  $\theta = \frac{-1+\sqrt{5}}{2}$  and  $q$  be a projection in  $A_\theta$  such that  $\tau_\theta(q) = \frac{3-\sqrt{5}}{2}$ . Then the  $K$ -groups have the following form:

$$K_0(\mathcal{O}_{qA_\theta, \phi}) \cong \ker(B_{\theta, q, \phi}), \quad K_1(\mathcal{O}_{qA_\theta, \phi}) \cong \mathbb{Z}^2/\text{Im}(B_{\theta, q, \phi}).$$

**Example 4.8.** Let  $\theta = \sqrt{5} - 2$  and  $q$  be a projection in  $A_\theta$  such that  $\tau_\theta(q) = \sqrt{5} - 2 = \frac{-4+\sqrt{20}}{2}$ . Then the  $K$ -groups have the following form:

$$K_0(\mathcal{O}_{qA_\theta, \phi}) \cong \mathbb{Z}/4\mathbb{Z} \oplus \ker(B_{\theta, q, \phi}), \quad K_1(\mathcal{O}_{qA_\theta, \phi}) \cong \mathbb{Z}^2/\text{Im}(B_{\theta, q, \phi}).$$

Therefore  $\mathcal{O}_{qA_\theta, \phi}$  is not Morita equivalent to the  $C^*$ -algebra in Example 4.4.

The following corollary is an extension of Proposition 4.5.

**Corollary 4.9.** Let  $\mathcal{E}_{\theta_1, q_1, \phi_1}$  be an  $A_{\theta_1}$ - $A_{\theta_1}$ -equivalence bimodule and  $\mathcal{E}_{\theta_2, q_2, \phi_2}$  an  $A_{\theta_2}$ - $A_{\theta_2}$ -equivalence bimodule such that  $D_{\theta_1} = D_{\theta_2}$  and  $\tau_{\theta_1}(q_1) = \tau_{\theta_2}(q_2)$ . Assume that  $\mathbb{Z}^2/\text{Im}(B_{\theta_1, q_1, \phi_1})$  is isomorphic to  $\mathbb{Z}^2/\text{Im}(B_{\theta_2, q_2, \phi_2})$ . Then  $\mathcal{O}_{q_1 A_{\theta_1}, \phi_1}$  is Morita equivalent to  $\mathcal{O}_{q_2 A_{\theta_2}, \phi_2}$ .

*Remark 4.10.* Let  $\theta_1 = \sqrt{10} - 3$  and  $\theta_2 = \frac{2+\sqrt{10}}{3}$ . Then  $D_{\theta_1} = D_{\theta_2} = 40$  and  $A_{\theta_1}$  is not Morita equivalent to  $A_{\theta_2}$ . (This fact is related to the fact that the class number of  $\mathbb{Q}(\sqrt{10})$  is two.) Therefore Corollary 4.9 is an extension of Proposition 4.5. Since there exists no relation between  $A_{\theta_1}$  and  $A_{\theta_2}$ , this result is very surprising.

## REFERENCES

- [1] B. Blackadar, *Operator Algebras: Theory of  $C^*$ -algebras and von Neumann Algebras*, Encyclopaedia of Mathematical Sciences, **122**, Springer, 2006. MR2188261 (2006k:46082)
- [2] L. G. Brown, P. Green and M. A. Rieffel, *Stable isomorphism and strong Morita equivalence of  $C^*$ -algebras*, Pacific J. Math. **71** (1977), no. 2, 349–363. MR0463928 (57:3866)
- [3] J. Buchmann and U. Vollmer, *Binary quadratic forms. An algorithmic approach*, Algorithms and Computation in Mathematics, **20**. Springer, Berlin, 2007. MR2300780 (2008b:11046)
- [4] A. Connes, *Noncommutative Geometry*, Academic Press, Inc., San Diego, CA, 1994. MR1303779 (95j:46063)
- [5] G. Elliott and M. Rørdam, *The automorphism group of the irrational rotation  $C^*$ -algebra*, Comm. Math. Phys. **155** (1993), no. 1, 3–26. MR1228523 (94j:46059)
- [6] D.C.F. Gauss, *Disquisitiones Arithmeticae*, 1801.
- [7] M. Jacobson and H. Williams, *Solving the Pell Equation*, Springer, 2009. MR2466979 (2009i:11003)
- [8] T. Katsura, *On  $C^*$ -algebras associated with  $C^*$ -correspondences*, J. Funct. Anal. **217** (2004), no. 2, 366–401. MR2102572 (2005e:46099)
- [9] E. Kirchberg, *The classification of purely infinite  $C^*$ -algebras using Kasparov's theory*, preprint, 1994.
- [10] K. Kodaka, *Endomorphisms of certain irrational rotation  $C^*$ -algebras*, Illinois J. Math. **36** (1992), no. 4, 643–658. MR1215799 (94b:46086)
- [11] K. Kodaka, *Picard groups of irrational rotation  $C^*$ -algebras*, J. London Math. Soc. (2) **56** (1997), no. 1, 179–188. MR1462834 (98e:46071)

- [12] Y. Manin, *Real multiplication and noncommutative geometry*, in: The legacy of Niels Henrik Abel, Springer-Verlag, Berlin, 2004, 685–727. MR2077591 (2006e:11077)
- [13] N. Nawata, *Morita equivalent subalgebras of irrational rotation algebras and real quadratic fields*, C. R. Math. Acad. Sci. Soc. R. Can. **31** (2009), no. 3, 87–96. MR2555388 (2010j:46107)
- [14] N. Nawata and Y. Watatani, *Fundamental group of simple C\*-algebras with unique trace*, Adv. Math. **225** (2010), no. 1, 307–318. MR2669354
- [15] W. L. Paschke, *The crossed product of a C\*-algebra by an endomorphism*, Proc. Amer. Math. Soc. **80** (1980), no. 1, 113–118. MR574518 (81m:46081)
- [16] N. C. Phillips, *A classification theorem for nuclear purely infinite simple C\*-algebras*, Doc. Math. **5** (2000), 49–114. MR1745197 (2001d:46086b)
- [17] M. Pimsner, *A class of C\*-algebra generalizing both Cuntz-Krieger algebras and crossed products by  $\mathbb{Z}$* , in: Free probability theory, AMS, 1997, 189–212. MR1426840 (97k:46069)
- [18] M. Pimsner and D. Voiculescu, *Imbedding the irrational rotation C\*-algebra into an AF-algebra*, J. Operator Theory **4** (1980), no. 2, 201–210. MR595412 (82d:46086)
- [19] M. A. Rieffel, *C\*-algebras associated with irrational rotations*, Pacific J. Math **93** (1981), no. 2, 415–429. MR623572 (83b:46087)
- [20] M. A. Rieffel, *Morita equivalence for operator algebras*, in: Operator algebras and applications, Part I (Kingston, Ont., 1980), Proc. Sympos. Pure Math., vol. **38**, AMS, Providence, RI, 1982, 285–298. MR679708 (84k:46045)
- [21] M. A. Rieffel, *Applications of strong Morita equivalence to transformation group C\*-algebras*, in: Operator algebras and applications, Part I (Kingston, Ont., 1980), Proc. Sympos. Pure Math., vol. **38**, AMS, Providence, RI, 1982, 299–310. MR679709 (84k:46046)
- [22] M. A. Rieffel, *The cancellation theorem for projective modules over irrational rotation C\*-algebras*, Proc. London Math. Soc. (3) **47** (1983), no. 2, 285–302. MR703981 (85g:46085)
- [23] M. A. Rieffel, *Projective modules over higher-dimensional noncommutative tori*, Canad. J. Math. **40** (1988), no. 2, 257–338. MR941652 (89m:46110)
- [24] M. Rørdam, *Classification of certain infinite simple C\*-algebras*, J. Funct. Anal. **131** (1995), no. 2, 415–458. MR1345038 (96e:46080a)

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