SEMICROSSED PRODUCTS OF THE DISK ALGEBRA

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Abstract. If \( \alpha \) is the endomorphism of the disk algebra, \( A(D) \), induced by composition with a finite Blaschke product \( b \), then the semicrossed product \( A(D) \times_{\alpha} \mathbb{Z}^+ \) imbeds canonically, completely isometrically into \( C(T) \times_{\alpha} \mathbb{Z}^+ \). Hence in the case of a non-constant Blaschke product \( b \), the C*-envelope has the form \( C(S_b) \times_s \mathbb{Z} \), where \( (S_b, s) \) is the solenoid system for \( (T, b) \). In the case where \( b \) is a constant, the C*-envelope of \( A(D) \times_{\alpha} \mathbb{Z}^+ \) is strongly Morita equivalent to a crossed product of the form \( C_0(S_e) \times_s \mathbb{Z} \), where \( e : T \times N \to T \times N \) is a suitable map and \( (S_e, s) \) is the solenoid system for \( (T \times N, e) \).

1. Introduction

If \( A \) is a unital operator algebra and \( \alpha \) is a completely contractive endomorphism, the semicrossed product is an operator algebra \( A \times_{\alpha} \mathbb{Z}^+ \) which encodes the covariant representations of \( (A, \alpha) \): namely completely contractive unital representations \( \rho : A \to B(H) \) and contractions \( T \) satisfying

\[
\rho(a)T = T\rho(\alpha(a)) \quad \text{for all } a \in A.
\]

Such algebras were defined by Peters \cite{Peters} when \( A \) is a C*-algebra.

One can readily extend Peter’s definition \cite{Peters} of the semicrossed product of a C*-algebra by a \( * \)-endomorphism to unital operator algebras and unital completely contractive endomorphisms. One forms the polynomial algebra \( P(A, t) \) of formal polynomials of the form \( p = \sum_{i=0}^{n} t^i a_i \), where \( a_i \in A \), with multiplication determined by the covariance relation \( at = t\alpha(a) \) and the norm

\[
\|p\| = \sup_{(\rho, T) \text{ covariant}} \left\| \sum_{i=0}^{n} T^i \rho(a_i) \right\|.
\]

This supremum is clearly dominated by \( \sum_{i=0}^{n} \|a_i\| \); so this norm is well defined. The completion is the semicrossed product \( A \times_{\alpha} \mathbb{Z}^+ \). Since this is the supremum of operator algebra norms, it is also an operator algebra norm. By construction, for each covariant representation \( (\rho, T) \), there is a unique completely contractive
representation $\rho \times T$ of $A \times_{\alpha} \mathbb{Z}^+$ into $\mathcal{B}(\mathcal{H})$ given by

$$\rho \times T(p) = \sum_{i=0}^{n} T_i \rho(a_i).$$

This is the defining property of the semicrossed product.

In this paper, we examine semicrossed products of the disk algebra by an endomorphism which extends to a $*$-endomorphism of $C(\mathbb{T})$. In the case where the endomorphism is injective, these have the form $\alpha(f) = f \circ b$ where $b$ is a non-constant Blaschke product. We show that every covariant representation of $(A(\mathbb{D}), \alpha)$ dilates to a covariant representation of $(C(\mathbb{T}), \alpha)$. This is readily dilated to a covariant representation $(\sigma, V)$, where $\sigma$ is a $*$-representation of $C(\mathbb{T})$ (so $\sigma(z)$ is unitary) and $V$ is an isometry. To go further, we use the recent work of Kakariadis and Katsoulis [6] to show that $C(\mathbb{T}) \times_{\alpha} \mathbb{Z}^+$ imbeds completely isometrically into a $C^*$-crossed product $C(S_b) \times_{\alpha} \mathbb{Z}$. In fact,

$$C^*_e(C(\mathbb{T}) \times_{\alpha} \mathbb{Z}^+) = C(S_b) \times_{\alpha} \mathbb{Z},$$

and as a consequence, we obtain that $(\rho, T)$ dilates to a covariant representation $(\tau, W)$, where $\tau$ is a $*$-representation of $C(\mathbb{T})$ (so $\sigma(z)$ is unitary) and $W$ is a unitary.

In contrast, if $\alpha$ is induced by a constant Blaschke product, we can no longer identify $C^*_e(C(\mathbb{T}) \times_{\alpha} \mathbb{Z}^+)$ up to isomorphism. In that case, $\alpha$ is evaluation at a boundary point. Even though every covariant representation of $(A(\mathbb{D}), \alpha)$ dilates to a covariant representation of $(C(\mathbb{T}), \alpha)$, the theory of [6] is not directly applicable since $\alpha$ is not injective. Instead, we use the process of “adding tails to $C^*$-correspondences” [8], as modified in [3, 7], and we identify $C^*_e(C(\mathbb{T}) \times_{\alpha} \mathbb{Z}^+)$ up to strong Morita equivalence as a crossed product. In Theorem [2,6] we show that $C^*_e(C(\mathbb{T}) \times_{\alpha} \mathbb{Z}^+)$ is strongly Morita equivalent to a $C^*$-algebra of the form $C_0(S_e) \times_s \mathbb{Z}$, where

$$e: \mathbb{T} \times \mathbb{N} \rightarrow \mathbb{T} \times \mathbb{N}$$

is a suitable map and $(S_e, s)$ is the solenoid system for $(\mathbb{T} \times \mathbb{N}, e)$.

Semicrossed products of the the disc algebra were introduced and first studied by Buske and Peters in [1], following relevant work of Hoover, Peters and Wogen [5]. The algebras $A(\mathbb{D}) \times_{\alpha} \mathbb{Z}^+$, where $\alpha$ is an arbitrary endomorphism, where classified up to algebraic endomorphism in [2]. Results associated with their $C^*$-envelope can be found in [1 Proposition III.13] and [10 Theorem 2]. The results of the present paper subsume and extend these earlier results.

2. The Disk Algebra

The $C^*$-envelope of the disk algebra $A(\mathbb{D})$ is $C(\mathbb{T})$, the space of continuous functions on the unit circle. Suppose that $\alpha$ is an endomorphism of $C(\mathbb{T})$ which leaves $A(\mathbb{D})$ invariant. We refer to the restriction of $\alpha$ to $A(\mathbb{D})$ as $\alpha$ as well. Then $\rho = \alpha(z) \in A(\mathbb{D})$ and has spectrum

$$\sigma_{A(\mathbb{D})}(\rho) \subset \sigma_{A(\mathbb{D})}(z) = \mathbb{D}$$

and

$$\sigma_{C(\mathbb{T})}(\rho) \subset \sigma_{C(\mathbb{T})}(z) = \mathbb{T}.$$ 

Thus $\|b\| = 1$ and $b(T) \subset \mathbb{T}$. It follows that $b$ is a finite Blaschke product. Therefore $\alpha(f) = f \circ b$ for all $f \in C(\mathbb{T})$. When $b$ is not constant, $\alpha$ is completely isometric.

A (completely) contractive representation $\rho$ of $A(\mathbb{D})$ is determined by $\rho(z) = A$, which must be a contraction. The converse follows from the matrix von Neumann...
inequality and shows that \( \rho(f) = f(A) \) is a complete contraction. A covariant representation of \((A(D), \alpha)\) is thus determined by a pair of contractions \((A, T)\) such that
\[
AT = Tb(A).
\]
The representation of \(A(D) \rtimes_{\alpha} \mathbb{Z}^+\) is given by
\[
\rho \times T\left( \sum_{i=0}^{n} t^i f_i \right) = \sum_{i=0}^{n} T^i f_i(A),
\]
which extends to a completely contractive representation of the semicrossed product by the universal property.

A contractive representation \(\sigma\) of \(C(T)\) is a \(*\)-representation and is likewise determined by \(U = \sigma(z)\), which must be unitary, and all unitary operators yield such a representation by the functional calculus. A covariant representation of \((C(T), \alpha)\) is given by a pair \((U, T)\) where \(U\) is unitary and \(T\) is a contraction satisfying \(UT = Tb(U)\). To see this, multiply on the left by \(U^*\) and on the right by \(b(U)^*\) to obtain the identity
\[
U^*T = Tb(U)^* = Tb(U) = T\alpha(\bar{z})(U).
\]
The set of functions \(\{f \in C(T) : f(U)T = T\alpha(f)(U)\}\) is easily seen to be a norm closed algebra. Since it contains \(z\) and \(\bar{z}\), it is all of \(C(T)\). So the covariance relation holds.

**Theorem 2.1.** Let \(b\) be a finite Blaschke product, and let \(\alpha(f) = f \circ b\). Then \(A(D) \rtimes_{\alpha} \mathbb{Z}^+\) is (canonically completely isometrically isomorphic to) a subalgebra of \(C(T) \rtimes_{\alpha} \mathbb{Z}^+\).

**Proof.** To establish that \(A(D) \rtimes_{\alpha} \mathbb{Z}^+\) is completely isometric to a subalgebra of \(C(T) \rtimes_{\alpha} \mathbb{Z}^+\), it suffices to show that each \((A, T)\) with \(AT = Tb(A)\) has a dilation to a pair \((U, S)\) with \(U\) unitary and \(S\) a contraction such that
\[
US = Sb(U) \quad \text{and} \quad PH^mS^nU^m|_H = T^nA^m\quad \text{for all} \quad m, n \geq 0.
\]
This latter condition is equivalent to \(H\) being semi-invariant for the algebra generated by \(U\) and \(S\).

The covariance relation can be restated as
\[
\begin{bmatrix} A & 0 \\ 0 & b(A) \end{bmatrix} \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & b(A) \end{bmatrix}.
\]
Dilate \(A\) to a unitary \(U\) which leaves \(H\) semi-invariant. Then \(\begin{bmatrix} A & 0 \\ 0 & b(A) \end{bmatrix}\) dilates to \(\begin{bmatrix} U & 0 \\ 0 & b(U) \end{bmatrix}\). By the Sz. Nagy-Foiaş Commutant Lifting Theorem, we may dilate \(\begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}\) to a contraction of the form \(\begin{bmatrix} * & S \\ * & \alpha(U) \end{bmatrix}\) which commutes with \(\begin{bmatrix} U & 0 \\ 0 & \alpha(U) \end{bmatrix}\) and has \(H \oplus H\) as a common semi-invariant subspace. Clearly, we may take the \(*\) entries to all equal 0 without changing things. So \((U, S)\) satisfies the same covariance relations \(US = Sb(U)\). Therefore we have obtained a dilation to the covariance relations for \((C(T), \alpha)\).

Once we have a covariance relation for \((C(T), \alpha)\), we can try to dilate further. Extending \(S\) to an isometry \(V\) follows a well-known path. Observe that
\[
b(U)S^*S = S^*US = S^*Sb(U).
\]
Thus $D = (I - S^*S)^{1/2}$ commutes with $b(U)$. Write $b^{(n)}$ for the composition of $b$ with itself $n$ times. Hence we can now use the standard Schaeffer dilation of $S$ to an isometry $V$ and simultaneously dilate $U$ to $U_1$ as follows:

$$V = \begin{bmatrix} S & 0 & 0 & 0 & \cdots \\ D & 0 & 0 & 0 & \cdots \\ 0 & I & 0 & 0 & \cdots \\ 0 & 0 & I & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \quad \text{and} \quad U_1 = \begin{bmatrix} U & 0 & 0 & 0 & \cdots \\ 0 & b(U_1) & 0 & 0 & \cdots \\ 0 & 0 & b^{(2)}(U_1) & 0 & \cdots \\ 0 & 0 & 0 & b^{(3)}(U_1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$ 

A simple calculation shows that $U_1V = Vb(U_1)$. So as above, $(U, V)$ satisfies the covariance relations for $(C(T), \alpha)$. 

We would like to make $V$ a unitary as well. This is possible in the case where $b$ is non-constant but the explicit construction is not obvious. Instead, we use the theory of $C^*$-envelopes and maximal dilations. First we need the following.

**Lemma 2.2.** Let $b$ be a finite Blaschke product, and let $\alpha(f) = f \circ b$. Then $C^*_e(A(\mathbb{D}) \times_\alpha \mathbb{Z}^+) \simeq C^*_e(C(T) \times_\alpha \mathbb{Z}^+)$. 

**Proof.** The previous theorem identifies $A(\mathbb{D}) \times_\alpha \mathbb{Z}^+$ completely isometrically as a subalgebra of $C(T) \times_\alpha \mathbb{Z}^+$. The $C^*$-envelope $\mathcal{C}$ of $C(T) \times_\alpha \mathbb{Z}^+$ is a Cuntz-Pimsner algebra containing a copy of $C(T)$ which is invariant under gauge actions. Now $\mathcal{C}$ is a $C^*$-cover of $C(T) \times_\alpha \mathbb{Z}^+$, so it is easy to see that it is also a $C^*$-cover of $A(\mathbb{D}) \times_\alpha \mathbb{Z}^+$. Since $A(\mathbb{D}) \times_\alpha \mathbb{Z}^+$ is invariant under the same gauge actions, its Shilov ideal $\mathcal{J} \subseteq \mathcal{C}$ will be invariant by these actions as well. If $\mathcal{J} \neq 0$, then by gauge invariance $\mathcal{J} \cap C(T) \neq 0$. Since the quotient map

$$A(\mathbb{D}) \longrightarrow C(T)/(\mathcal{J} \cap C(T))$$

is completely isometric, we obtain a contradiction. Hence $\mathcal{J} = 0$ and the conclusion follows. 

We now recall some of the theory of semicrossed products of $C^*$-algebras. When $\mathfrak{A}$ is a $C^*$-algebra, the completely isometric endomorphisms are the faithful $*$-endomorphisms. In this case, Peters shows [4] Prop. 1.8 that there is a unique $C^*$-algebra $\mathfrak{B}$, a $*$-automorphism $\beta$ of $\mathfrak{B}$ and an injection $j$ of $\mathfrak{A}$ into $\mathfrak{B}$ so that $\beta \circ j = j\alpha$ and $\mathfrak{B}$ is the closure of $\bigcup_{n \geq 0} \beta^{-n}(j(\mathfrak{A}))$. It follows [4] Prop. II.4] that $\mathfrak{A} \times_\alpha \mathbb{Z}^+$ is completely isometrically isomorphic to the subalgebra of the crossed product algebra $\mathfrak{B} \times_\beta \mathbb{Z}$ generated as a non-selfadjoint algebra by an isomorphic copy $j(\mathfrak{A})$ of $\mathfrak{A}$ and the unitary $u$ implementing $\beta$ in the crossed product. Actually, Kakariadis and the second author [6] Thm. 2.5] show that $\mathfrak{B} \times_\beta \mathbb{Z}$ is the $C^*$-envelope of $\mathfrak{A} \times_\alpha \mathbb{Z}^+$. This last result is valid for semicrossed products of not necessarily unital $C^*$-algebras by (non-degenerate) injective $*$-endomorphisms, and we will use it below in that form.

In the case where $\mathfrak{A} = C_0(X)$ is commutative and $\alpha$ is induced by an injective self-map of $X$, the pair $(\mathfrak{B}, \beta)$ has an alternative description.

**Definition 2.3.** Let $X$ be a topological space and $\phi$ a surjective self-map of $X$. We define the solenoid system of $(X, \phi)$ to be the pair $(S_\phi, s)$, where

$$S_\phi = \{(x_n)_{n \geq 1} : x_n = \phi(x_{n+1}), x_n \in X, n \geq 1\}$$
equipped with the relative topology inherited from the product topology on \( \prod_{i=1}^{\infty} X_i \), where \( X_i = X \), \( i = 1, 2, \ldots \), and \( s \) is the backward shift on \( S_\phi \).

It is easy to see that if \( A = C_0(X) \), with \( X \) a locally compact Hausdorff space, and \( \alpha \) is induced by a surjective self-map \( \phi \) of \( X \), then the pair \((\mathcal{B}, \beta)\) for \((A, \alpha)\) described above is conjugate to the solenoid system \((S_\phi, s)\). Therefore, we obtain

**Corollary 2.4.** Let \( b \) be a non-constant finite Blaschke product, and let \( \alpha(f) = f \circ b \) on \( C(\mathbb{T}) \). Then

\[
C^*_e(A(\mathbb{D}) \times_\alpha \mathbb{Z}^+) = C(S_b) \times_s \mathbb{Z},
\]

where \((S_b, s)\) is the solenoid system of \((\mathbb{T}, b)\).

The above corollary leads to a dilation result, for which a direct proof is far from obvious.

**Corollary 2.5.** Let \( \alpha \) be an endomorphism of \( A(\mathbb{D}) \) induced by a non-constant finite Blaschke product and let \( \lambda \in \mathbb{T} \) be a contraction satisfying

\[
AT = T\alpha(A).
\]

Then there exist unitary operators \( U \) and \( W \) on a Hilbert space \( K \supset \mathcal{H} \) which simultaneously dilate \( A \) and \( T \), in the sense that

\[
P_H W^m U^n |_{\mathcal{H}} = T^m A^n \quad \text{for all } m, n \geq 0,
\]

so that

\[
UW = W\alpha(U).
\]

**Proof.** Every covariant representation \((A, T)\) of \((A(\mathbb{D}), \alpha)\) dilates to a covariant representation \((U_1, V)\) of \((C(\mathbb{T}), \alpha)\). This in turn dilates to a maximal dilation \( \tau \) of \( C(\mathbb{T}) \times_\alpha \mathbb{Z}^+ \), in the sense of Dritschel and McCullough \[4\]. The maximal dilations extend to \(*\)-representations of the \( C^* \)-envelope. Then \( A \) dilated to \( \tau(j(z)) = U \) is unitary and \( T \) dilates to the unitary \( W \) which implements the automorphism \( \beta \) on \( \mathcal{B} \) and restricts to the action of \( \alpha \) on \( C(\mathbb{T}) \). \( \square \)

The situation changes when we move to non-injective endomorphisms \( \alpha \) of \( A(\mathbb{D}) \). Indeed, let \( \lambda \in \mathbb{T} \) and consider the endomorphism \( \alpha_\lambda \) of \( A(\mathbb{D}) \) induced by evaluation on \( \lambda \), i.e.,

\[
\alpha_\lambda(f)(z) = f(\lambda) \quad \text{for all } z \in \mathbb{D}.
\]

(Thus \( \alpha_\lambda \) is the endomorphism of \( A(\mathbb{D}) \) corresponding to a constant Blaschke product.) If two contractions \( A, T \) satisfy

\[
AT = T\alpha_\lambda(A) = \lambda T,
\]

then the existence of unitary operators \( U, W \), dilating \( A \) and \( T \) respectively, implies that \( A = \lambda I \). It is easy to construct a pair \( A, T \) satisfying \( AT = \lambda T \) and yet \( A \neq \lambda I \). This shows that the analogue Corollary 2.5 fails for \( \alpha = \alpha_\lambda \) and therefore one does not expect \( C^*_e(A(\mathbb{D}) \times_\alpha \mathbb{Z}^+) \) to be isomorphic to the crossed product of a commutative \( C^* \)-algebra, at least under canonical identifications. However, as we have seen, a weakening of Corollary 2.5 is valid for \( \alpha = \alpha_\lambda \) if one allows \( W \) to be an isometry instead of a unitary operator. In addition, we can identify \( C^*_e(A(\mathbb{D}) \times_\alpha \mathbb{Z}^+) \) as being strongly Morita equivalent to a crossed product \( C^* \)-algebra. Indeed, if

\[
e: \mathbb{T} \times \mathbb{N} \rightarrow \mathbb{T} \times \mathbb{N}
\]
is defined as
\[
e(z, n) = \begin{cases} 
(1, 1) & \text{if } n = 1, \\
(z, n - 1) & \text{otherwise},
\end{cases}
\]
then

**Theorem 2.6.** Let \( \alpha = \alpha_\lambda \) be an endomorphism of \( A(\mathbb{D}) \) induced by evaluation at a point \( \lambda \in \mathbb{T} \). Then \( C^*_e(A(\mathbb{D}) \times_\alpha \mathbb{Z}^+) \) is strongly Morita equivalent to \( C_0((S_e) \times_\alpha \mathbb{Z}) \), where \( e: \mathbb{T} \times \mathbb{N} \to \mathbb{T} \times \mathbb{N} \) is defined above and \((S_e, s)\) is the solenoid system of \((\mathbb{T} \times \mathbb{N}, e)\).

**Proof.** In light of Lemma 2.2, it suffices to identify the C*-envelope of \( C(\mathbb{T}) \times_\alpha \mathbb{Z}^+ \). As \( \alpha \) is no longer an injective endomorphism of \( C(\mathbb{T}) \), we invoke the process of adding tails to C*-correspondences [8], as modified in [3, 7].

Indeed, [7, Example 4.3] implies that the C*-envelope of the tensor algebra associated with the dynamical system \((C(\mathbb{T}), \alpha)\) is strongly Morita equivalent to the Cuntz-Pimsner algebra associated with the injective dynamical system \((\mathbb{T} \times \mathbb{N}, e)\) defined above. Therefore by invoking the solenoid system of \((\mathbb{T} \times \mathbb{N}, e)\), the conclusion follows from the discussion following Lemma 2.2. \( \square \)

**References**


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