3-MANIFOLDS
WITH POSITIVE FLAT CONFORMAL STRUCTURE

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ABSTRACT. In this paper, we consider a closed 3-manifold \( M \) with flat conformal structure \( C \). We will prove that if the Yamabe constant of \((M, C)\) is positive, then \((M, C)\) is Kleinian.

1. Introduction and Main Theorem

In 1988, Schoen and Yau [19] gave a final resolution for the Yamabe Problem (cf. [3, 15, 18]). In [19, Proposition 3.3], they also proved that any closed \( n \)-manifold with flat conformal structure of positive Yamabe constant is Kleinian, provided that \( n \geq 4 \). Moreover, under the assumption that an extended Positive Mass Theorem holds (but a proof has not yet appeared), they showed that the above assertion still holds even when \( n = 3 \) (see [19, Proposition 4.4’] and the paragraph just before it).

On the other hand, there are enormous examples of closed 3-manifolds with flat conformal structures which are not Kleinian (see [8, Remark 7.4]).

The purpose of this brief note is to prove the above assertion for the remaining case \( n = 3 \).

**Theorem 1.1.** Let \( M \) be a closed 3-manifold with flat conformal structure \( C \). If its Yamabe constant is positive, then \((M, C)\) is Kleinian.

This assertion can be obtained by an argument in the proof of [1, the second assertion of Theorem 1.4], which is a combination of a result [19, Proposition 4.2], a positive mass theorem [1 the first assertion of Theorem 1.4] (different from the one Schoen and Yau mentioned in [19]) and a classification of 3-manifolds with positive scalar curvature [7, 10, 11]. Here, we will explicitly give a proof of it (see also Remark 2.2 below).

The remaining sections are organized as follows. Section 2 contains some necessary definitions and preliminary geometric results. Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminaries

Let \( M \) be a closed 3-manifold, that is, a compact 3-manifold without boundary. To simplify the presentation and the argument, we always assume that \( \dim M = 3 \).
which is of constant scalar curvature case of \[1\], the first assertion of Theorem 1.4: hold (see \[1\], Remark 1.5-(2) for instance). Mass theorem for asymptotically flat 3-manifolds with singularities does not always hold (see \[1\], Remark 1.5-(2) for instance).

Assume that \( (M, g) \) is an asymptotically flat 3-manifold \((M, g) \) can be defined in the usual way (cf. \[4\]). Note also that the positive Yamabe, Trudinger, Aubin and Schoen asserts that each conformal class \( g \in C \) of \( C \) has a minimizer \( \tilde{g} \) of \( E[\cdot] \), called a Yamabe metric (or a solution of the Yamabe Problem), which is of constant scalar curvature

\[ R_{\tilde{g}} = Y(M, C) \cdot \text{Vol}_g(M)^{-2/3}. \]

Let \( M_{\infty} \) be an infinite covering of \( M \). We shall say that the fundamental group \( \pi_1(M) \) of \( M \) has a descending chain of finite index subgroups tending to \( \pi_1(M_{\infty}) \) if it satisfies the following: There exists a family of subgroups \( \{\Gamma_i\}_{i \geq 1} \) of \( \pi_1(M) \) such that

(i) each \( \Gamma_i \) is of finite index in \( \pi_1(M) \) with \( \Gamma_i \supset \pi_1(M_{\infty}) \),
(ii) \( \pi_1(M) = \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_i \supset \Gamma_{i+1} \supset \cdots \),
(iii) \( \bigcap_{i=1}^{\infty} \Gamma_i = \pi_1(M_{\infty}) \).

Assume that \( Y(M, C) > 0 \). Take a positive scalar curvature metric \( g \in C \) and any point \( p \in M \). Then, there exists the normalized Green’s function \( G_p \) for \( L_g \) with a pole at \( p \), that is,

\[ L_g G_p = c_0 \cdot \delta_p \quad \text{on} \quad M \quad \text{and} \quad \lim_{q \to p} \text{dist}(q, p) G_p(q) = 1. \]

Here, \( L_g := -8\Delta_g + R_g, c_0 > 0 \) and \( \delta_p \) stand respectively for the conformal Laplacian, a specific universal positive constant and the Dirac \( \delta \)-function at \( p \). Assume also that the covering \( P_{\infty} : M_{\infty} \to M \) is normal. Let \( g_{\infty} \) denote the lift of \( g \) to \( M_{\infty} \), and \( p_{\infty} \) a point in \( M_{\infty} \) with \( P_{\infty}(p_{\infty}) = p \). Then, there exists uniquely also a normalized minimal positive Green’s function \( G_{\infty} \) on \( M_{\infty} \) for \( L_{g_{\infty}} := -8\Delta_{g_{\infty}} + R_{g_{\infty}} \) with pole at \( p_{\infty} \) (cf. \[13\]), which satisfies the following:

\[ (P_{\infty})^* G_p = \sum_{\gamma \in \mathcal{G}} G_{\infty} \circ \gamma \quad \text{on} \quad M_{\infty}. \]

Here, \( \mathcal{G} \) stands for the group of deck transformations for the normal covering \( M_{\infty} \to M \). Set

\[ g_{\infty,AF} := G_{\infty}^4 \cdot g_{\infty} \quad \text{on} \quad M_{\infty}^* := M_{\infty} - \{p_{\infty}\}. \]

Then, \( g_{\infty,AF} \) defines a scalar-flat, asymptotically flat metric on \( M_{\infty}^* \) (cf. \[15\]). Note that this asymptotically flat 3-manifold \((M_{\infty}^*, g_{\infty,AF})\) has infinitely many singularities created by the ends of \( M_{\infty}^* \). However, the mass \( m_{ADM}(g_{\infty,AF}) \) of \((M_{\infty}^*, g_{\infty,AF})\) can be defined in the usual way (cf. \[4\]). Note also that the positive mass theorem for asymptotically flat 3-manifolds with singularities does not always hold (see \[1\] Remark 1.5-(2) for instance).

Once this is understood, the following positive mass theorem holds as a special case of \[1\] the first assertion of Theorem 1.4:
Proposition 2.1. Let \((M, C)\) be a closed 3-manifold with \(Y(M, C) > 0\). Let \((M_\infty, g_\infty)\) be a normal infinite Riemannian covering of \((M, g)\) such that \(\pi_1(M)\) has a descending chain of finite index subgroups tending to \(\pi_1(M_\infty)\), where \(g \in C\) is a positive scalar curvature metric and \(g_\infty\) is its lift to \(M_\infty\). For any point \(p_\infty \in M_\infty\), let \(G_\infty\) denote the normalized minimal positive Green's function on \(M_\infty^*\) with pole at \(p_\infty\). Then, the asymptotically flat 3-manifold \((M_\infty^*, g_\infty, AF)\) has nonnegative mass
\[
m_{\text{ADM}}(g_\infty, AF) \geq 0.
\]

Remark 2.2. Assume that \(M = \# \ell(S^1 \times S^2)\) for \(\ell \geq 2\) and \(M_\infty\) is its universal covering. Note that \(M_\infty\) is spin. For each small \(\sigma > 0\), consider the complete metric \(g_{\sigma, AF} := (G_\infty + \sigma)^4 \cdot g_\infty\) with \(R_{g_{\sigma, AF}} \geq 0\) on \(M_\infty^*\) (cf. [19] Proposition 4.4'). Then, only one end of \((M_\infty^*, g_{\sigma, AF})\) is asymptotically flat and the other infinitely many ends are merely complete. Gilles Carron and the referee kindly pointed out that Witten’s approach [21] (cf. [16]) to the Positive Mass Theorem is still valid for the family \((M_\infty^*, g_{\sigma, AF})\) of \(0 < \sigma < 1\). It implies that a more general positive mass theorem than Proposition 2.1 is a folk theorem for experts in this field, and Theorem 1.11 is too. But Proposition 2.1 itself is a complete form, and hence, by using it, we will give here an explicit and self-contained proof of Theorem 1.11.

A conformal 3-manifold \((M, C)\) is said to be locally conformally flat if, for any point \(p \in M\), there exists a metric \(\overline{g} \in C\) such that \(\overline{g}\) is flat on some neighborhood of \(p\). A conformal class \(C\) on \(M\) is called a flat conformal structure if \((M, C)\) is locally conformally flat. In [14], Kuiper proved that, for a simply connected locally conformally flat \(3\)-manifold \((X, C')\), there is a conformal immersion into \((S^3, C_0)\) called a developing map, which is unique up to composition with a Möbius transformation of \((S^3, C_0)\). Therefore, the universal covering of a locally conformally flat manifold \((M, C)\) admits a developing map. Here, \((S^3, C_0)\) denotes the 3-sphere \(S^3\) with the conformal class \(C_0 := [g_0]\) of the standard metric \(g_0\) of constant curvature one. \((M, C)\) is called Kleinian if \((M, C)\) is conformal to \(\Omega/\Gamma\) for some open set \(\Omega\) of \(S^3\) and some discrete subgroup \(\Gamma\) of the conformal transformation group Conf\((S^3, C_0)\), which leaves \(\Omega\) invariant and acts freely and properly discontinuously on \(\Omega\). Note that, if the developing map of the universal covering of a locally conformally flat manifold \((M, C)\) is injective, then \((M, C)\) is Kleinian.

With this understanding, the following criterion also holds as a special case of [19] Proposition 4.2: 

Proposition 2.3. Let \((M, C)\) be a closed 3-manifold with \(Y(M, C) > 0\), and \((\widetilde{M}, \mathcal{G})\) be the universal Riemannian covering of \((M, g)\), where \(g \in C\) is a positive scalar curvature metric. For any point \(\overline{p} \in \widetilde{M}\), let \(G\) denote the normalized minimal positive Green’s function on \(\widetilde{M}\) for \(L_{\overline{g}}\) with pole at \(\overline{p}\), and \((\widetilde{M} - \{\overline{p}\}, \mathcal{G}_{\overline{p}}^4 \cdot \overline{g})\) the asymptotically flat 3-manifold as above. If the mass \(m_{\text{ADM}}(\mathcal{G}_{\overline{p}}^4 \cdot \overline{g})\) is nonnegative, then the developing map of \((M, [\overline{g}])\) is injective. In particular, \((M, C)\) is Kleinian.

Remark 2.4. We remark that the mass \(m_{\text{ADM}}(\mathcal{G}_{\overline{p}}^4 \cdot \overline{g})\) is equal to the ADM energy \(E\) of \((\widetilde{M} - \{\overline{p}\}, \mathcal{G}_{\overline{p}}^4 \cdot \overline{g})\) appearing in [19] page 64 up to a positive constant.

3. Proof of main theorem

Proof of Theorem 1.11 Consider the universal covering \(\widetilde{M}\) of \(M\) and denote the lift of the flat conformal structure \(C\) by \(\widetilde{C}\). If \(|\pi_1(M)| < \infty\), then \((\widetilde{M}, \widetilde{C})\) is conformal
to \((S^3, C_0)\) by Kuiper’s Theorem \cite{kuiper}. Hence, \((M, C)\) is Kleinian. From now on, we assume that \(|\pi_1(M)| = \infty\), that is, the degree of the covering map \(P : \tilde{M} \to M\) is infinite.

Take a unit-volume Yamabe metric \(g \in C\), and consider its lift \(\tilde{g} \in \tilde{C}\) to \(\tilde{M}\). Note that \(R_{\tilde{g}} = R_{\tilde{g}} = Y(M, C) > 0\). Take any base points \(p \in M, \tilde{p} \in \tilde{M}\) satisfying \(P(\tilde{p}) = p\), and fix them. Then, let \(\tilde{G}\) denote the normalized minimal positive Green function on \(\tilde{M}\) for \(L_{\tilde{g}}\) with pole at \(\tilde{p}\), and the mass \(m_{\text{ADM}}(\tilde{g}_{AF})\) of the asymptotically flat 3-manifold \((\tilde{M} - \{\tilde{p}\}, \tilde{g}_{AF} := G^4 \cdot \tilde{g})\).

Suppose that

\[
m_{\text{ADM}}(\tilde{g}_{AF}) \geq 0.
\]

Recall that we can choose the base point \(\tilde{p} \in \tilde{M}\) arbitrarily. It then follows from Proposition \[2.3\] that the developing map of \((\tilde{M}, \tilde{C})\) is injective, and hence \((M, C)\) is Kleinian. In this case, especially \(m_{\text{ADM}}(\tilde{g}_{AF}) = 0\). Therefore, it is enough to show that \(m_{\text{ADM}}(\tilde{g}_{AF}) \geq 0\).

By combining \[7\] Theorem \[8.1\] (cf. \[9\]) with \(Y(M, C) > 0\) (replacing \(M\) by its orientable double covering if necessary), \(M\) can be decomposed uniquely into prime closed 3-manifolds

\[
M = N_1 \# \cdots \# N_i \# \ell_2(S^1 \times S^2),
\]

where \(\pi_1(N_i)\) is finite for \(i = 1, \ldots, \ell_1\) and \(\ell_1, \ell_2\) are nonnegative integers. By applying the \(C\)-prime decomposition theorem for closed 3-manifolds with flat conformal structures \[10, 11\] to \((M, C)\), there exists a flat conformal structure \(C_i\) on each \(N_i (i = 1, \ldots, \ell_1)\). Then, Kuiper’s Theorem \[14\] again implies that each \((N_i, C_i)\) is a nontrivial quotient of \((S^3, C_0)\). After taking an appropriate finite covering \(M'\) of \(M\), we have

\[
M' = \#(S^1 \times S^2) \quad \text{for some} \quad \ell \geq 1.
\]

Recall that \(\tilde{M}\) is the infinite universal covering of \(M\). Then, there exists (uniquely) an infinite universal covering \(\tilde{M} \to M'\). Moreover, since \(\pi_1(M')\) is a finitely generated free group, it has a descending chain of finite index subgroups tending to \(\pi_1(M) = \{e\}\). Let \(g'\) be the lifting of \(g\) to \(M'\). Applying Proposition \[2.1\] to the normal infinite Riemannian covering \((\tilde{M}, \tilde{g}) \to (M', g')\), we have that

\[
m_{\text{ADM}}(\tilde{g}_{AF}) \geq 0.
\]

This completes the proof of Theorem \[1.1\].

\[\square\]

Remark 3.1. Even if we replace the positivity \(Y(M, C) > 0\) in Theorem \[1.1\] by the nonnegativity \(Y(M, C) \geq 0\), it seems that the same conclusion still holds. More precisely, we propose the following (cf. \[5, 13\]).

**Conjecture.** Let \(M\) be a closed 3-manifold with flat conformal structure \(C\). If its Yamabe constant is zero, then either (1) or (2) holds:

1. There exists a flat metric \(g \in C\).
2. There exists a smooth family \(\{g_t\}_{0 \leq t \leq 1}\) of locally conformally flat metrics on \(M\) such that \(g_0 \in C\) and \(Y(M, [g_t]) > 0\) (possibly \(Y(M, [g_t]) < 0\) for some \(t \in (0, 1)\)).

In the case (1), the universal covering \((\tilde{M}, \tilde{C})\) of \((M, C)\) is conformal to \((S^3 - \{p_N\}, C_0)\) where \(p_N := (1, 0, 0, 0) \in S^3\), and hence \((M, C)\) is Kleinian. In the case (2), Theorem \[1.1\] implies that \((M, [g_1])\) is Kleinian. The argument in the proof of
Theorem 1.1 also implies that there exists a torsion free subgroup \( \Gamma \) of finite index in \( \pi_1(M) \) such that \( \Gamma \) is either a trivial group or a nontrivial finitely generated free group. Then, the virtual cohomological dimension \( \text{vcd} \pi_1(M) \) of \( \pi_1(M) \) is either 0 or 1 (see [6]). Therefore, \((M,[g_1])\) is a closed Kleinian 3-manifold with \( \text{vcd} \pi_1(M) < 3 \). The quasi-conformal stability of Kleinian groups [12] Theorem 2 implies that any flat conformal structure on \( M \) which is a smooth deformation of \([g_1]\) is also Kleinian; in particular \( C \) is too.

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