BEREZIN TRANSFORM
AND WEYL-TYPE UNITARY OPERATORS
ON THE BERGMAN SPACE

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Abstract. For \( \mathbb{D} \) the open complex unit disc with normalized area measure, we consider the Bergman space \( L^2_a(\mathbb{D}) \) of square-integrable holomorphic functions on \( \mathbb{D} \). Induced by the group \( \text{Aut}(\mathbb{D}) \) of biholomorphic automorphisms of \( \mathbb{D} \), there is a standard family of Weyl-type unitary operators on \( L^2_a(\mathbb{D}) \). For all bounded operators \( X \) on \( L^2_a(\mathbb{D}) \), the Berezin transform \( \tilde{X} \) is a smooth, bounded function on \( \mathbb{D} \). The range of the mapping \( \text{Ber}: X \mapsto \tilde{X} \) is invariant under \( \text{Aut}(\mathbb{D}) \). The “mixing properties” of the elements of \( \text{Aut}(\mathbb{D}) \) are visible in the Berezin transforms of the induced unitary operators. Computations involving these operators show that there is no real number \( M > 0 \) with \( M\|\tilde{X}\|_{\infty} \geq \|X\| \) for all bounded operators \( X \) and are used to check other possible properties of \( \tilde{X} \). Extensions to other domains are discussed.

1. Introduction

For \( H \) a complex Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \|v\| = \langle v, v \rangle^{1/2} \), we consider the algebra of all bounded linear operators \( \text{Op}(H) \). For the unit sphere \( S = \{ v \in H : \langle v, v \rangle = 1 \} \) and for \( X \) in \( \text{Op}(H) \), we have the usual operator norm \( \|X\| = \sup\{ \|Xv\| : v \in S \} \). We will also be concerned with the numerical range \( W(X) = \{ \langle Xv, v \rangle : v \in S \} \) and the numerical radius \( w(X) = \sup\{ |\langle Xv, v \rangle| : v \in S \} \). The set \( W(X) \) is convex (Hausdorff’s Theorem) and its closure contains the spectrum of \( X \). There is a standard norm estimate

\[ w(X) \leq \|X\| \leq 2w(X) \]

and a not-so-standard power estimate (Berger’s Theorem [5])

\[ w(X^n) \leq w(X)^n. \]

In the case that \( H \) is the Bergman Hilbert space \( L^2_a(\mathbb{D}) \) or one of a large family of “reproducing kernel Hilbert spaces”, we have, for each \( c \) in \( \mathbb{D} \), a reproducing kernel function \( K(\cdot, c) \) so that, for any \( f \) in \( L^2_a(\mathbb{D}) \),

\[ f(c) = \langle f, K(\cdot, c) \rangle. \]

The normalized kernel functions \( k_c(\cdot) \equiv K(\cdot, c)K(c, c)^{-1/2} \) play an important role in the analysis of operators on \( L^2_a(\mathbb{D}) \) as well as on other reproducing kernel spaces. In particular, for every bounded linear operator, we define the Berezin transform by \( \tilde{X}(c) = \langle Xk_c, k_c \rangle \). The map \( \text{Ber}(X) = \tilde{X} \) is one-to-one and \( \tilde{X}(\cdot) \) is known to be
real-analytic \cite{4} as well as Lipschitz with respect to the Bergman metric distance function on $D$ \cite{6}.

It is not hard to check that the range of $Ber$ contains all bounded holomorphic functions on $D$. Clearly, the range of $\tilde{X}$ is contained in $W(X)$ and, for $\|f\|_\infty = \sup\{|f(z)| : z \in D\}$, we have $\|\tilde{X}\|_\infty \leq w(X)$ so that $\tilde{X}$ is in the Banach space $BC(D)$ of bounded continuous functions $BC(D)$.

Convexity of range($\tilde{X}$) is easily seen to fail (take $X$ to be multiplication by a suitable holomorphic function). It is a natural problem to determine to what extent $\|\tilde{X}\|_\infty$ imitates $w(X)$. Is there a real number $M > 0$ with $M \|\tilde{X}\|_\infty \geq \|X\|$ for all $X$? Is $\|\tilde{X}^2\|_\infty \leq \|\tilde{X}\|_\infty^2$ for all $X$? The unitary operators $V_{[\lambda, c]}$ and related operators discussed in the next section provide examples to show that these estimates do not hold for $Op(L^2_a(D))$. Similar constructions yield the same result for the Segal-Bargmann space of Gaussian square-integrable entire functions on complex $n$-space $C^n$. The extension of our analysis to general bounded symmetric domains is plausible but presents significant difficulties.

2. Weyl-type unitary operators on $L^2_a(D)$

We consider the full group $Aut(D)$, given for $\lambda$ in $C$ with $|\lambda| = 1$ and $c, z$ in $D$, by

$$[\lambda, c](z) = \lambda \frac{z - c}{1 - \overline{c}z}.$$  

The group $Aut(D)$ acts on $L^2_a(D)$ by

$$(\lambda c f)(z) = f(\lambda \frac{z - c}{1 - \overline{c}z}),$$

and it is standard that

$$(V_{[\lambda, c]} f)(z) = k_c(z) f(\lambda \frac{z - c}{1 - \overline{c}z})$$

is a unitary transformation from $L^2_a(D)$ to itself, where

$$k_c(z) = \frac{1 - |c|^2}{(1 - \overline{c}z)^2}$$

is the normalized Bergman kernel for evaluation at $c$.

Multiplication on $Aut(D)$ is given by

$$(\lambda c) [\mu, d] = \left[ \frac{\mu \lambda}{1 + \overline{c}\alpha}, \frac{\lambda d + c}{1 + \overline{c}\alpha} \right],$$

and it is easy to check that the map $[\lambda, c] \rightarrow V_{[\lambda, c]}$ is a projective unitary representation of $Aut(D)$ with

$$V_{[\lambda, c]} V_{[\mu, d]} = \frac{1 + \overline{c}\alpha}{1 + d\alpha} V_{[\lambda, c][\mu, d]}.$$ 

We will be interested in the Berezin transform of $V_{[\lambda, c]}$. We first check that

$$(\star \star)$$

$$V_{[\lambda, c]} k_a = \frac{1 + a\overline{c}\alpha}{1 + \overline{c}\alpha} k_{[\lambda, c]}(a).$$

It follows, using the defining property of the reproducing kernel, that

$$(\star \star \star)$$

$$\langle V_{[\lambda, c]} k_a, k_a \rangle = \frac{(1 - |a|^2)^2(1 - |c|^2)}{(1 - \lambda |a|^2) - (a\overline{c} - \overline{c}a|c|)^2}. $$
Remarks. It follows from the above discussion that
\[ V_{[\lambda,c]}^* = V_{[\lambda,c]}^{-1} = V_{[\lambda,c]} \]
The involutive unitary operators \( V_{[-1,c]} \) are standard objects in the analysis of \( D \) as the prototypical bounded symmetric domain. We will use the elementary formula \( V_{[1,c]}^2 = V_{[1, \frac{2c}{1+c^2}]} \) in our analysis.

Using the above remark, we first have

**Theorem 1.** *The range of Ber: Op\([L^2_0(D)]\) \( \rightarrow BC(D) \) is invariant under Aut\( (D) \).*

**Proof.** Using (***) we can check that
\[ \tilde{X}\{[\lambda,c](a)\} = \langle XV_{[\lambda,c]}k_a, \tilde{V}_{[\lambda,c]}k_a \rangle = \langle V_{[\lambda,c]}XV_{[\lambda,c]}^*k_a, k_a \rangle. \]

**Corollary.** *The projective unitary representation \([\lambda,c] \rightarrow V_{[\lambda,c]} \) of Aut\( (D) \) on \( L^2_0(D) \) is irreducible.*

**Proof.** For \( a,b \) arbitrary in \( D \), \([[-1,a],[-1,b]](a) = b \). For \( X \) in Op\([L^2_0(D)]\) with \( XV_{[\lambda,c]} = V_{[\lambda,c]}X \) for all \([\lambda,c] \) in Aut\( (D) \), we have \( \tilde{X}\{[\lambda,c](a)\} = \tilde{X}(a) \) for all \( a \) in \( D \). Taking \([\lambda,c] = [-1,a][-1,b] \) gives \( \tilde{X}(a) = \tilde{X}(b) \) for arbitrary \( b \). Thus, \( \tilde{X} \) must be a constant function so that \( X \) is a scalar multiple of the identity operator. \( \square \)

**Remark.** The irreducibility is certainly “well known”.

Next, we explicitly calculate \( \|\tilde{V}_{[\lambda,c]}\|_\infty \) for \( \lambda = \pm 1 \).

**Theorem 2.** *We have \( \|\tilde{V}_{[1,c]}\|_\infty = 1 - |c|^2 \) and \( \|\tilde{V}_{[-1,c]}\|_\infty = 1 \) for all \( c \) in \( D \).*

**Proof.** First, for \( \tilde{V}_{[1,c]} \), we note that (***) gives
\[ \langle V_{[1,c]}k_a, k_a \rangle = \frac{(1 - |a|^2)(1 - |c|^2)}{((1 - |a|^2) + (\bar{a}c - ac))}. \]

Since \( i(\bar{a}c - ac) \) is real, we see that
\[ |(1 - |a|^2) + (\bar{a}c - ac)|^2 = (1 - |a|^2)^2 + |\bar{a}c - ac|^2, \]
so
\[ |\langle V_{[1,c]}k_a, k_a \rangle| = \frac{(1 - |a|^2)(1 - |c|^2)}{2(1 - |a|^2)^2 + |\bar{a}c - ac|^2}. \]

Since \( \tilde{V}_{[1,c]}(0) = 1 - |c|^2 \), it follows that \( \|\tilde{V}_{[1,c]}\|_\infty = 1 - |c|^2 \).

For \( \tilde{V}_{[-1,c]} \), we have from (***) that
\[ \tilde{V}_{[-1,c]}k_a = \left( \frac{1 - a\bar{c}}{1 - c\bar{a}} \right) k_{[-1,c](a)}. \]

We note that, for each \( c \) in \( D \), the equation \([-1,c](a) = a \) has a unique solution in \( D \), namely \( a(0) = 0 \) and
\[ a(c) = \frac{1 - \sqrt{1 - |c|^2}}{c} \]
for \( c \neq 0 \). Thus, we have \( \tilde{V}_{[-1,c]}(a(c)) = 1 \). Unitarity of \( V_{[-1,c]} \) now implies that \( \|\tilde{V}_{[-1,c]}\|_\infty = 1 \) for all \( c \) in \( D \). \( \square \)
Corollary 1. There is no real number $M > 0$ so that $M \| \tilde{X} \|_{\infty} \geq \| X \|$ for all $X$ in $Op(L^2_0(D))$. Equivalently, range$(Ber)$ is a non-closed linear subspace of $BC(D)$.

Proof. Since $V_{[1,c]}$ is unitary, we have $\|V_{[1,c]}\| = 1$ for all $c$ in $D$. But $\| \tilde{V}_{[1,c]} \|_{\infty} = 1 - |c|^2$ can be made arbitrarily small for $c$ in $D$. Thus, there is no real number $M > 0$ so that $M \| \tilde{X} \|_{\infty} \geq \| X \|$ for all $X$ in $Op(L^2_0(D))$.

By the Schwarz inequality, $Ber$ is a bounded linear transformation from the Banach space $Op(L^2_0(D))$ into the Banach space $BC(D)$. If range$(Ber)$ were closed in $BC(D)$, Ber would be a 1-1 bounded linear mapping onto the Banach space $\{\text{range}(Ber), \| \cdot \|_{\infty} \}$. The open mapping theorem would then give a norm estimate for $Ber^{-1} \tilde{X} = X$ of the form $\|X\| \leq M \| \tilde{X} \|_{\infty}$ for all $X$ in $Op(L^2_0(D))$. Conversely, if $\|X\| \leq M \| \tilde{X} \|_{\infty}$ for all $X$ in $Op(L^2_0(D))$, then a standard argument shows that range$(Ber)$ is closed in $BC(D)$. □

Remark. An unpublished proof of this result, using Toeplitz operators, is due to Fedor Nazarov.

Corollary 2. The range of $\tilde{V}_{[-1,c]}$ is exactly the interval $(0,1]$.

Proof. Note that $(* * * *)$ gives

$$\tilde{V}_{[-1,c]}(a) = \left[\frac{1 - |c|^2}{(1 - |c|^2) + |c - a|^2}\right]^2 (1 - |c|^2).$$

Hence, the range of $\tilde{V}_{[-1,c]}$ is a connected subset of the positive real line which includes $\{1\}$, is bounded by 1, and, by taking $|a|$ near 1, has points arbitrarily close to 0.

Remark. The unitary operator $V_{[-1,c]}$ has spectrum $\{+1,-1\}$ and is certainly not a positive operator despite the positivity of $\tilde{V}_{[-1,c]}$.

A modification of the $V_{[1,c]}$ shows that Berger’s Theorem fails for $\| \tilde{X} \|_{\infty}$.

Theorem 3. For $X_c = V_{[1,c]} + V_{[1,-c]}$, we have $\| \tilde{X}_c \|_{\infty} = 2(1 - |c|^2)$ and

$$\| \tilde{X}_c^2 \|_{\infty} = 4 \left( \frac{(1 + |c|^4)}{(1 + |c|^2)^2} \right)$$

for all $c$ in $D$. Thus, $\| \tilde{X}_c^2 \|_{\infty} > \| \tilde{X}_c \|_{\infty}^2$ for all $c$ with $1 > |c| > 0$.

Proof. This is a direct calculation using the facts that $V_{[1,c]} = V_{[1,\frac{2c}{1+|c|^2}]}$ and that $\tilde{V}_{[1,c]}(0) = 1 - |c|^2 = \| \tilde{V}_{[1,c]} \|_{\infty}$. Note that $X_c^2 = V_{[1,c]} + 2I + V_{[1,-c]}$. □

3. WEYL-TYPE UNITARY OPERATORS ON THE SEGAL-BARGMANN SPACE

We next briefly consider a space which is a model for Bergman spaces on bounded symmetric domains, even though the domain here is all of $C^n$. The Segal-Bargmann space $H^2(C^n, d\mu)$ is a Bergman space which consists of all entire functions which are square-integrable with respect to the normalized Gaussian measure $d\mu(z) = \exp(-|z|^2/2(2\pi)^n) d\mu(z)$. Here, $d\mu(z)$ is the standard Lebesgue volume measure on $C^n$. The Bergman kernel for evaluation at $c$ is just $K(z,c) = \exp(z \cdot c/2)$, where we take $z \cdot c = z_1c_1 + z_2c_2 + \cdots + z_n c_n$. Thus, $k_c(z) = \exp(z \cdot c/2 - |c|^2/4)$ is the
normalized kernel function. We limit our attention to the analogs of the \(V_{[1,c]}\) and \(V_{[-1,c]}\). These are the Weyl unitary operators acting on \(H^2(C^n, d\mu)\) by
\[
(W_c f)(z) = k_c(z) f(z - c)
\]
and the involutive unitary operators
\[
(U_c f)(z) = k_c(z) f(c - z).
\]

It is well known \([1]\) that the map \(c \to W_c\) gives a strongly continuous projective irreducible representation of \((C^n, +)\) which extends to a unitary representation of the Heisenberg group. For \(\chi_c(z) = \exp(i\text{Im}(z \cdot c))\), we have
\[
W_a W_b = \chi_a(b/2) W_{a+b}.
\]
It follows that \(W_c^* = W_c^{-1} = W_{-c}\). It is also easy to check that
\[
W_c k_a = \chi_c(a/2) k_{a+c}.
\]

For \(U_c\), it is easy to check that \(U_c^{-1} = U_c^* = U_c\) but the multiplicative structure is not evident. A direct calculation shows that
\[
U_c k_a = \chi_a(c/2) k_{c-a}.
\]

We can now establish results analogous to Theorem 2.

**Theorem 4.** We have \(\|\widetilde{W}_c\|_\infty = \exp(-|c|^2/4)\) and \(\|\widetilde{U}_c\|_\infty = 1\) for all \(c\) in \(C^n\).

*Proof.* We check first that
\[
\langle W_c k_a, k_a \rangle = \chi_c(a) \exp(-|c|^2/4).
\]
It follows immediately that \(\|\widetilde{W}_c\|_\infty = \exp(-|c|^2/4)\). We also have
\[
\langle U_c k_a, k_a \rangle = \exp(-|c - 2a|^2/4)
\]
and, taking \(a = c/2\), it follows that \(\|\widetilde{U}_c\|_\infty = 1\). \(\square\)

The method of Theorem 3 shows that Berger’s Theorem fails for \(\|\widetilde{X}\|_\infty\) with \(X\) in \(\text{Op}[H^2(C^n, d\mu)]\).

**Theorem 5.** For \(Y_c = W_c + W_{-c}\), we have \(\|\widetilde{Y}_c\|_\infty = 2 \exp(-|c|^2/4)\) and \(\|\widetilde{Y}_c^2\|_\infty = 2(1 + \exp(-|c|^2))\) for all \(c\) in \(C^n\). Thus, \(\|\widetilde{Y}_c^2\|_\infty > \|\widetilde{Y}_c\|_\infty^2\) for all \(c\) with \(|c| \neq 0\).

*Proof.* This is a direct calculation using the fact that \(W_c^2 = W_{2c}\). \(\square\)

4. Extensions to General Bounded Symmetric Domains

It is natural to try to give general versions of our results for operators on the Bergman space \(L^2_a(\Omega)\) of square-integrable holomorphic functions on \(\Omega\), a general bounded symmetric domain (BSD) in \(C^n\). Here we use normalized Lebesgue measure on \(\Omega\). We do not have a complete picture, but there is enough to justify a brief discussion.

BSD’s are Hermitian symmetric spaces of the non-compact type \([2, 7, 9]\). There is a standard classification of BSD’s going back to H. Cartan. We work in the Harish-Chandra realization of BSD’s as bounded convex domains \(\Omega\) containing the origin 0 of \(C^n\) and invariant under the map \(z \to \lambda z\) for \(\lambda\) in \(C\) and \(|\lambda| = 1\). The group \(\text{Aut}(\Omega)\) of biholomorphic automorphisms of \(\Omega\) is transitive. In particular, for each \(c\) in \(\Omega\), there is an automorphism \(\varphi_c\) so that: (1) \(\varphi_c \circ \varphi_{c'} = \text{identity}\), (2) \(\varphi_c(0) = c\), and (3) \(\varphi_c(a(c)) = a(c)\), for \(a(c)\) the midpoint, in the Bergman metric,
of the unique geodesic segment joining \( c \) to 0. Note that, on \( D \), \( \varphi_c = [-1, c] \) and \( a(c) \) is determined by (†).

For the Bergman kernel functions \( K(z, a) \) on \( \Omega \), we have \( K(z, a) = \overline{K(a, z)} \) and \( K(z, 0) = 1 \). It is also known that \( K(z, a) \neq 0 \); see [10]. For \( k_a(z) = K(z, a) \{ K(a, a) \}^{-1/2} \), we have \( \|k_a\| = 1 \) in \( L_a^2(\Omega) \). It is known that \( K(\lambda a, \lambda b) = K(a, b) \) for \( |\lambda| = 1 \) with \( \lambda \) in \( C \) and there are transformation laws [3] pp. 926-928

\[
(\dagger) \quad K(\varphi_c(z), \varphi_c(a))k_c(z)\overline{k_c(a)} = K(z, a).
\]

Considering the involutive unitary operators

\[
(U_c f)(z) = k_c(z)f(\varphi_c(z))
\]

on \( L_a^2(\Omega) \), we know [3] that \( U_c^* = U_c^{-1} = U_c \) and we can partially extend Theorem 2.

**Theorem 6.** For arbitrary \( a, c \) in \( \Omega \), we have

\[
(U_c k_a)(z) = \lambda(c, a)k_{\varphi_c(a)}(z)
\]

for all \( \lambda(c, a) \) in \( C \) with \( |\lambda(c, a)| = 1 \). Taking \( a = a(c) \) to be the fixed point of \( \varphi_c \) described above, we find that \( \|U_c\|_\infty = 1 \).

**Proof.** Using (\( \dagger \dagger \)), we have

\[
(\dagger \dagger \dagger)
K(\varphi_c(z), a)
= K(\varphi_c(z), \varphi_c(a))

= K(z, \varphi_c(a))K(c, c)
\overline{K(z, c)K(c, \varphi_c(a))}.
\]

It follows from the definition of \( U_c \) that

\[
(U_c k_a)(z) = k_c(z)k_{\varphi_c(a)}(z)
\]

\[
(\dagger \dagger \dagger \dagger)
= K(z, c)K(\varphi_c(z), a)
\overline{K(c, c)^{1/2}K(a, a)^{1/2}}.
\]

Combining (\( \dagger \dagger \dagger \)) and (\( \dagger \dagger \dagger \dagger \)) gives

\[
(\dagger \ast)
(U_c k_a)(z) = \lambda(c, a)k_{\varphi_c(a)}(z)
\]

and, since \( U_c \) is unitary, we must have \( |\lambda(c, a)| = 1 \).

Now taking \( a = a(c) \), the fixed point of \( \varphi_c \) discussed above, we have

\[
(U_c k_{\varphi_c(c)})(z) = \lambda(c, a(c))k_{\varphi_c(c)}(z)
\]

so that \( |\overline{U_c(a(c))}| = 1 \) and \( \|U_c\|_\infty = 1 \).

**Remarks.** It is not hard to check that

\[
\langle U_c k_a, k_a \rangle = \frac{K(\varphi_c(a), a)}{K(\varphi_c(a), c)}\frac{K(c, c)^{1/2}}{K(a, a)^{1/2}}.
\]

Since \( U_c^* = U_c \), it follows that

\[
(\dagger \dagger \ast)
\frac{K(\varphi_c(a), a)}{K(\varphi_c(a), c)}
\]

must be real-valued. In the case \( \Omega = D \), the expression in (\( \dagger \dagger \ast \)) is always positive. We do not know whether positivity persists in general.
For a general BSD Ω, the analysis of the Weyl-type operator
\[(V_c f)(z) = k_c(z) f(-\varphi_c(z))\]
is non-trivial. We can check that \(V_c\) is unitary, with
\[V_c k_a = \mu(c, a) k_{\varphi_c(-a)}\]
for \(|\mu(c, a)| = 1\) and all \(c, a\) in \(\Omega\). It remains difficult to determine \(\|\tilde{V}_c\|_\infty\).

5. PROBLEMS

The most obvious problems left open are:

**Problem 1.** For \(\Omega\) a BSD in \(\mathbb{C}^n\) with boundary \(\partial\Omega\), is
\[
\lim_{c \to \partial\Omega} \|\tilde{V}_c\|_\infty = 0
\]
for \(V_c\) defined by \((\dagger \star \dagger)\)?

**Problem 2.** Is there a bounded domain (perhaps not BSD) where \(\|\tilde{X}\|_\infty\) is an equivalent norm to \(\|X\|\) on \(Op(L^2_0(\Omega))\)?

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