BEREZIN TRANSFORM
AND WEYL-TYPE UNITARY OPERATORS
ON THE BERGMAN SPACE

L. A. COBURN

(Communicated by Richard Rochberg)

Abstract. For $D$ the open complex unit disc with normalized area measure, we consider the Bergman space $L^2_a(D)$ of square-integrable holomorphic functions on $D$. Induced by the group $\text{Aut}(D)$ of biholomorphic automorphisms of $D$, there is a standard family of Weyl-type unitary operators on $L^2_a(D)$. For all bounded operators $X$ on $L^2_a(D)$, the Berezin transform $\tilde{X}$ is a smooth, bounded function on $D$. The range of the mapping $\text{Ber}: X \mapsto \tilde{X}$ is invariant under $\text{Aut}(D)$. The “mixing properties” of the elements of $\text{Aut}(D)$ are visible in the Berezin transforms of the induced unitary operators. Computations involving these operators show that there is no real number $M > 0$ with $M \parallel \tilde{X} \parallel_{\infty} \geq \parallel X \parallel$ for all bounded operators $X$ and are used to check other possible properties of $\tilde{X}$. Extensions to other domains are discussed.

1. Introduction

For $H$ a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|v\| = \langle v, v \rangle^{1/2}$, we consider the algebra of all bounded linear operators $\text{Op}(H)$. For the unit sphere $S = \{v \in H : \langle v, v \rangle = 1\}$ and for $X$ in $\text{Op}(H)$, we have the usual operator norm $\|X\| = \sup\{\|Xv\| : v \in S\}$. We will also be concerned with the numerical range $W(X) = \{\langle Xv, v \rangle : v \in S\}$ and the numerical radius $w(X) = \sup\{|\langle Xv, v \rangle| : v \in S\}$. The set $W(X)$ is convex (Hausdorff’s Theorem) and its closure contains the spectrum of $X$. There is a standard norm estimate

$$w(X) \leq \|X\| \leq 2w(X)$$

and a not-so-standard power estimate (Berger’s Theorem [5])

$$w(X^n) \leq w(X)^n.$$

In the case that $H$ is the Bergman Hilbert space $L^2_a(D)$ or one of a large family of “reproducing kernel Hilbert spaces”, we have, for each $c$ in $D$, a reproducing kernel function $K(\cdot, c)$ so that, for any $f$ in $L^2_a(D)$,

$$f(c) = \langle f, K(\cdot, c) \rangle.$$

The normalized kernel functions $k_c(\cdot) \equiv K(\cdot, c)K(c, c)^{-1/2}$ play an important role in the analysis of operators on $L^2_a(D)$ as well as on other reproducing kernel spaces. In particular, for every bounded linear operator, we define the Berezin transform by $\tilde{X}(c) = \langle Xk_c, k_c \rangle$. The map $\text{Ber}(X) = \tilde{X}$ is one-to-one and $\tilde{X}(\cdot)$ is known to be
real-analytic [4] as well as Lipschitz with respect to the Bergman metric distance function on $D$ [6].

It is not hard to check that the range of $Ber$ contains all bounded holomorphic functions on $D$. Clearly, the range of $\tilde{X}$ is contained in $W(X)$ and, for $\|f\|_\infty = \sup\{|f(z)| : z \in D\}$, we have $\|\tilde{X}\|_\infty \leq w(X)$ so that $\tilde{X}$ is in the Banach space [8] p. 121 of bounded continuous functions $BC(D)$.

Convexity of range($\tilde{X}$) is easily seen to fail (take $X$ to be multiplication by a suitable holomorphic function). It is a natural problem to determine to what extent $\|\tilde{X}\|_\infty$ imitates $w(X)$. Is there a real number $M > 0$ with $M \|\tilde{X}\|_\infty \geq \|X\|$ for all $X$? Is $\|\tilde{X}^2\|_\infty \leq \|\tilde{X}\|_\infty^2$ for all $X$? The unitary operators $V_{\lambda,c}$ and related operators discussed in the next section provide examples to show that these estimates do not hold for $Op(L^2_e(D))$. Similar constructions yield the same result for the Segal-Bargmann space of Gaussian square-integrable entire functions on complex $n$-space $\mathbf{C}^n$. The extension of our analysis to general bounded symmetric domains is plausible but presents significant difficulties.

2. WEYL-TYPE UNITARY OPERATORS ON $L^2_e(D)$

We consider the full group $Aut(D)$, given for $\lambda$ in $\mathbf{C}$ with $|\lambda| = 1$ and $c, z$ in $D$, by

$$[\lambda, c](z) = \lambda \frac{z - c}{1 - \bar{c}z}.$$  

The group $Aut(D)$ acts on $L^2_e(D)$ by

$$([\lambda, c] f)(z) = f(\lambda \frac{z - c}{1 - \bar{c}z}),$$

and it is standard that

$$(V_{\lambda,c} f)(z) = k_c(z) f(\lambda \frac{z - c}{1 - \bar{c}z})$$

is a unitary transformation from $L^2_e(D)$ to itself, where

$$k_c(z) = \frac{1 - |c|^2}{(1 - \bar{c}z)^2}$$

is the normalized Bergman kernel for evaluation at $c$.

Multiplication on $Aut(D)$ is given by

$$(*) \quad [\lambda, c][\mu, d] = [\mu + \frac{1 + \bar{c}d}{1 + \bar{d}\lambda}, \frac{\lambda d + c}{1 + \bar{d}\lambda}],$$

and it is easy to check that the map $[\lambda, c] \to V_{\lambda,c}$ is a projective unitary representation of $Aut(D)$ with

$$(**) \quad V_{\lambda,c} V_{[\mu, d]} = \frac{1 + d\bar{\lambda}}{1 + c\bar{d} \lambda} V_{[\lambda, c][\mu, d]}.$$

We will be interested in the Berezin transform of $V_{\lambda,c}$. We first check that

$$(* *) \quad V_{[\lambda, c]} k_a = \frac{1 + \bar{\lambda}\bar{a}}{1 + a\lambda} k_{[\lambda, -c](a)}.$$

It follows, using the defining property of the reproducing kernel, that

$$(* * *) \quad \langle V_{[\lambda, c]} k_a, k_a \rangle = \frac{(1 - |a|^2)^2(1 - |c|^2)}{(1 - \lambda |a|^2)(a\bar{\lambda} - \bar{a}c)^2}.$$
Remarks. It follows from the above discussion that
\[ V^*_\lambda = V_{\overline{\lambda} - \lambda} = V^\dagger_{\overline{\lambda}}. \]
The involutive unitary operators \( V^*[a,c] \) are standard objects in the analysis of \( D \) as
the prototypical bounded symmetric domain. We will use the elementary formula
\[ V^2[a,c] = V_{[1, \frac{2x+2y}{4}]} \] in our analysis.

Using the above remark, we first have

**Theorem 1.** The range of Ber: \( \text{Op}\{L^2(D)\} \rightarrow BC(D) \) is invariant under \( \text{Aut}(D) \).

**Proof.** Using (***), we can check that
\[ \tilde{X} \{ \lambda, c \}(a) = \langle Xk_{[\lambda, c]}(a), k_{[\lambda, c]}(a) \rangle = \langle XV_{[\lambda, c]}k_a, V_{[\lambda, c]}k_a \rangle = \langle (V_{[\lambda, c]}X)k_a, k_a \rangle. \]

**Corollary.** The projective unitary representation \( [\lambda, c] \rightarrow V_{[\lambda, c]} \) of \( \text{Aut}(D) \) on \( L^2(D) \) is irreducible.

**Proof.** For \( a, b \) arbitrary in \( D \), \((-1, a)[-1, b]) = b \). For \( X \) in \( \text{Op}\{L^2(D)\} \) with \( XV_{[\lambda, c]} = V_{[\lambda, c]}X \) for all \( \lambda, c \) in \( \text{Aut}(D) \), we have \( \tilde{X} \{ \lambda, c \}(a) = \tilde{X}(a) \) for all \( a \) in \( D \). Taking \( [\lambda, c] = [-1, a][-1, b] \) gives \( \tilde{X}(a) = \tilde{X}(b) \) for arbitrary \( b \). Thus, \( \tilde{X} \) must be a constant function so that \( X \) is a scalar multiple of the identity operator. \( \square \)

**Remark.** The irreducibility is certainly “well known”.

Next, we explicitly calculate \( \|\tilde{V}_{[1,c]}\|_\infty \) for \( \lambda = \pm 1 \).

**Theorem 2.** We have \( \|\tilde{V}_{[1,c]}\|_\infty = 1 - |c|^2 \) and \( \|\tilde{V}_{[-1,c]}\|_\infty = 1 \) for all \( c \) in \( D \).

**Proof.** First, for \( \tilde{V}_{[1,c]} \), we note that (****) gives
\[ \langle \tilde{V}_{[1,c]}k_a, k_a \rangle = \frac{(1 - |a|^2)^2(1 - |c|^2)}{|(1 - |a|^2) + (\overline{ac} - a\overline{c})|^2}. \]

Since \( i(\overline{ac} - a\overline{c}) \) is real, we see that
\[ |(1 - |a|^2) + (\overline{ac} - a\overline{c})|^2 = (1 - |a|^2)^2 + |\overline{ac} - a\overline{c}|^2, \]
so
\[ |\langle \tilde{V}_{[1,c]}k_a, k_a \rangle| = \frac{(1 - |a|^2)^2(1 - |c|^2)}{(1 - |a|^2)^2 + |\overline{ac} - a\overline{c}|^2}. \]

Since \( \tilde{V}_{[1,c]}(0) = 1 - |c|^2 \), it follows that \( \|\tilde{V}_{[1,c]}\|_\infty = 1 - |c|^2 \).

For \( \tilde{V}_{[-1,c]} \), we have from (****) that
\[ \tilde{V}_{[-1,c]}k_a = \frac{1 - \alpha\overline{c}}{1 - \overline{c}\alpha} \tilde{V}_{[-1,c]}(a). \]
We note that, for each \( c \) in \( D \), the equation \([-1, c](a) = a \) has a unique solution in \( D \), namely \( a(0) = 0 \) and
\[ a(c) = \frac{1 - \sqrt{1 - |c|^2}}{c} \]
for \( c \neq 0 \). Thus, we have \( \tilde{V}_{[-1,c]}(a) = 1 \). Unitarity of \( \tilde{V}_{[-1,c]} \) now implies that \( \|\tilde{V}_{[-1,c]}\|_\infty = 1 \) for all \( c \) in \( D \). \( \square \)
Corollary 1. There is no real number \( M > 0 \) so that \( M \| \tilde{X} \|_{\infty} \geq \| X \| \) for all \( X \) in \( \text{Op}(L^2_\alpha(D)) \). Equivalently, \( \text{range}(\text{Ber}) \) is a non-closed linear subspace of \( BC(D) \).

Proof. Since \( V_{[1,c]} \) is unitary, we have \( \| V_{[1,c]} \| = 1 \) for all \( c \) in \( D \). But \( \| \tilde{V}_{[1,c]} \|_{\infty} = 1 - |c|^2 \) can be made arbitrarily small for \( c \) in \( D \). Thus, there is no real number \( M > 0 \) so that \( M \| \tilde{X} \|_{\infty} \geq \| X \| \) for all \( X \) in \( \text{Op}(L^2_\alpha(D)) \).

By the Schwarz inequality, \( \text{Ber} \) is a bounded linear transformation from the Banach space \( \text{Op}(L^2_\alpha(D)) \) into the Banach space \( BC(D) \). If \( \text{range}(\text{Ber}) \) were closed in \( BC(D) \), \( \text{Ber} \) would be a 1-1 bounded linear mapping onto the Banach space \( \{ \text{range}(\text{Ber}), \| \cdot \|_{\infty} \} \). The open mapping theorem would then give a norm estimate for \( \text{Ber}^{-1} \tilde{X} = X \) of the form \( \| X \| \leq M \| \tilde{X} \|_{\infty} \) for all \( X \) in \( \text{Op}(L^2_\alpha(D)) \). Conversely, if \( \| X \| \leq M \| \tilde{X} \|_{\infty} \) for all \( X \) in \( \text{Op}(L^2_\alpha(D)) \), then a standard argument shows that \( \text{range}(\text{Ber}) \) is closed in \( BC(D) \).

Remark. An unpublished proof of this result, using Toeplitz operators, is due to Fedor Nazarov.

Corollary 2. The range of \( \tilde{V}_{[-1,c]} \) is exactly the interval \( (0,1] \).

Proof. Note that \((***)\) gives

\[
\tilde{V}_{[-1,c]}(a) = \left( \frac{1 - |a|^2}{1 - |c|^2 + |c - a|^2} \right)^2 (1 - |c|^2).
\]

Hence, the range of \( \tilde{V}_{[-1,c]} \) is a connected subset of the positive real line which includes \( \{1\} \), is bounded by 1, and, by taking \( |a| \) near 1, has points arbitrarily close to 0.

Remark. The unitary operator \( V_{[-1,c]} \) has spectrum \( \{+1,-1\} \) and is certainly not a positive operator despite the positivity of \( \tilde{V}_{[-1,c]} \).

A modification of the \( V_{[1,c]} \) shows that Berger’s Theorem fails for \( \| \tilde{X} \|_{\infty} \).

Theorem 3. For \( X_c = V_{[1,c]} + V_{[-1,c]} \), we have \( \| \tilde{X}_c \|_{\infty} = 2(1 - |c|^2) \) and

\[
\| \tilde{X}_c^2 \|_{\infty} = 4 \left( \frac{1 + |c|^4}{1 + |c|^2} \right)
\]

for all \( c \) in \( D \). Thus, \( \| \tilde{X}_c^2 \|_{\infty} > \| \tilde{X}_c \|_{\infty}^2 \) for all \( c \) with \( 1 > |c| > 0 \).

Proof. This is a direct calculation using the facts that \( V_{[1,c]} = V_{[1,\frac{2c}{1+|c|^2}]} \) and that \( \tilde{V}_{[1,c]}(0) = 1 - |c|^2 = \| \tilde{V}_{[1,c]} \|_{\infty} \). Note that \( X_c^2 = V_{[1,c]}^2 + 2I + V_{[-1,c]}^2 \).

3. WEYL-TYPE UNITARY OPERATORS ON THE SEGAL-BARGMANN SPACE

We next briefly consider a space which is a model for Bergman spaces on bounded symmetric domains, even though the domain here is all of \( \mathbb{C}^n \). The Segal-Bargmann space \( H^2(C^n, d\mu) \) is a Bergman space which consists of all entire functions which are square-integrable with respect to the normalized Gaussian measure \( d\mu(z) = \exp[-|z|^2/2](2\pi)^{-n} dv(z) \). Here, \( dv(z) \) is the standard Lebesgue volume measure on \( \mathbb{C}^n \). The Bergman kernel for evaluation at \( c \) is just \( K(z,c) = \exp(z \cdot c/2 - |c|^2/4) \), where we take \( z \cdot c = z_1c_1 + z_2c_2 + \cdots + z_n c_n \). Thus, \( k_c(z) = \exp(z \cdot c/2 - |c|^2/4) \) is the
normalized kernel function. We limit our attention to the analogs of the $V_{[1,c]}$ and $V_{[-1,c]}$. These are the Weyl unitary operators acting on $H^2(\mathbb{C}^n, d\mu)$ by
\[(W_c f)(z) = k_c(z) f(z - c)\]
and the involutive unitary operators
\[(U_c f)(z) = k_c(z) f(c - z).\]

It is well known \[1\] that the map $c \to W_c$ gives a strongly continuous projective irreducible representation of $(\mathbb{C}^n, +)$ which extends to a unitary representation of the Heisenberg group. For $\chi_c(z) = \exp(i \text{Im}(z \cdot c))$, we have
\[W_a W_b = \chi_a(b/2) W_{a+b}.\]
It follows that $W_c^{-1} = W_{-c}^* = U_c$ but the multiplicative structure is not evident. A direct calculation shows that
\[U_c k_a = \chi_a(c/2) k_{c-a}.\]

We can now establish results analogous to Theorem 2.

**Theorem 4.** We have $\|\tilde{W}_c\|_\infty = \exp(-|c|^2/4)$ and $\|\tilde{U}_c\|_\infty = 1$ for all $c$ in $\mathbb{C}^n$.

**Proof.** We check first that
\[\langle W_c k_a, k_a \rangle = \chi_c(a) \exp(-|c|^2/4).\]
It follows immediately that $\|\tilde{W}_c\|_\infty = \exp(-|c|^2/4)$. We also have
\[\langle U_c k_a, k_a \rangle = \exp(-|c - 2a|^2/4)\]
and, taking $a = c/2$, it follows that $\|\tilde{U}_c\|_\infty = 1$. \qed

The method of Theorem 3 shows that Berger’s Theorem fails for $\|X\|_\infty$ with $X$ in $\text{Op}(H^2(\mathbb{C}^n, d\mu))$.

**Theorem 5.** For $Y_c = W_c + W_{-c}$, we have $\|\tilde{Y}_c\|_\infty = 2 \exp(-|c|^2/4)$ and $\|\tilde{Y}_c^2\|_\infty = 2(1 + \exp(-|c|^2))$ for all $c$ in $\mathbb{C}^n$. Thus, $\|\tilde{Y}_c^2\|_\infty > \|\tilde{Y}_c^2\|_\infty$ for all $c$ with $|c| \neq 0$.

**Proof.** This is a direct calculation using the fact that $W_c^2 = W_{2c}$. \qed

4. Extensions to General Bounded Symmetric Domains

It is natural to try to give general versions of our results for operators on the Bergman space $L^2_a(\Omega)$ of square-integrable holomorphic functions on $\Omega$, a general bounded symmetric domain (BSD) in $\mathbb{C}^n$. Here we use normalized Lebesgue measure on $\Omega$. We do not have a complete picture, but there is enough to justify a brief discussion.

BSD’s are Hermitian symmetric spaces of the non-compact type \[2, 7, 9\]. There is a standard classification of BSD’s going back to H. Cartan. We work in the Harish-Chandra realization of BSD’s as bounded convex domains $\Omega$ containing the origin 0 of $\mathbb{C}^n$ and invariant under the map $z \to \lambda z$ for $\lambda$ in $\mathbb{C}$ and $|\lambda| = 1$. The group $\text{Aut}(\Omega)$ of biholomorphic automorphisms of $\Omega$ is transitive. In particular, for each $c$ in $\Omega$, there is an automorphism $\varphi_c$ so that: (1) $\varphi_c \circ \varphi_c = \text{identity}$, (2) $\varphi_c(0) = c$, and (3) $\varphi_c(a(c)) = a(c)$, for $a(c)$ the midpoint, in the Bergman metric,
of the unique geodesic segment joining \( c \) to 0. Note that, on \( D \), \( \varphi_c = [-1, c] \) and \( a(c) \) is determined by (†).

For the Bergman kernel functions \( K(z, a) \) on \( \Omega \), we have \( K(z, a) = \overline{K(a, z)} \) and \( K(z, 0) = 1 \). It is also known that \( K(z, a) \neq 0 \); see [10]. For \( k_a(z) = K(z, a)\{K(a, a)\}^{-1/2} \), we have \( \|k_a\| = 1 \) in \( L^2_a(\Omega) \). It is known that \( K(\lambda a, \lambda b) = K(a, b) \) for \( |\lambda| = 1 \) with \( \lambda \) in \( C \) and there are transformation laws [3 pp. 926-928]

(††)

\[
K(\varphi_c(z), \varphi_c(a))k_c(z)k_c(a) = K(z, a).
\]

Considering the involutive unitary operators

\[
(U_c f)(z) = k_c(z)f(\varphi_c(z))
\]
on \( L^2_a(\Omega) \), we know [3] that \( U^*_c = U^{-1}_c = U_c \) and we can partially extend Theorem 2.

**Theorem 6.** For arbitrary \( a, c \) in \( \Omega \), we have

\[
(U_c k_a)(z) = \lambda(c, a)k_{\varphi_c(a)}(z)
\]

for all \( \lambda(c, a) \) in \( C \) with \( |\lambda(c, a)| = 1 \). Taking \( a = a(c) \) to be the fixed point of \( \varphi_c \) described above, we find that \( \|U_c\|_\infty = 1 \).

**Proof.** Using (†††), we have

\[
K(\varphi_c(z), a) = K(\varphi_c(z), \varphi_c(a))k_c(z)k_c(a)
\]

\[
(\dagger \dagger \dagger)
\]

\[
K(\varphi_c(z), a) = \frac{K(z, \varphi_c(a))K(c, c)}{K(z, c)K(c, \varphi_c(a))}.
\]

It follows from the definition of \( U_c \) that

\[
(U_c k_a)(z) = k_c(z)k_{\varphi_c(a)}(z)
\]

\[
(\dagger \dagger \dagger \dagger)
\]

\[
= \frac{K(z, c)K(\varphi_c(z), a)}{K(c, c)^{1/2}K(a, a)^{1/2}}.
\]

Combining († † † †) and († † † † †) gives

\[
(\dagger *)
\]

\[
(U_c k_a)(z) = \lambda(c, a)k_{\varphi_c(a)}(z)
\]

and, since \( U_c \) is unitary, we must have \( |\lambda(c, a)| = 1 \).

Now taking \( a = a(c) \), the fixed point of \( \varphi_c \) discussed above, we have

\[
(\dagger * *)
\]

\[
(U_c k_{\varphi(a(c))})(z) = \lambda(c, a(c))k_{\varphi(a(c))}(z)
\]

so that \( |\tilde{U}_c(a(c))| = 1 \) and \( \|\tilde{U}_c\|_\infty = 1 \).

\( \square \)

**Remarks.** It is not hard to check that

\[
\langle U_c k_a, k_a \rangle = \frac{K(\varphi_c(a), a)}{K(\varphi_c(a), c)} \frac{K(c, c)^{1/2}}{K(a, a)}.
\]

Since \( U^*_c = U_c \), it follows that

\[
(\dagger * \dagger)
\]

\[
\frac{K(\varphi_c(a), a)}{K(\varphi_c(a), c)}
\]

must be real-valued. In the case \( \Omega = D \), the expression in († * †) is always positive.

We do not know whether positivity persists in general.
For a general BSD Ω, the analysis of the Weyl-type operator
\((V_c f)(z) = k_c(z) f(-\varphi_c(z))\)
is non-trivial. We can check that \(V_c\) is unitary, with
\[ V_c k_a = \mu(c, a) k_{-\varphi_c(-a)} \]
for \(|\mu(c, a)| = 1\) and all \(c, a\) in \(\Omega\). It remains difficult to determine \(\|\tilde{V}_c\|_\infty\).

5. PROBLEMS

The most obvious problems left open are:

Problem 1. For \(\Omega\) a BSD in \(\mathbb{C}^n\) with boundary \(\partial \Omega\), is
\[ \lim_{c \to \partial \Omega} \|\tilde{V}_c\|_\infty = 0 \]
for \(V_c\) defined by \((\dagger \ast \dagger)\)?

Problem 2. Is there a bounded domain (perhaps not BSD) where \(\|\tilde{X}\|_\infty\) is an equivalent norm to \(\|X\|\) on \(Op(L^2_\alpha(\Omega))\)?

ACKNOWLEDGMENTS

The author thanks his muse and companion Kathryn Grossman Cohen for encouraging him to complete this project. The author also thanks Adam Koranyi for a useful discussion.

REFERENCES


DEPARTMENT OF MATHEMATICS, THE STATE UNIVERSITY OF NEW YORK AT BUFFALO, BUFFALO, NEW YORK 14260
E-mail address: lcoburn@buffalo.edu