

## COPIES OF $c_0(\Gamma)$ IN $C(K, X)$ SPACES

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ABSTRACT. We extend some results of Rosenthal, Cembranos, Freniche, E. Saab-P. Saab and Ryan to study the geometry of copies and complemented copies of  $c_0(\Gamma)$  in the classical Banach spaces  $C(K, X)$  in terms of the cardinality of the set  $\Gamma$ , of the density and caliber of  $K$  and of the geometry of  $X$  and its dual space  $X^*$ . Here are two sample consequences of our results:

- (1) If  $C([0, 1], X)$  contains a copy of  $c_0(\aleph_1)$ , then  $X$  contains a copy of  $c_0(\aleph_1)$ .
- (2)  $C(\beta\mathbb{N}, X)$  contains a complemented copy of  $c_0(\aleph_1)$  if and only if  $X$  contains a copy of  $c_0(\aleph_1)$ .

Some of our results depend on set-theoretic assumptions. For example, we prove that it is relatively consistent with ZFC that if  $C(K)$  contains a copy of  $c_0(\aleph_1)$  and  $X$  has dimension  $\aleph_1$ , then  $C(K, X)$  contains a complemented copy of  $c_0(\aleph_1)$ .

### 1. INTRODUCTION

Let  $K$  be a compact Hausdorff space and  $X$  a Banach space. By  $C(K, X)$  we denote the Banach space of all continuous  $X$ -valued functions defined on  $K$  and equipped with the supremum norm. This space will be denoted by  $C(K)$  in the case  $X = \mathbb{R}$ . For a set  $\Gamma$ ,  $c_0(\Gamma)$  is the Banach space of all scalar-valued maps  $f$  on  $\Gamma$  with the property that for every  $\varepsilon > 0$ , the set  $\{\gamma \in \Gamma : |f(\gamma)| \geq \varepsilon\}$  is finite and equipped with the supremum norm. We will often refer to  $c_0(\Gamma)$  as  $c_0(\tau)$  when the cardinality of  $\Gamma$  (denoted by  $|\Gamma|$ ) is equal to  $\tau$ . As usual, this space will be denoted by  $c_0$  when  $\tau = \aleph_0$ .

Since  $C(K, X)$  contains complemented copies of both  $C(K)$  and  $X$ , it is clear that if  $X$  or  $C(K)$  contains a copy of  $c_0(\tau)$ , then so does  $C(K, X)$ . Of course, the same can be said about complemented copies of  $c_0(\tau)$ . The goal of this paper is to study whether or not the converse is true. To be specific, we consider first the following.

**Problem 1.1.** *Suppose that  $C(K, X)$  contains a copy of  $c_0(\tau)$ . Must at least one of  $C(K)$  or  $X$  contain a copy of  $c_0(\tau)$ ?*

When  $\tau = \aleph_0$ , the answer is well known and easily seen to be affirmative. For  $\tau \geq \aleph_1$ , Theorem 3.1 provides a partial affirmative answer to this question. But it turns out that, even for  $\tau = \aleph_1$ , set-theoretic axioms are needed to determine

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the answer. It is negative in the presence of the Continuum Hypothesis (CH) (see Proposition 3.2) and affirmative when Martin's Axiom (MA) and the negation of CH are assumed (see Corollary 3.4).

A more satisfying answer to Problem 1.1 can be given if we know more about the compact space  $K$ . In Theorem 3.3, which is the first main result of this paper, we prove that if  $\tau \geq \aleph_0$  is a caliber of  $K$  and  $C(K, X)$  contains a copy of  $c_0(\tau)$ , then  $X$  contains a copy of  $c_0(\tau)$ . For example, either this result or Theorem 3.6, the second of our main results, yields that if  $K$  is a compact metric space and  $C(K, X)$  contains a copy of  $c_0(\aleph_1)$ , then  $X$  must contain a copy of  $c_0(\aleph_1)$ . In Section 3, we give precise statements and proofs of these results, their consequences and relevant definitions. The key combinatorial tools that we use are some results of Haskell Rosenthal. Since these results are so fundamental to our work, we state them in a very brief Section 2.

Section 4 is motivated by the following problem.

**Problem 1.2.** *What assumptions on  $K$  and  $X$  yield a complemented copy of  $c_0(\tau)$  in  $C(K, X)$ ?*

A central concept used here is the Josefson-Nissenzweig- $\alpha$  property (in brief, the  $JN_\alpha$  property), which is a natural extension of the result of the Josefson-Nissenzweig theorem to a nonseparable setting (see Definition 4.3). The main result of Section 4, Theorem 4.5, is the third of the main results of this paper. It yields a complemented copy of  $c_0(\aleph_\alpha)$  in the injective tensor product of two Banach spaces  $X$  and  $Y$  if  $X$  has the  $JN_\alpha$  property and  $Y$  contains a copy of  $c_0(\aleph_\alpha)$ . As an application, we prove in Corollary 4.7 that it is relatively consistent with ZFC that for all compact  $K$  and Banach spaces  $X$  with dimension  $\aleph_1$ , if  $C(K)$  contains a copy of  $c_0(\aleph_1)$ , then  $C(K, X)$  contains a complemented copy of  $c_0(\aleph_1)$ .

In Section 5, we use the results of Sections 3 and 4 to solve Problem 1.2 completely if  $K$  is the Cantor cube  $\mathbf{2}^m = \{0, 1\}^m$  where  $m \geq \aleph_0$ , or if  $K$  is the Stone-Ćech compactification  $\beta I$  of a discrete set  $I$  with  $|I| = 2^m$ . More precisely, under these conditions, Corollary 5.4 states that a Banach space  $X$  contains a copy of  $c_0(2^m)$  if and only if  $C(K, X)$  contains a complemented copy of  $c_0(2^m)$ .

For the most part, our notation and terminology are standard and introduced as needed. (Undefined terms may be found in [10] and [12].) Some that have already been used in this introduction or are needed shortly are given now. We write  $X \sim Y$  when the Banach spaces  $X$  and  $Y$  are isomorphic,  $Y \hookrightarrow X$  when  $X$  contains a copy of  $Y$ , that is, contains a subspace isomorphic to  $Y$ ,  $Y \overset{c}{\hookrightarrow} X$  when  $X$  contains a complemented copy of  $Y$ , and  $X \twoheadrightarrow Y$  when  $Y$  is isomorphic to a quotient of  $X$ . The *density character* of a topological space  $S$  (denoted by  $\text{dens}(S)$ ) is the smallest cardinality of a dense subset of  $S$ . Following standard practice, we refer to the density character of a Banach space  $X$  as its *dimension* and denote it by  $\dim(X)$ .

## 2. SOME RESULTS OF HASKELL ROSENTHAL

Before turning to the study of copies of  $c_0(\tau)$  in  $C(K, X)$  spaces we state some important results of H. Rosenthal on  $c_0(\tau)$  spaces, as they play a central role in the sequel.

Let  $\tau > \aleph_0$ . We recall that a compact Hausdorff space  $K$  satisfies the  $\tau$ -*chain condition* if every disjoint family of open subsets of  $K$  has cardinality less than  $\tau$ .

As is customary, we refer to the  $\aleph_1$ -chain condition as the *countable chain condition* or just the ccc [15, p. 226].

**Theorem 2.1** ([15, p. 230]). *Let  $K$  be a compact Hausdorff space and  $\tau > \aleph_0$ . Then  $C(K)$  contains no copy of  $c_0(\tau)$  if and only if  $K$  satisfies the  $\tau$ -chain condition.*

**Theorem 2.2** ([15, p. 227]). *Let  $S$  be a topological space satisfying the  $\tau$ -chain condition with  $\tau > \aleph_0$  and suppose that  $\mathcal{F}$  is a family of open subsets of  $S$  with  $|\mathcal{F}| = \tau$ . Then there exists an infinite sequence  $F_1, F_2, \dots$  of distinct members of  $\mathcal{F}$  with  $\bigcap_{i=1}^\infty F_i \neq \emptyset$ .*

Let  $\tau \geq \aleph_0$ . Let  $(e_i)_{i \in \tau}$  be the usual unit-vector basis of  $c_0(\tau)$ . That is, if  $i, j \in \tau$ , then  $e_i(j) = 1$  if  $i = j$  and  $e_i(j) = 0$  if  $i \neq j$ . Given a subset  $\Gamma$  of  $\tau$ , we identify  $c_0(\Gamma)$  with the subspace of  $c_0(\tau)$  consisting of those elements  $g$  such that  $g(\gamma) = 0$  for every  $\gamma \notin \Gamma$ .

**Theorem 2.3** ([14, Remark following Theorem 3.4]). *Let  $X$  be a Banach space and  $\tau \geq \aleph_0$ . Suppose that there exists an operator  $T : c_0(\tau) \rightarrow X$  such that  $\inf\{\|T(e_i)\| : i \in \tau\} > 0$ . Then there exists a subset  $\Gamma$  of  $\tau$  with  $|\Gamma| = \tau$  such that  $T|_{c_0(\Gamma)}$  is an isomorphism onto its image.*

### 3. ON COPIES OF $c_0(\tau)$ IN $C(K, X)$ SPACES

The results in this section follow from the following simple idea. Suppose that  $(f_i)_{i \in \tau} \subseteq C(K, X)$  is equivalent to the usual unit vector basis of  $c_0(\tau)$ . That is, there are constants  $\delta > 0$  and  $M < \infty$  such that, for all finite subsets  $F$  of  $\tau$  and for all scalars  $(t_i)_{i \in F}$ ,

$$\delta \max_{i \in F} |t_i| \leq \left\| \sum_{i \in F} t_i f_i \right\| \leq M \max_{i \in F} |t_i|.$$

We seek a point  $k_0 \in K$  with the property that, for a “large” subset  $\sigma$  of  $\tau$ , the set  $(f_i(k_0))_{i \in \sigma}$  is equivalent to the usual unit vector basis of  $c_0(\sigma)$  in  $X$ . The results of Rosenthal quoted in Section 2 provide the combinatorial mechanisms for achieving this.

Our first result is a vector-valued extension of Theorem 2.1.

**Theorem 3.1.** *Let  $K$  be a compact Hausdorff space,  $X$  a Banach space and  $\tau > \aleph_0$ . Then*

$$c_0(\tau) \hookrightarrow C(K, X) \iff c_0(\tau) \hookrightarrow C(K) \text{ or } c_0 \hookrightarrow X.$$

*Proof.* Suppose that  $C(K, X)$  contains a copy of  $c_0(\tau)$ . Then there exists an isomorphism  $T$  from  $c_0(\tau)$  onto a subspace of  $C(K, X)$ . Let  $\delta > 0$  be such that  $\|T(x)\| \geq \delta$  for all  $x \in c_0(\tau)$  with  $\|x\| = 1$ .

For each  $i \in \tau$ , let

$$U_i = \{k \in K : \|T(f_i)(k)\| > \delta/2\}.$$

Now assume that  $C(K)$  contains no copy of  $c_0(\tau)$ . According to Theorem 2.1,  $K$  satisfies the  $\tau$ -chain condition. Since  $(U_\gamma)_{\gamma \in \tau}$  is a  $\tau$ -family of open sets of  $K$ , there exists by Theorem 2.2 an infinite sequence  $\gamma_1, \gamma_2, \dots$  of distinct elements of  $\tau$  with  $\bigcap_{i=1}^\infty U_{\gamma_i} \neq \emptyset$ .

Pick  $k_0 \in \bigcap_{i=1}^\infty U_{\gamma_i}$ . Then, for all  $i \in \mathbb{N}$ ,

$$\|T(f_{\gamma_i})(k_0)\| > \delta/2.$$

Let  $P_{k_0}$  be the operator from  $C(K, X)$  to  $X$  defined by  $P_{k_0}(g) = g(k_0)$  for  $g \in C(K, X)$ . Put  $I = \{\gamma_i : i \in \mathbb{N}\}$ . Let us consider the operator  $L = P_{k_0}T|_{c_0(I)} : c_0(I) \rightarrow X$ . Consequently

$$\inf \{\|L(f_{\gamma_i})\| : i \in \mathbb{N}\} > \delta/2.$$

So, according to Theorem 2.3, there exists an infinite subset  $J$  of  $I$  such that  $L|_{c_0(J)}$  is an isomorphism onto its image. This completes the proof of the theorem.  $\square$

Theorem 3.1 is optimal. That is, even in the case  $\tau = \aleph_1$ , we cannot replace  $c_0$  by  $c_0(\aleph_1)$ . Indeed, assuming CH, Laver and Galvin (see [7]) independently constructed a compact Hausdorff space  $K_1$  satisfying the ccc such that the topological product  $K_1 \times K_1$  does not satisfy the ccc. Translating this result into the present context by using Theorem 2.1, we conclude the following.

**Proposition 3.2.** *Assume CH. Then there is a compact Hausdorff space  $K_1$  such that*

$$c_0(\aleph_1) \hookrightarrow C(K_1 \times K_1) \sim C(K_1, C(K_1)) \text{ but } c_0(\aleph_1) \not\hookrightarrow C(K_1).$$

Proposition 3.2 suggests that stronger positive results require either more assumptions on  $K$  or the Banach space  $X$  (in Proposition 3.2,  $X = C(K_1)$ ) or other models of set theory. Theorem 3.3, the first of the main results of this paper, is exactly of this sort.

Recall that a cardinal  $\tau > \aleph_0$  is said to be a *caliber* [3] of the topological space  $S$  if for any family  $\{U_i : i \in \tau\}$  of nonempty open sets in  $S$ , there exists a subset  $\Gamma$  of  $\tau$  such that  $|\Gamma| = \tau$  and  $\bigcap_{i \in \Gamma} U_i \neq \emptyset$ .

**Theorem 3.3.** *Let  $K$  be a compact Hausdorff space,  $X$  a Banach space and  $\tau > \aleph_0$ . If  $\tau$  is a caliber of  $K$ , then*

$$c_0(\tau) \hookrightarrow C(K, X) \implies c_0(\tau) \hookrightarrow X.$$

*Proof.* Assume that  $C(K, X)$  contains a copy of  $c_0(\tau)$  and  $\tau$  is a caliber of  $K$ . Let  $T$  and  $(U_i)_{i \in \tau}$  be as in the proof of Theorem 3.1. Since  $(U_i)_{i \in \tau}$  is a family of open sets of  $K$ , there exists a subset  $\Gamma$  of  $\tau$  with  $|\Gamma| = \tau$  such that  $\bigcap_{\gamma \in \Gamma} U_\gamma \neq \emptyset$ . Fix  $k_1$  in  $\bigcap_{\gamma \in \Gamma} U_\gamma$ . Then for all  $\gamma \in \Gamma$ ,

$$\|T(f_\gamma)(k_1)\| > M/2.$$

Let  $Q_{k_1}$  be the operator from  $C(K, X)$  to  $X$  defined by  $Q_{k_1}(g) = g(k_1)$  for  $g \in C(K, X)$ . Finally, consider the operator  $R = Q_{k_1}T|_{c_0(\Gamma)} : c_0(\Gamma) \rightarrow X$ . Hence we have

$$\inf \{\|R(f_\gamma)\| : \gamma \in \Gamma\} > M/2.$$

Therefore by Theorem 2.3 there exists a subset  $\Gamma_1$  of  $\Gamma$  with  $|\Gamma_1| = |\Gamma|$  such that  $R|_{c_0(\Gamma_1)}$  is an isomorphism onto its image.  $\square$

As mentioned above, the set-theoretic assumptions of Proposition 3.2 raise the question whether there are other models of set theory for which its conclusions can be improved by replacing  $c_0$  by  $c_0(\aleph_1)$ . Of course, in such a model, the topological product of two compact Hausdorff spaces satisfying ccc must also satisfy ccc. To this end, we note (see [7, p. 34]) that Kunen, Rowbottom and Solovay proved independently that ZFC has this property when MA and the negation of CH are assumed. With this assumption and Theorem 3.3 in hand, we have the following.

**Corollary 3.4.** *Assume  $MA + \neg CH$ . Let  $K$  be a compact Hausdorff space and  $X$  a Banach space. Then*

$$c_0(\aleph_1) \hookrightarrow C(K, X) \implies c_0(\aleph_1) \hookrightarrow C(K) \text{ or } c_0(\aleph_1) \hookrightarrow X.$$

*Proof.* If  $K$  does not satisfy the ccc, then by Theorem 2.1,  $C(K)$  contains a copy of  $c_0(\aleph_1)$ . Otherwise,  $K$  satisfies the ccc and by [17, p. 16],  $\aleph_1$  is a caliber of  $K$ . According to Theorem 3.3,  $X$  contains a copy of  $c_0(\aleph_1)$ .  $\square$

Theorem 3.3 suggests the following more general problem.

**Problem 3.5.** *Let  $K$  be a compact Hausdorff space. Characterize the cardinals  $\tau$  such that for every Banach space  $X$  we have*

$$c_0(\tau) \hookrightarrow C(K, X) \implies c_0(\tau) \hookrightarrow X.$$

Theorem 3.6, the second of our main results, begins to address this problem. We recall that the *cofinality*  $\text{cf}(\tau)$  of  $\tau$  is the smallest cardinal  $\mu$  such that there exists a family of ordinals  $\{\alpha_i : i < \mu\}$  with  $\sup\{\alpha_i : i < \mu\} = \tau$ . We also note that this theorem applies to every compact metric space  $K$ , since such a  $K$  has  $\text{dens}(K) \leq \aleph_0$ .

**Theorem 3.6.** *Let  $K$  be a compact Hausdorff space,  $X$  a Banach space and  $\tau > \aleph_0$ . If  $\text{cf}(\tau) > \text{dens}(K)$ , then*

$$c_0(\tau) \hookrightarrow C(K, X) \implies c_0(\tau) \hookrightarrow X.$$

*Proof.* Suppose that  $C(K, X)$  contains a copy of  $c_0(\tau)$  and  $\text{cf}(\tau) > \text{dens}(K)$ . Let  $T$  and  $\delta$  be as in the proof of Theorem 3.1. Let  $D$  be a dense set in  $K$  with  $|D| = \text{dens}(K)$ . For each  $d \in D$ , let

$$I_d = \{\gamma \in \tau : \|T(f_\gamma)(d)\| > \delta/2\}.$$

Therefore,  $\tau = \bigcup_{d \in D} I_d$ . Consequently there is a  $d_1 \in D$  such that  $|I_{d_1}| = \tau$ . Let  $P_{d_1} : C(K, X) \rightarrow X$  be the natural projection; that is,  $P_{d_1}(g) = g(d_1)$  for  $g \in C(K, X)$ . Next, consider the operator  $S = P_{d_1}T|_{c_0(I_{d_1})} : c_0(I_{d_1}) \rightarrow X$ . Then  $\inf\{\|S(f_i)\| : i \in I_{d_1}\} > 0$ . By Theorem 2.3 there exists  $J \subset I_{d_1}$  with  $|J| = \tau$  such that  $S|_{c_0(J)}$  is an isomorphism onto its image, and the theorem is proved.  $\square$

#### 4. ON COMPLEMENTED COPIES OF $c_0(\tau)$ IN $C(K, X)$ SPACES

We now turn our attention to complemented copies of  $c_0(\tau)$  in  $C(K, X)$  spaces. We begin by recalling the following result, which was obtained independently by Cembranos [2, Main Theorem] and Freniche [6, Corollary 2.5].

**Theorem 4.1.** *Let  $K$  be a compact Hausdorff space and let  $X$  be an infinite dimensional Banach space. Then*

$$c_0 \hookrightarrow C(K) \implies c_0 \overset{c}{\hookrightarrow} C(K, X).$$

The next result shows that Theorem 4.1 does not extend in a natural way to nonseparable  $c_0(\tau)$  spaces.

**Proposition 4.2.** *There exists a compact Hausdorff space  $K$  such that*

$$c_0(\aleph_1) \hookrightarrow C(K) \text{ but } c_0(\aleph_1) \not\overset{c}{\hookrightarrow} C(K, X),$$

*for every separable Banach space  $X$ .*

*Proof.* Let  $K_2$  be the compact Hausdorff spaces constructed recently by Dow, Junnila and Pelant [5, Example 2.16]. Since the set of isolated points of  $K_2$  is uncountable, it follows that  $K_2$  does not satisfy the  $\aleph_1$ -chain condition. Hence Theorem 2.1 implies that  $C(K_2)$  contains a copy of  $c_0(\aleph_1)$ . Now let  $X$  be a separable Banach space and  $(x_n)_{n \in \mathbb{N}}$  a dense sequence in  $X$ . We know that  $C(K_2, X)$  is the closed linear span of the family of functions of the form  $\sum_{i=1}^n f(\cdot)x_i$ , where  $f_1, \dots, f_n \in C(K_2)$  [4, p. 225]. Furthermore, it is easy to see that for each  $x \in X, x \neq 0$ , the subspace of  $C(K_2, X)$  given by  $E_x = \{f(\cdot)x : f \in C(K_2)\}$  is isomorphic to  $C(K_2)$ . Therefore, since every operator  $T : C(K_2) \rightarrow c_0(\aleph_1)$  has separable range [5, Theorem 1.6], it follows that every operator  $T : C(K_2, X) \rightarrow c_0(\aleph_1)$  also has separable range. In particular,  $C(K_2, X)$  has no complemented copy of  $c_0(\aleph_1)$ .  $\square$

Before stating and proving Theorem 4.5, we introduce the following concept, which is central to our results in this section.

**Definition 4.3.** Let  $\alpha$  be an ordinal number. A Banach space  $X$  has the *Josefson-Nissenzweig- $\alpha$  property* (in short,  $X$  has the  $JN_\alpha$  property or just  $X$  has  $JN_\alpha$ ) if there exists a family of elements  $(x_\gamma^*)_{\gamma \in \aleph_\alpha}$  in the unit sphere of  $X^*$  such that  $(x_\gamma^*(x))_{\gamma \in \aleph_\alpha}$  belongs to  $c_0(\aleph_\alpha)$  for every  $x \in X$ .

*Remark 4.4.* The classical Josefson-Nissenzweig theorem states that every infinite dimensional Banach space has  $JN_0$  [11], [13]. (See also [8] for another proof of this result.) Of course, the classical  $l_p(\aleph_\alpha)$  spaces, with  $1 \leq p < \infty$ , have  $JN_\alpha$ . Indeed, it is enough to consider  $(e_\gamma)_{\gamma \in \aleph_\alpha}$ , the usual unit-vector basis of the dual of  $l_p(\aleph_\alpha)$ . It is also easy to check that if  $X$  has  $JN_\alpha$  and  $X$  is a quotient space of  $Y$ , then  $Y$  also has  $JN_\alpha$ .

Our immediate goal is to prove Theorem 4.5, the third of the main results of this paper. This is a nonseparable version of a theorem of Ryan [18] which generalizes the main result of E. Saab and P. Saab in [20], which in turn is a generalization of Theorem 4.1. Using the definition of the  $JN_\alpha$  property, our proof may be viewed as a slight modification of Ryan's proof.

Given Banach spaces  $X$  and  $Y$ ,  $\mathcal{K}(X, Y)$  is the Banach space of compact operators from  $X$  to  $Y$  and  $X \hat{\otimes} Y$  is the injective tensor product of  $X$  and  $Y$  [19, p. 355].

**Theorem 4.5.** *Let  $Y$  be a Banach space containing a copy of  $c_0(\aleph_\alpha)$  and let  $X$  be a Banach space having the  $JN_\alpha$  property. Then  $Y \hat{\otimes} X$  contains a copy of  $c_0(\aleph_\alpha)$  which is complemented in  $\mathcal{K}(Y^*, X)$ .*

*Proof.* There exists a family  $(y_i)_{i \in \aleph_\alpha}$  in  $Y$  and a constant  $M > 0$  such that

$$\max_{i \in F} |a_i| \leq \left\| \sum_{i \in F} a_i y_i \right\| \leq M \max_{i \in F} |a_i|$$

for every family  $(a_i)_{i \in \aleph_\alpha}$  of scalars and for every finite subset  $F$  of  $\aleph_\alpha$ .

Let  $(y_i^*)_{i \in \aleph_\alpha}$  be a family of Hahn-Banach extensions of the coordinate functionals associated with the family  $(y_i)_{i \in \aleph_\alpha}$ . That is,  $\|y_i^*\| \leq 1$ ,  $y_i^*(y_i) = 1$  and  $y_i^*(y_j) = 0$  if  $i \neq j$ .

By hypothesis there exists a family  $(x_i^*)_{i \in \aleph_\alpha}$  in the unit sphere of  $X^*$  such that  $(x_i^*(x))_{i \in \aleph_\alpha}$  belongs to  $c_0(\aleph_\alpha)$  for every  $x \in X$ . For each  $i \in \aleph_\alpha$ , choose  $x_i$  in  $X$  such that  $\|x_i\| \leq 2$  and  $x_i^*(x_i) = 1$ .

We claim that the family  $(y_i \otimes x_i)_{i \in \aleph_\alpha}$  in  $Y \hat{\otimes} X$  is equivalent to the unit vector basis of  $c_0(\aleph_\alpha)$ . To establish this, first observe that since  $\|y_i^* \otimes x_i^*\| = 1$ , we have for every finite subset  $F$  of  $\aleph_\alpha$  with  $i \in F$  that

$$\left\| \sum_{j \in F} a_j y_j \otimes x_j \right\| \geq \| (y_i^* \otimes x_i^*) \left( \sum_{j \in F} a_j y_j \otimes x_j \right) \| = |a_i|.$$

Also we have for every finite subset  $F$  of  $\aleph_\alpha$  that

$$\begin{aligned} \left\| \sum_{j \in F} a_j y_j \otimes x_j \right\| &= \sup_{x^* \in B_{X^*}} \left\| \sum_{j \in F} a_j x^*(x_j) y_j \right\| \\ &\leq M \sup_{x^* \in B_{X^*}} \max_{j \in F} |a_j| |x^*(x_j)| \\ &\leq 2 M \max_{j \in F} |a_j|. \end{aligned}$$

Therefore

$$\max_{j \in F} |a_j| \leq \left\| \sum_{j \in F} a_j y_i \otimes x_i \right\| \leq 2 M \max_{j \in F} |a_j|,$$

which establishes our claim.

Let  $W$  denote the closed linear span of the set  $\{y_i \otimes x_i : i \in \aleph_\alpha\}$ . We finish the argument by constructing a projection from  $\mathcal{K}(Y^*, X)$  onto  $W$ . If  $u \in \mathcal{K}(Y^*, X)$ , then the closure of  $(u(y_i^*))_{i \in \aleph_\alpha}$  is a compact subset  $C$  of  $X$ . Since  $(x_i^*(x))_{i \in \aleph_\alpha}$  belongs to  $c_0(\aleph_\alpha)$  for every  $x \in X$ , it follows that for every  $\varepsilon > 0$ , there exists a finite subset  $F_\varepsilon$  of  $\aleph_\alpha$  such that  $|x_i^*(c)| \leq \varepsilon$  for every  $c \in C$  and for every  $i \notin F_\varepsilon$ . In particular,  $x_i^*(u(y_i^*)) \in c_0(\aleph_\alpha)$ . Hence the series

$$\sum_{i \in \aleph_\alpha} x_i^* u(y_i^*) y_i \otimes x_i$$

converges to an element of  $W$ . We define

$$P(u) = \sum_{i \in \aleph_\alpha} x_i^* u(y_i^*) y_i \otimes x_i.$$

It is easy to see that  $P$  is a bounded projection of  $\mathcal{K}(Y^*, X)$  onto  $W$ . □

As an immediate consequence of Theorem 4.5, we obtain the following, which is a generalization of Cembranos-Freniche's result (the case  $\alpha = 0$  and  $Y = C(K)$ ).

**Corollary 4.6.** *Let  $K$  be a compact Hausdorff space,  $X$  a Banach space and  $\alpha$  an ordinal. Then*

$$c_0(\aleph_\alpha) \hookrightarrow C(K) \text{ and } X \text{ has } JN_\alpha \implies c_0(\aleph_\alpha) \overset{c}{\hookrightarrow} C(K, X).$$

Thus in contrast with Proposition 4.2 we have the following.

**Corollary 4.7.** *Let  $K$  be a compact Hausdorff space. Then it is relatively consistent with ZFC that for all Banach spaces  $X$  with  $\dim(X) = \aleph_1$ ,*

$$c_0(\aleph_1) \hookrightarrow C(K) \implies c_0(\aleph_1) \overset{c}{\hookrightarrow} C(K, X).$$

*Proof.* Recall that a bounded fundamental biorthogonal system for a Banach space  $X$  is a family  $(x_i, x_i^*)$  of pairs from  $X \times X^*$  such that  $x_i^*(x_j) = \delta_{ij}$  and the closed linear subspace of  $X$  generated to  $(x_i)$  is  $X$ . By a result of Todorćević [21, Corollary 6] it is relatively consistent with ZFC that every Banach space  $X$  with  $\dim(X) = \aleph_1$  admits a bounded fundamental biorthogonal system. Therefore  $X$  has  $JN_1$ . Otherwise, according to [21, Lemma 1] every bounded linear operator  $T : X \rightarrow c_0(\aleph_1)$  has separable range. Thus a theorem of Holický, Smídek and Zajíček [9] (see also

[5, Corollary 1.15]) would imply that  $\dim(X) = \aleph_0$ , a contradiction. Since  $X$  has  $JN_1$ , we may apply Corollary 4.6 to finish the proof.  $\square$

As an immediate consequence of Theorem 3.1 and Corollary 4.6 we also obtain the following result.

**Corollary 4.8.** *Let  $K$  be a compact Hausdorff space and  $X$  a Banach space having  $JN_\alpha$  and containing no copy of  $c_0$ . Then*

$$c_0(\aleph_\alpha) \hookrightarrow C(K) \iff c_0(\aleph_\alpha) \xhookrightarrow{c} C(K, X).$$

5. ON COMPLEMENTED COPIES OF  $c_0(\tau)$  IN  $C(\mathbf{2}^m, X)$  AND  $C(\beta I, X)$  SPACES

In this section we provide further consequences of Theorems 3.1, 3.3, 3.6 and Corollary 4.6. We recall that a cardinal  $\tau$  is said to be *regular* if  $\text{cf}(\tau) = \tau$  and *singular* if  $\text{cf}(\tau) < \tau$ .

**Theorem 5.1.** *Let  $m$  be a cardinal satisfying  $\aleph_0 < \text{cf}(\aleph_\alpha) \leq \aleph_\alpha \leq 2^m$ , for some ordinal  $\alpha$ . If  $X$  is a Banach space, then*

$$c_0(\aleph_\alpha) \hookrightarrow X \iff c_0(\aleph_\alpha) \xhookrightarrow{c} C(\mathbf{2}^m, X).$$

*Proof.* Assume that  $X$  contains a copy of  $c_0(\aleph_\alpha)$ . Let  $\gamma$  be an ordinal such that  $2^m = \aleph_\gamma$ . By [14, Proposition 5.2],  $C(\mathbf{2}^m)^*$  contains a copy of  $L_1([0, 1]^m)$ . Since  $L_1([0, 1]^m)$  contains a copy of  $l_2(2^m)$ , it follows from [15, Remark 2, p. 181] that

$$C(\mathbf{2}^m) \twoheadrightarrow l_2(\aleph_\gamma).$$

Hence by Remark 4.4,  $C(\mathbf{2}^m)$  has  $JN_\gamma$ . Since  $\alpha \leq \gamma$ , it follows that  $C(\mathbf{2}^m)$  also has  $JN_\alpha$ . Then by Corollary 4.6 we conclude that

$$c_0(\aleph_\alpha) \xhookrightarrow{c} C(\mathbf{2}^m, X).$$

Suppose now that  $C(\mathbf{2}^m, X)$  contains a complemented copy of  $c_0(\aleph_\alpha)$ . By our hypothesis,  $\aleph_\alpha$  is a regular ordinal or a singular ordinal with  $\aleph_0 < \text{cf}(\aleph_\alpha)$ . Hence by [3, Corollary 3.7.a and Theorem 3.18.a],  $\aleph_\alpha$  is a caliber of  $\mathbf{2}^m$ . So Theorem 3.3 implies that  $X$  contains a copy of  $c_0(\aleph_\alpha)$ .  $\square$

It is well known that  $\mathbf{2}^m$  with  $m$  infinite does not have caliber  $\tau$  whenever  $\tau$  is singular with  $\text{cf}(\tau) = \aleph_0$  [3, Theorem 2.3.a and Corollary 2.12]. However, we also obtain the following.

**Theorem 5.2.** *Let  $m$  be a cardinal satisfying  $\aleph_\omega \leq 2^m$ . If  $X$  is a Banach space isomorphic to a complemented subspace of some  $C(K)$  space, then*

$$c_0(\aleph_\omega) \hookrightarrow X \iff c_0(\aleph_\omega) \xhookrightarrow{c} C(\mathbf{2}^m, X).$$

*Proof.* For the sufficiency of the statement of Theorem 5.2 it is enough to proceed as in the proof of Theorem 5.1. Next we will prove the necessity of this statement. By hypothesis, for every  $n < \omega$ , we have

$$c_0(\aleph_n) \hookrightarrow C(\mathbf{2}^m, X).$$

Since  $\aleph_n$  is a caliber of  $\mathbf{2}^m$  [3, Theorem 3.18.a], it follows by Theorem 3.3 that  $X$  contains a copy of  $c_0(\aleph_n)$ . Moreover, according to [1, Proposition 4.2] there exists

a totally disconnected compact Hausdorff space  $G$  such that  $C(G)$  contains  $X$  as a complemented subspace. Consequently

$$c_0(\aleph_n) \hookrightarrow C(G), \quad \forall n < \omega.$$

Hence by [1, Theorem 4.3], we conclude that  $X$  contains a copy of  $c_0(\aleph_\omega)$ .  $\square$

**Theorem 5.3.** *Let  $I$  be a set satisfying  $\aleph_0 \leq |I| < \text{cf}(\aleph_\alpha) \leq \aleph_\alpha \leq 2^{|I|}$ , for some ordinal  $\alpha$ . If  $X$  is a Banach space, then*

$$c_0(\aleph_\alpha) \hookrightarrow X \iff c_0(\aleph_\alpha) \overset{c}{\hookrightarrow} C(\beta I, X).$$

*Proof.* First suppose that  $X$  contains a copy of  $c_0(\aleph_\alpha)$ . Let  $\gamma$  be an ordinal such that  $2^{|I|} = \aleph_\gamma$ . By [15, Remark 2, p. 203] we know that

$$C(\beta I) \twoheadrightarrow l_2(\aleph_\gamma).$$

Therefore according to Remark 4.4  $C(\beta I)$  has  $JN_\gamma$ . Since  $\alpha \leq \gamma$ , it follows that  $C(\beta I)$  also has  $JN_\alpha$ . So by Corollary 4.6,

$$c_0(\aleph_\alpha) \overset{c}{\hookrightarrow} C(\beta I, X).$$

Conversely, assume that  $C(\beta I, X)$  contains a complemented copy of  $c_0(\aleph_\alpha)$ . Since  $\text{dens}(\beta I) = |I|$  and by hypothesis  $|I| < \text{cf}(\aleph_\alpha)$ , it follows from Theorem 3.6 that  $X$  contains a copy of  $c_0(\aleph_\alpha)$ .  $\square$

**Corollary 5.4.** *Let  $I$  be a set of cardinality  $\mathfrak{m} \geq \aleph_0$  and  $X$  a Banach space. Then the following statements are equivalent:*

- (a)  $C(\mathbf{2}^{\mathfrak{m}}, X)$  contains a complemented copy of  $c_0(\mathbf{2}^{\mathfrak{m}})$ .
- (b)  $C(\beta I, X)$  contains a complemented copy of  $c_0(\mathbf{2}^{\mathfrak{m}})$ .
- (c)  $X$  contains a copy of  $c_0(\mathbf{2}^{\mathfrak{m}})$ .

*Proof.* Since  $\mathfrak{m} \geq \aleph_0$ , by a well-known theorem of König [10, p. 158] we conclude that  $\mathfrak{m} < \text{cf}(\mathbf{2}^{\mathfrak{m}})$ . So by Theorems 5.1 and 5.3 with  $\aleph_\alpha = \mathbf{2}^{\mathfrak{m}}$  we are done.  $\square$

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