AN INDUCTIVE ANALYTIC CRITERION FOR FLATNESS

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Abstract. We present a constructive criterion for flatness of a morphism of analytic spaces \( \varphi : X \to Y \) (over \( K = \mathbb{R} \) or \( \mathbb{C} \)) or, more generally, for flatness over \( O_Y \) of a coherent sheaf of \( O_X \)-modules \( \mathcal{F} \). The criterion is a combination of a simple linear-algebra condition “in codimension zero” and a condition “in codimension one” which can be used together with the Weierstrass preparation theorem to inductively reduce the fibre-dimension of the morphism \( \varphi \).

1. Introduction

The main result of this article is a constructive criterion for flatness of a morphism of analytic spaces \( \varphi : X \to Y \) (over \( K = \mathbb{R} \) or \( \mathbb{C} \)) or, more generally, for flatness over \( O_Y \) of a coherent sheaf of \( O_X \)-modules \( \mathcal{F} \).

In the special case that \( X = Y \) and \( \varphi = \text{id}_X \) (the identity morphism of \( X \)), our criterion reduces to the following “linear algebra criterion”. In a neighbourhood of a point \( a \in X \), an \( O_X \)-module \( \mathcal{F} \) can be presented as

\[
O_X^p \xrightarrow{\Phi} O_X^q \to \mathcal{F} \to 0,
\]

where \( \Phi \) is given by multiplication by a \( q \times p \)-matrix of analytic functions. Let \( r = \text{rank} \Phi(a) \). Then \( \mathcal{F}_a \) is \( O_{X,a} \)-flat if and only if all minors of order \( r + 1 \) of \( \Phi \) vanish near \( a \).

Our flatness criterion, in general, is a combination of a condition “in codimension zero” similar to the preceding and a condition “in codimension one” which can be used together with the Weierstrass preparation theorem to inductively reduce the fibre-dimension of the morphism \( \varphi \).

To justify the criterion, we use it to give natural constructive proofs of several classical results — Hironaka’s existence of the local flattener [7], Douady’s openness of flatness [4], and Frisch’s generic flatness theorem [5]. The proofs are essentially a mix of linear algebra and appropriate applications of the Weierstrass preparation theorem.

For example, in the case \( X = Y \), the linear algebra criterion above provides an immediate construction of the local flattener of \( \mathcal{F} \) at \( a \) (i.e., the largest germ of an analytic subspace \( T \) of \( X \) at \( a \) such that \( \mathcal{F}_a \) is \( O_T \)-flat). We can simply

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\]

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\]
take \( O_T = O_X/I \), where the ideal \( I \) is generated by the minors of order \( r+1 \) of \( \Phi \). Hironaka’s local flattener, in general, can be described using a similar linear algebra construction and the Weierstrass preparation theorem.

**Algebraic formulation of the flatness criterion.** Let \( \varphi : Z \to W \) and \( \lambda : T \to W \) denote morphisms of analytic space-germs, where \( W \) is regular, and let \( F \) denote a finite \( O_Z \)-module. We are concerned with \( O_T \)-flatness of the module \( F \otimes_{O_W} O_T \), where \( \otimes_{O_W} \) denotes the analytic tensor product; i.e., the tensor product in the category of local analytic \( O_W \)-algebras. (For a review of the analytic tensor product and its right-derived functor \( \text{Tor} \), which is used below in the proof of Lemma 3.2, see [1, §2] or [2, §2].) Via the embedding \( (\phi, \text{id}_Z) : Z \to W \times Z \) and the natural projection \( \pi : W \times Z \to W \), we can view \( F \) as an \( O_{W \times Z} \)-module and therefore as an \( O_W \)-module. Via an embedding \( Z \to \mathbb{K}^n \) we can also replace \( Z \) by \( \mathbb{K}^n \) without changing the \( O_W \)-module structure of \( F \). In particular, then \( O_Z = \mathbb{K}\{x\} = \mathbb{K}\{x_1, \ldots, x_m\} \), \( O_T = R/J \) for an appropriate ideal \( J \) in \( R := \mathbb{K}\{y\} = \mathbb{K}\{y_1, \ldots, y_n\} \), and \( O_{W \times Z} = R\{x\} := \mathbb{K}\{y, x_1, \ldots, x_m\} \). Let \( A := O_{W \times Z} \). Let \( m \) denote the maximal ideal \((y_1, \ldots, y_n) \) of \( R \), and let \( n = m + (x_1, \ldots, x_m) \subset A \). Then \( n \) is the maximal ideal of \( A \). Given a power series \( f = f(y, x) \in A \), we denote by \( f(0) \) or by \( f(0, x) \) its evaluation at \( y = 0 \), i.e., the image of \( f \) under the homomorphism \( A \to A(0) := A\otimes_R R/m \) of \( R \)-modules. Similarly, given an \( A \)-submodule \( M \) of \( A^n \), we denote by \( M(0) \) the evaluation of \( M \) at \( y = 0 \), i.e., \( M(0) = \{m(0) \in A(0)^n : m \in M \} \). In particular, \( A(0) \cong \mathbb{K}\{x\} \).

We are thus interested in flatness of \( F \otimes_R R/J \) over \( R/J \), where \( F \) is a finitely generated \( A \)-module and \( J \) is an ideal in \( R \).

**Theorem 1.1.** Let \( R, A, F \) and \( J \) be as above. Then:

(A) There exist \( g \in A, l \in \mathbb{N} \) and a homomorphism \( \psi : A^l \to F \) of \( A \)-modules such that \( g(0, x) \neq 0, g \cdot F \subset \ker \psi \) and \( \ker \psi \subset m \cdot A^l \).

(B) \( F \otimes_R R/J \) is a flat \( R/J \)-module if and only if, for any \( g, l \) and \( \psi \) as in (A), the following two conditions hold:

1. \( \ker \psi \subset J \cdot A^l \);
2. \((F/\ker \psi) \otimes_R R/J \) is a flat \( R/J \)-module.

**Remark 1.2.** The above theorem allows one to study flatness of a module \( F \) by repeated reduction of the fibre-dimension over \( R \). Indeed, consider \( g \) and \( \psi \) as in (A). First suppose that \( g(0, 0) = 0 \). Since \( g(0, x) \neq 0 \), we can apply the Weierstrass division theorem (after a generic linear coordinate change in \( x \)) to conclude that \( A/(g \cdot A) \) is a finite \( R\{\tilde{x}\} \)-module, where \( \tilde{x} = (x_1, \ldots, x_{m-1}) \). Then \( F/\ker \psi \) is a finite \( R\{\tilde{x}\} \)-module too, since \( g \cdot F \subset \ker \psi \). On the other hand, if \( g(0, 0) \neq 0 \) (which is the case when the number of \( x \)-variables is 0), then condition (2) of (B) in the theorem is vacuous and no fibre dimension reduction is needed.

**Proof of Theorem 1.1 (A).** Consider a presentation of \( F \) as an \( A \)-module

\[
A^p \xrightarrow{\Phi} A^q \xrightarrow{\Psi} F \to 0.
\]

By applying \( \otimes_R R/J \) and \( \otimes_R R/\mathfrak{m} \) to (1.1), we get presentations

\[
A^p/J \cdot A^q \xrightarrow{\Phi_J} A^q/J \cdot A^q \xrightarrow{\Psi_J} F \otimes_R R/J \to 0
\]

and

\[
A^p/\mathfrak{m} \cdot A^q \xrightarrow{\Phi_m} A^q/\mathfrak{m} \cdot A^q \xrightarrow{\Psi_m} F \otimes_R R/\mathfrak{m} \to 0
\]
of $F \otimes_R R/J$ and $F \otimes_R R/\mathfrak{m}$ respectively. Notice that identifying $\Phi$ with a matrix (with entries in $A$), $\Phi_{\mathfrak{m}}$ becomes the matrix with entries obtained by evaluating the corresponding entries of $\Phi$ at $y = 0$.

Let $r_{\mathfrak{m}} := \text{rank}(\Phi_{\mathfrak{m}})$. Choose an ordering of the columns and rows of $\Phi$ so that $\Phi$ can be written in block form as

$$
\Phi = \begin{bmatrix}
\alpha & \beta \\
\gamma & \delta
\end{bmatrix},
$$

where the matrix $\alpha$ is of size $r_{\mathfrak{m}} \times r_{\mathfrak{m}}$ and $(\det \alpha)(0) = (\det \alpha)(0, x) \neq 0$ in $A(0)$.

Let $\alpha^#$ denote the adjoint matrix of $\alpha$, i.e., an $r_{\mathfrak{m}} \times r_{\mathfrak{m}}$ matrix with $\alpha^# \cdot \alpha = (\det \alpha) \cdot \text{Id}_{r_{\mathfrak{m}}}.$

Now, take $q := \det \alpha, l := q - r_{\mathfrak{m}}$, and let $\psi$ be the restriction of $\Psi : A^{r_{\mathfrak{m}}+1} \to F$ to $\{0\}^{r_{\mathfrak{m}}} \oplus A^l \cong A^l$. Then $g(0, x) \neq 0$. The condition $g \cdot F \subseteq \text{im} \psi$ is equivalent to saying that, for every vector $(\rho, \sigma) \in A^{r_{\mathfrak{m}}} \oplus A^l$, there exists $\sigma' \in A^l$ such that $\Psi(g \cdot (\rho, \sigma)) = \Psi((0, \sigma'))$ or, equivalently, that $g \cdot A^l \subseteq \ker \Psi + \{0\}^{r_{\mathfrak{m}}} + A^l = \text{im} \Phi + \{0\}^{r_{\mathfrak{m}}} + A^l$. But the latter follows from the fact that $g \cdot A^l \subseteq \text{im} \alpha$.

Finally, by the choice of $\psi, \sigma \in \ker \psi$ if and only if $(0, \sigma) \in \text{im} \Phi \cap \{0\}^{r_{\mathfrak{m}}} + A^l$. Then $(0, \sigma) = \Phi((\zeta, \eta))$ for some $(\zeta, \eta) \in A^{r_{\mathfrak{m}}} \oplus A^{r_{\mathfrak{m}}-1}$ with $\alpha \zeta + \beta \eta = 0$. By the choice of $r_{\mathfrak{m}}$, every row of $[\gamma, \delta]$ is an $A(0)$-linear combination of the rows of $[\alpha, \beta]$ modulo $\mathfrak{m}$. Hence $\gamma \xi + \beta \eta = 0$ implies that $\gamma \xi + \beta \eta \in \mathfrak{m} \cdot A^l$, i.e., that $\sigma \in \mathfrak{m} \cdot A^l$. □

Theorem 1.3 is the main result of this article. We will prove it in Section 3.

Remark 2.2. Suppose that $\lambda : T \to W$ is a morphism such that $F \otimes_{O_W} O_T$ is $O_T$-flat. Since flatness is preserved by base change (see [7], Prop.6.8), it follows that $(F \otimes_{O_W} O_T) \otimes_{O_T} S$ is $S$-flat, for every subring $S$ of $O_T$. In particular, identifying $O_W/\ker \lambda^*$ with $\text{im} \lambda^*$, we get that $F \otimes_{O_W} (O_W/\ker \lambda^*) \cong (F \otimes_{O_W} O_T) \otimes_{O_T} (O_W/\ker \lambda^*)$ is $(O_W/\ker \lambda^*)$-flat. Therefore, in Theorem 2.1 it suffices to consider an embedding $\lambda : T \to W$ and to show that there is an ideal $I(F)$ in $O_W$ such that $F \otimes_{O_W} (O_W/I(J))$ is $O_W/I(J)$-flat if and only if $I(F) \subseteq J.$
The germ \( P \) is called the **local flattener** of \( F \) (with respect to \( \varphi \)), and \( I(F) \) is the **ideal of the local flattener**.

**Proof of Theorem 2.1**. The uniqueness of \( P \) is automatic, since \( \lambda_P : \mathcal{O}_W \to \mathcal{O}_P \) is surjective.

By regularity of \( W \), we can identify \( \mathcal{O}_W \) with the ring \( R = \mathbb{K}\{y\} \) of convergent power series in \( y = (y_1, \ldots, y_n) \). Assume that \( Z \) is a subgerm of \( \mathbb{K}^m_0 \). Using the graph of \( \varphi \) to embed \( Z \) in \( W \times \mathbb{K}^m \), we can think of \( \mathcal{O}_Z \) as a quotient ring of \( A = R\{x\} \), where \( x = (x_1, \ldots, x_m) \). Then \( F \) is a finitely generated \( A \)-module. We will proceed by induction on \( m \), the number of the \( x \)-variables.

Choose \( g \in A \) and \( \psi : A^l \to F \) satisfying Theorem 1.1(A). Let \( J(F) \) be the ideal in \( R \) generated by the coefficients of (the expansions in \( x \) of) the elements in \( \ker \psi \), i.e., the unique minimal ideal \( J \) in \( R \) satisfying \( \ker \psi \subseteq J \cdot A^l \). If \( F = \im \psi \) (which is the case if \( m = 0 \), since then \( g \) is invertible in \( A \)), then Theorem 1.1(B) implies that \( J(F) \) is the ideal of the local flattener of \( F \). If \( F \neq \im \psi \), then \( m > 0 \) and we may assume by the inductive hypothesis (see Remark 1.2) that there is a local flattening ideal \( I(F/\im \psi) \) in \( \mathcal{O}_W \). It follows that \( I(F) = J(F) + I(F/\im \psi) \) is the ideal of the local flattener of \( F \).

Let \( X \) and \( Y \) be analytic spaces over \( \mathbb{K} \), and let \( \varphi : Y \times X \to Y \) be the canonical projection. Let \( F \) be a coherent \( \mathcal{O}_{Y \times X} \)-module. For \((\eta, \xi) \in Y \times X\), let \( I_{\eta, \xi}(F) \) denote the ideal in \( \mathcal{O}_{Y,\eta} \) of the local flattener of the stalk \( F_{(\eta, \xi)} \) (with respect to \( \varphi \)).

Given any ideal \( J \) in \( \mathcal{O}_{Y,\eta} \), we let \( J_\eta' \) denote the ideal generated by \((\text{system of generators of}) \ J \) at nearby points \( \eta' \in Y \). Then Theorem 1.1 implies the following.

**Theorem 2.3** (Openness of flatness). For every \((\eta, \xi)\) in a sufficiently small open neighbourhood of \((\eta_0, \xi_0)\) in \( Y \times X \), with \( \eta \) in a representative of the zero-set germ \( \mathcal{V}(I_{\eta_0, \xi_0}(F)) \), we have

\[
I_{\eta, \xi}(F) \subset (I_{\eta_0, \xi_0}(F))_\eta.
\]

**Remark 2.4** (Douady’s openness of flatness [3]). Let \( \varphi : X \to Y \) be a morphism of analytic spaces, and let \( F \) be a coherent sheaf of \( \mathcal{O}_X \)-modules. Let \( J \) be a coherent sheaf of ideals in \( \mathcal{O}_X \), and let \( Z \) be the closed analytic subspace of \( Y \) defined by \( J \) (i.e., \( \mathcal{O}_Z = \mathcal{O}_Y / J \) and \( \#(\mathcal{O}_Y / J) = \supp(\mathcal{O}_Y / J) \)). Then Theorem 2.3 implies that

\[
N_X(Z) = \{ \xi \in \varphi^{-1}(\{Z\}) : F_{(\varphi(\xi), \xi)} \otimes_{\mathcal{O}_{Y, \varphi(\xi)}} \mathcal{O}_{Z, \varphi(\xi)} \text{ is not } \mathcal{O}_{Z, \varphi(\xi)}\text{-flat} \}
\]

is a closed subset of \( |X| \). In particular, for \( Z = Y \), the latter implies openness of the set of points \( \xi \in X \) with the property that \( F_{(\varphi(\xi), \xi)} \) is a flat \( \mathcal{O}_{Y, \varphi(\xi)} \)-module. This result is due to Douady [3] and is the classical form of “openness of flatness”.

**Proof of Theorem 2.3**. As in the proof of Theorem 2.1 we proceed by induction on the fibre-dimension \( m \) of \( \varphi : X \times Y \to Y \). Using Theorem 1.1(A) with \( F = F_{(\eta_0, \xi_0)} \), we can choose neighbourhoods \( U \) of \( \eta_0 \) and \( V \) of \( \eta_0 \), a function \( g \) analytic on \( V \times U \), and a morphism \( \psi : \mathcal{O}_{V \times U} \to F_{|V \times U} \) of \( \mathcal{O}_{V \times U} \)-modules, such that \( g(\eta_0, x) \neq 0 \), \( g(\eta_0, \xi_0) \cdot F_{(\eta_0, \xi_0)} \subseteq (im \psi)_{(\eta_0, \xi_0)} \) and \( (\ker \psi)_{(\eta_0, \xi_0)} \subseteq m_{V \times U \cdot \mathcal{O}_{V \times U}(\eta_0, \xi_0)} \). Since our problem is local, we can assume that \( U \) (resp. \( V \)) is an open polydisc in \( \mathbb{C}^m \) (resp. \( \mathbb{C}^n \)) centred at \( \eta_0 \) (resp. \( \eta_0 \)). (After shrinking \( V \) if necessary) let \( J \) be a coherent \( \mathcal{O}_V \)-ideal such that \( J_{\eta_0} = I_{\eta_0, \xi_0}(F) \); we can assume that \( J_{\eta} = (I_{\eta_0, \xi_0}(F))_{\eta} \) for all \( \eta \in V \). Let \( Z \) denote the closed analytic subspace of \( V \) defined by \( J \); i.e., \( |Z| \) is a representative in \( V \) of the zero-set germ \( \mathcal{V}(I_{\eta_0, \xi_0}(F)) \). Then Theorem 1.1(B)
implies that
\begin{align}
(2.2) & \quad (\ker \psi)_{(\eta_0, \xi_0)} \subset J_{\eta_0} \cdot \mathcal{O}_V^l \cdot \mathcal{O}_{U \setminus \eta_0, \xi_0}, \\
(2.3) & \quad (\mathcal{F}/\ker \psi)_{(\eta_0, \xi_0)} \otimes_{\mathcal{O}_V, \eta_0} \mathcal{O}_{Z, \eta_0} \text{ is } \mathcal{O}_{Z, \eta_0}-flat.
\end{align}
It follows (after shrinking $U$ and $V$ if needed) that $g(\eta, x) \neq 0$ for all $\eta \in V$ and $g \cdot \mathcal{F} \subset \ker \psi$. Then (2.2) implies that
\begin{align}
(2.4) & \quad (\ker \psi)_{(\eta, \xi)} \subset J_{\eta} \cdot \mathcal{O}_V^l \cdot \mathcal{O}_{U \setminus \eta, \xi} \subset m_{\mathcal{V}, \eta} \cdot \mathcal{O}_V^l \cdot \mathcal{O}_{U \setminus \eta, \xi},
\end{align}
for all $(\eta, \xi) \in V \times U$ with $\eta \in |Z|$.

If $g(\eta, \xi) \neq 0$ (which is the case if $m = 0$), then, by Theorem [11] B, the first inclusion of (2.4) implies that $I_{\eta, \xi}(\mathcal{F}) \subset J_{\eta} = (I_{\eta_0, \xi_0}(\mathcal{F})), $ as required.

Otherwise $g(\eta_0, \xi_0) = 0$ (and $m > 0$). By Theorem [11] it suffices to show that $(\mathcal{F}/\ker \psi)_{(\eta_0, \xi_0)} \otimes_{\mathcal{O}_V, \eta_0} \mathcal{O}_{Z, \eta_0} \text{ is } \mathcal{O}_{Z, \eta_0}-flat$, provided $\eta \in |Z|$ and $g(\eta, \xi) = 0$. After a linear change of the $x$-variables, we can assume that $U = U' \times U''$, where $U'$ is spanned by the variables $\tilde{x} = (x_1, \ldots, x_{m-1})$ and $U''$ is spanned by $x_m$, and that $g(\eta_0, \xi_0)$ is regular in $x_m - \xi_0m$, where $\xi_0m$ is the last coordinate of $\xi_0$. By Remark [12] after shrinking $U$ if needed, we can consider $\mathcal{F}/\ker \psi$ as a coherent $\mathcal{O}_V \times U'$-module; we denote it $\tilde{\mathcal{F}}$. Let $\tilde{\xi}_0$ denote the $\tilde{x}$-coordinates of $\xi_0$. Then $\tilde{\mathcal{F}}_{(\eta_0, \xi_0)} \subset (\mathcal{F}/\ker \psi)_{(\eta_0, \xi_0)}$ (since $g(\eta_0, \tilde{\xi}_0, \cdot)$ vanishes only at $\xi_0m$), and hence $\tilde{\mathcal{F}}_{(\eta_0, \xi_0)} \otimes_{\mathcal{O}_V, \eta_0} \mathcal{O}_{Z, \eta} \text{ is } \mathcal{O}_{Z, \eta_0}$-flat, by (2.3). By the inductive hypothesis, $\tilde{\mathcal{F}}_{(\eta, \xi)} \otimes_{\mathcal{O}_V, \eta} \mathcal{O}_{Z, \eta}$ is $\mathcal{O}_{Z, \eta}$-flat for every $(\eta, \xi) \in |Z| \times U'$. To complete the proof, observe that for any $(\eta, \xi) \in |Z| \times U$ with $g(\eta, \xi) = 0$, $(\mathcal{F}/\ker \psi)_{(\eta, \xi)}$ is a direct summand of $\tilde{\mathcal{F}}_{(\eta, \xi)}$. Indeed, one can show this by a direct calculation based on ‘collecting into’ the remainder of Weierstrass Division by $g(\eta, \xi, \cdot)$ the remainders of division by the factors of $g(\eta, \xi, \cdot)$. Hence $(\mathcal{F}/\ker \psi)_{(\eta, \xi)} \otimes_{\mathcal{O}_V, \eta} \mathcal{O}_{Z, \eta}$ is $\mathcal{O}_{Z, \eta}$-flat, as a direct summand of $\tilde{\mathcal{F}}_{(\eta, \xi)} \otimes_{\mathcal{O}_V, \eta} \mathcal{O}_{Z, \eta}$, by [23] Ch. 1, §2.3, Prop. 2. \hfill \Box

Remark 2.5 (Frisch’s generic flatness theorem [5]). Let $\varphi : X \to Y$ denote a morphism of complex-analytic spaces and let $\mathcal{F}$ denote a coherent sheaf of $\mathcal{O}_X$-modules. Frisch’s generic flatness theorem asserts that the non-flat locus $\Sigma := \{ \xi \in X : \mathcal{F}_\xi \text{ is not } \mathcal{O}_{Y, \varphi(\xi)}-flat \}$ is a closed analytic subset of $X$ and that if $X$ is reduced, then $\varphi(\Sigma)$ is nowhere dense in $Y$. The first assertion follows from Theorem 2.3 above, together with the fact that $\Sigma$ is a constructible subset of $X$. See [2] Thm. 7.15 for a constructive elementary proof of the latter. The second assertion then follows in a simple way (as in [5] Prop. IV.14]) and, in fact, can also be proved using Theorem 1.1 and further development of 2.1.

3. Proof of the main theorem

We use the notation preceding Theorem 1.1. Consider a presentation 1.1 of $F$ as an $A$-module. Applying $\otimes_R R/\mathfrak{m}$, we get a homomorphism $\Phi_\mathfrak{m} : A(0)^p \to A(0)^q$ of $A(0)$-modules such that $F \otimes_R R/\mathfrak{m} \cong \ker(\Phi_\mathfrak{m})$.

Set $r_\mathfrak{m} := \det(\Phi_\mathfrak{m})$. We can assume that $\Phi$ is given by a block matrix 1.4 and $g := \det \alpha$ satisfies $g(0, x) \neq 0$. For an ideal $J$ in $R$, define

$$\ker_J \Phi := \{ \xi \in A^p : \Phi(\xi) \in J \cdot A^q \}$$

and

$$\operatorname{rank}_J \Phi := \min\{ r \geq 1 : \text{all } (r+1) \times (r+1) \text{ minors of } \Phi \text{ belong to } J \cdot A \}.$$
Our proof of Theorem 1.1(B) is based on showing that property (1) of the theorem is equivalent to equalities $g - l = \text{rank } \Phi = \text{rank } \Phi_m$ and that property (2) of the theorem is equivalent to $R/J$-flatness of $G \otimes_R R/J$, where

$$G := A^{r_m}/[g \cdot A^{r_m} + \text{im } (\alpha^\# \cdot \beta)].$$

The latter equivalence is obvious if $g$ is a unit in $A$, since both $F/\text{im } \psi$ and $G$ are zero in this case. Suppose then that $g$ is not invertible in $A$, that is, $g(0,0) = 0$. Since $g(0,x) \neq 0$, then after a (generic and linear) change of the $x$-coordinates to $(\tilde{x}, x_m)$, where $\tilde{x} = (x_1, \ldots, x_{m-1})$, we have $g(0,0, x_m) \neq 0$. By the Weierstrass Preparation Theorem, $g = u \cdot P$, where $u(0,0) \neq 0$ and $P(y,x) = x_m^d + \sum_{i=1}^d p_i(y, \tilde{x}) \cdot x_m^{d-i}$, with $p_i(0,0) = 0$.

The ring $A/g \cdot A$ is a finite free $R/\tilde{R}$-module. We shall describe the action of $\alpha^\# \cdot \beta : A^{p - r_m} \to A^{r_m}$ modulo $g$ as a linear mapping of finite $R/\tilde{R}$-modules. Given $\eta \in A^{p - r_m}$, Weierstrass division by $g$ gives $\eta \equiv \sum_{j=1}^d \eta_j x_m^{d-j} \mod g$, with $\eta_j \in R/\tilde{R}$.

Applying Weierstrass division by $g$ to the entries of $\alpha^\# \cdot \beta$, we form matrices $T_i = T_i(y, \tilde{x})$, $1 \leq i \leq d$, such that

$$\eta \equiv \eta_1 x_m^{d-1} + \eta_2 x_m^{d-2} + \cdots + \eta_d x_m^0 \mod g$$

(3.1) \hspace{1cm} $(\alpha^\# \cdot \beta)(\eta) \equiv (\sum_{i=1}^d T_i \cdot x_m^{d-i}) \cdot (\sum_{j=1}^d \eta_j x_m^{d-j}) \mod g$.

Applying Euclid division by $P(y,x)$ (as a monic polynomial in $x_m$) to the latter product, we obtain the matrix $G = (G_{ij})_{1 \leq i \leq d}$, with block-matrices $G_{ij}$ of size $r_m \times (p - r_m)$ and entries in $R/\tilde{R}$, such that all entries of the matrix

$$\left(\sum_{i=1}^d T_i \cdot x_m^{d-i}\right) \cdot \left(\sum_{j=1}^d \eta_j x_m^{d-j}\right) - \sum_{1 \leq i < j \leq d} G_{ij} \cdot \eta_j x_m^{d-i}$$

are linear in the $\eta_j$ with coefficients in the ideal generated by $P(y,x)$ in the ring $R/\tilde{R}[x_m]$. Then $G$ coincides with $R/\tilde{R}[x_m^{r_m}]$-modules. With these preparations and modulo Lemma 3.2 below, Theorem 1.1(B) is a consequence of the following.

**Proposition 3.1.** Let $G : R/\tilde{R}[x_m]^{(p-r_m)d} \to R/\tilde{R}[x_m]^{r_m d}$ be as above (or $G = 0$ if $g(0,0) \neq 0$). Then $\ker(\Phi_m) = (\ker_J \Phi)(0)$ if and only if $\text{rank } \Phi_m = \text{rank } \Phi$ and $\ker(G_m) = (\ker_J G)(0)$.

Before proving Proposition 3.1, let us note that the first equality of Proposition 3.1 expresses $R/J$-flatness of $F$:

**Lemma 3.2.** $F \otimes_R R/J$ is $R/J$-flat if and only if $(\ker_J \Phi)(0) = \ker(\Phi_m)$.

**Proof.** By definition of $\ker_J \Phi$, $\zeta \in \ker_J \Phi$ implies $\Phi(\zeta) \in \mathfrak{m} \cdot A^q$, and hence $\Phi_m(\zeta(0)) = 0$. Therefore, we always have $(\ker_J \Phi)(0) \subset \ker(\Phi_m)$. On the other hand, by a well-known criterion for flatness (see, e.g., [6, Prop. 6.2]), $F \otimes_R R/J$ is $R/J$-flat if and only if $\text{Tor}_1^{R/J}(F \otimes_R R/J, R/\mathfrak{m}) = 0$.

By (1.2), we have $F \otimes_R R/J \cong (A^q / J \cdot A^q) / \Phi_j(A^p / J \cdot A^p)$. Notice that $\ker(\Phi_j) = (\ker_J \Phi)/J \cdot A^p$. Hence $\Phi_j(A^p / J \cdot A^p) \cong (A^p / J \cdot A^p)/\ker(\Phi_j) \cong A^p / \ker_J \Phi$, and we get from (1.2) a short exact sequence

$$0 \to A^p / \ker_J \Phi \to A^q / J \cdot A^q \to F \otimes_R R/J \to 0.$$
The induced long exact sequence of $\text{Tor}^{R/J}$-modules ends with

$$0 \rightarrow \text{Tor}^{R/J}_1(F \otimes_R R/J, R/m) \rightarrow (A^p/\ker J \Phi) \otimes_R R/m \rightarrow (A^q/\ker J \Phi) \otimes_R R/m \rightarrow 0,$$

where the leftmost term is zero by $R$-flatness of $A^q$. Thus $F \otimes_R R/J$ is $R/J$-flat if and only if

$$A(0)^p/(\ker J \Phi)(0) \cong (A^p/\ker J \Phi) \otimes_R R/m \rightarrow A^q \otimes_R R/m \cong A(0)^q$$

is injective. By Lemma 3.3, the latter condition is equivalent to $(\ker J \Phi)(0) \supset \ker(\Phi_m)$, which completes the proof of the lemma.

The proof of Proposition 5.1 depends on several lemmas, which follow. First, we establish a useful cancellation law.

Lemma 3.3. Let $J$ be an ideal in $R$, and let $g, \zeta \in A$ be such that $g(0, x) \neq 0$ in $A(0) = \mathbb{K}\{x\}$ and $g \cdot \zeta \in J \cdot A$. Then $\zeta \in J \cdot A$.

Proof. Write $\zeta = \sum_{\nu \in N^m} \zeta_\nu x^\nu$, where $\zeta_\nu \in R$, and consider $g$ and $\zeta$ as elements of the ring $\hat{A} := R[[x]]$. By assumption, $g \not\in \mathfrak{m} \cdot \hat{A}$. Hence, after localizing in $\mathfrak{m} \cdot \hat{A}$, we get $\zeta_\mathfrak{m} \hat{A} \in (J \mathfrak{m} \cdot \hat{A})$, because $\mathfrak{m} \hat{A}$ is invertible in $\hat{A}$. Since $\hat{A}$ is a free $R$-module, we have $\hat{A} \mathfrak{m} \hat{A} \cong R \mathfrak{m} \{x\}$, and hence $\zeta_\mathfrak{m} \hat{A} \in (J \mathfrak{m} \cdot \hat{A})$ if and only if, for all $\nu \in N^m$, $(\zeta_\nu)_m \in J_m$, that is, $\zeta_\nu \in J$. Thus $\zeta \in J \cdot A$, as required.

Recall that $r_m$ denotes the rank of $\Phi_m$ (in the notation at the beginning of this section).

Lemma 3.4. Let $J$ be an ideal in $R$. Then the following conditions are equivalent:

(i) $\text{rank } \Phi_m = \text{rank } j \Phi$;

(ii) we can order the columns and rows of $\Phi$ so that $\Phi$ has block form (1.4) with $\alpha$ of size $r \times r$, $(\det \alpha)(0, x) \neq 0$ and $\text{rank } j \Phi = r$;

(iii) we can order the columns and rows of $\Phi$ so that $\Phi$ has block form (1.4), where $\alpha$ has size $r \times r$, $(\det \alpha)(0, x) \neq 0$, and all entries of $(\det \alpha) \cdot \delta - \gamma \cdot \alpha^\#: \beta$ are in $J \cdot A$;

(iv) if $\Phi$ is a block matrix (1.4), where $\alpha$ is of size $r_m \times r_m$ and $(\det \alpha)(0) \neq 0$, then all entries of $(\det \alpha) \cdot \delta - \gamma \cdot \alpha^# \cdot \beta$ are in $J \cdot A$;

(v) if $g \in A$, $g(0, x) \neq 0$, and $A^q = A^r \oplus A^t$, where $g \cdot A^q \subset \text{im } \Phi + \{0\}^r \oplus A^t$ and $\text{im } \Phi \cap \{0\}^r \oplus A^t \subset \{0\}^r \oplus \mathfrak{m} \cdot A^t$, then $\text{im } \Phi \cap (\{0\}^r \oplus A^t) \subset \{0\}^r \oplus J \cdot A$;

(vi) if $g \in A$, $g(0, x) \neq 0$ and $\psi : A^t \rightarrow F$ is a homomorphism of $A$-modules such that $g \cdot F \subset \text{im } \psi$ and $\ker \psi \subset \mathfrak{m} \cdot A^t$, then $\ker \psi \subset J \cdot A^t$.

Proof. (ii) $\Rightarrow$ (i): Clearly $r \leq \text{rank } \Phi_m$ and $\text{rank } \Phi_m \leq \text{rank } j \Phi$. Hence all three are equal if $\text{rank } j \Phi = r$.

(i) $\Rightarrow$ (iv): By Remark 3.3 all entries of $(\det \alpha) \cdot \delta - \gamma \cdot \alpha^# \cdot \beta$ are $(r_m + 1) \times (r_m + 1)$ minors of $\Phi$, and hence they belong to $J \cdot A$ if $\text{rank } j \Phi = r_m$.

(iv) $\Rightarrow$ (iii): Set $r = r_m$ and let $\alpha, \beta, \alpha^#$ be as in (iv).

(iii) $\Rightarrow$ (ii): Set $g = \det \alpha$. By the matrix identity of Remark 3.3 all $(r + 1) \times (r + 1)$ minors of $\Phi$ are combinations of the entries of $(\det \alpha) \cdot \delta - \gamma \cdot \alpha^# \cdot \beta$ with coefficients in $A$. Hence, if $\zeta$ is an $(r + 1) \times (r + 1)$ minor of $\Phi$, then $g^{r+1} \cdot \zeta \in J \cdot A$, which by Lemma 3.3 implies $\zeta \in J \cdot A$. 

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\[
\begin{align*}
(\text{v}) & \Rightarrow (\text{vi}): \text{The homomorphism } \psi : A^l \to F \text{ can be extended to a surjective homomorphism } \Psi : A^q \to F, \text{ which by Oka's coherence theorem extends to an exact sequence } A^p \xrightarrow{\Phi} A^q \xrightarrow{\Psi} F \to 0. \\
(\text{vi}) & \Rightarrow (\text{v}): \text{The assumptions in (v) imply the assumptions in (vi), with the same } g \text{ and } \psi \text{ being the restriction of } \Psi \text{ (from the above exact sequence) to } \{0\}^r \oplus A^l. \text{ Then } \ker \Phi \cap \{0\}^r \oplus A^l = \ker \psi \subset J \cdot A^l.
\end{align*}
\]

It remains to show that (iv) is equivalent to (v). Write \( \Phi \) in block form \((1.4)\), with \( \alpha \) of size \( r \times r \). We will use the fact that \((g, \sigma) \in A^q = A^r \oplus A^l\) belongs to \( \ker \Phi \cap \{0\}^r \oplus A^l \) if and only if \( \sigma = \gamma \xi + \delta \eta \) and \( \alpha \xi + \beta \eta = 0 \), for some \((\xi, \eta) \in A^r \oplus A^{r-l} \). Then \((\det \alpha) \cdot \xi = (\alpha \# \cdot \alpha)(\xi) = -\gamma \cdot \alpha \# \cdot \beta(\eta)\), and hence

\[
(\det \alpha) \cdot \sigma = \gamma((\det \alpha) \cdot \xi) + (\det \alpha) \cdot \delta(\eta) = -\gamma \cdot \alpha \# \cdot \beta(\eta) + (\det \alpha) \cdot \delta(\eta).
\]

It follows that

\[
(3.3) \quad (\det \alpha) \cdot [\ker \Phi \cap \{0\}^r \oplus A^l] \subset \{0\}^r \oplus \ker [\det \alpha] \cdot \delta - \gamma \cdot \alpha \# \cdot \beta \subset \ker [\Phi \cap \{0\}^r \oplus A^l],
\]

where the latter inclusion is a consequence of Remark 1.3.

\( (\text{v}) \Rightarrow (\text{iv}): \text{The assumptions of (iv) imply that all entries of } (\det \alpha) \cdot \delta - \gamma \cdot \alpha \# \cdot \beta \text{ are in } m \cdot A \text{ (by Remark 1.3), as } (r_m + 1) \times (r_m + 1) \text{ minors of } \Phi. \text{ Therefore the assumptions of (v) follow with } r := r_m, \ l := q - r \text{ and } g := \det \alpha. \text{ Indeed, } g \cdot \Id_r = \alpha \cdot \alpha \#, \text{ and so }
\]

\[
g \cdot A^l \subset \alpha(A^r) \oplus A^l \subset \ker \Phi \oplus \{0\}^r \oplus A^l.
\]

Also, by \((3.3)\), \( \zeta = (g, \sigma) \in \ker \Phi \cap \{0\}^r \oplus A^l \) implies \( g \cdot \sigma \in \ker [\det \alpha] \cdot \delta - \gamma \cdot \alpha \# \cdot \beta \). Hence \( g \cdot \zeta \in m \cdot A^q \), and therefore \( \zeta \in m \cdot A^q \), by Lemma 3.3.

Now, \((\text{v})\) implies \( \ker \Phi \cap \{0\}^r \oplus A^l \subset \{0\}^r \oplus J \cdot A^l \), which by \((3.3)\) means that \( \ker [\det \alpha] \cdot \delta - \gamma \cdot \alpha \# \cdot \beta \subset J \cdot A^l \), and hence the entries of \( (\det \alpha) \cdot \delta - \gamma \cdot \alpha \# \cdot \beta \) are in \( J \cdot A \).

\( (\text{iv}) \Rightarrow (\text{v}): \text{Let } \pi_1 : A^q = A^r \oplus A^l \to A^r \text{ denote the canonical projection to the first direct summand. By the assumptions of (v), there is a matrix } \Xi \text{ of size } p \times r \text{ with entries in } A \text{ such that } g \cdot \Id_r = \pi_1 \cdot \Phi \cdot \Xi. \text{ Since } g(0, x) \neq 0, \text{ it follows that } \ker [\pi_1] \cdot \Phi = r. \text{ Therefore there is an ordering of columns of } \Phi \text{ such that } \pi_1 \cdot \Phi = [\alpha, \beta], \text{ with } \alpha \text{ of size } r \times r \text{ and } (\det \alpha)(0, x) \neq 0. \text{ Then } \Phi \text{ has block form } (1.4) \text{ and } \{0\}^r \oplus \ker [\det \alpha] \cdot \delta - \gamma \cdot \alpha \# \cdot \beta \subset \ker [\Phi \cap \{0\}^r \oplus A^l]. \text{ Hence, by the assumptions of (v), all entries of } (\det \alpha) \cdot \delta - \gamma \cdot \alpha \# \cdot \beta \text{ are in } m \cdot A^l. \text{ Using the equivalence of (ii) and (iii) for } J = m, \text{ we see that } r = \rank m \Phi = \rank \Phi m; \text{ i.e., the assumptions of (iv) are satisfied. It follows that } J \cdot A^l \supset \ker [(\det \alpha) \cdot \delta - \gamma \cdot \alpha \# \cdot \beta]; \text{ hence } \{0\}^r \oplus J \cdot A^l \supset (\det \alpha) \cdot \ker [\Phi \cap \{0\}^r \oplus A^l] \text{, by } (3.3), \text{ and thus } \ker [\Phi \cap \{0\}^r \oplus A^l] \subset \ker [\Phi \cap \{0\}^r \oplus A^l], \text{ by Lemma 3.3.} \]

**Lemma 3.5.** Assume that \( \ker (\Phi_m) = (\ker_j \Phi)(0) \). Then \( \rank \Phi_m = \rank \Phi \).

**Proof:** Clearly, \( r_m = \rank \Phi_m \leq \rank \Phi \). For the opposite inequality, choose \( \xi_j(0) \in \ker \Phi_m \subset \mathbb{K}\{x\}^p, \ 1 \leq j \leq p - r_m, \) so that the \( p \times (p - r_m) \) matrix \( \xi(x) = [\xi_1(x), \ldots, \xi_{p-r_m}(x)] \) has rank \( p - r_m \). Then, by assumption, there is a matrix \( \Xi = \Xi(y, x) \) of size \( p \times (p - r_m) \) such that the entries of \( \Phi \cdot \Xi \) are in \( J \cdot A \) and \( \Xi(0, x) = \xi(x) \). It follows that \( \rank \Xi = p - r_m \). By Cramer's Rule (and after an
appropriate reordering of the columns of $\Phi$ and rows of $\Xi$), there exists a matrix $\Sigma$ of size $(p-r_m) \times (p-r_m)$ with entries in $A$ such that
\[
\Xi \cdot \Sigma = \begin{bmatrix} g \cdot \text{Id}_{p-r_m} \\
\Gamma \end{bmatrix},
\]
where $g \in A$ satisfies $g(0, x) \neq 0$ and $\Gamma$ is a matrix with entries in $A$ of size $r_m \times (p-r_m)$. Write $\Phi = [\Phi_1, \Phi_2]$, where $\Phi_1$ consists of the first $p-r_m$ columns of $\Phi$. It follows that $g \cdot \Phi_1 + \Phi_2 \cdot \Gamma$ is a matrix with entries in $J \cdot A$, and hence the entries of $g \cdot \Phi - [-\Phi_2 \cdot \Gamma, g \cdot \Phi_2]$ are also in $J \cdot A$. Since $\Phi_2$ is of size $q \times r_m$, then $\text{rank} [-\Phi_2 \cdot \Gamma, g \cdot \Phi_2] \leq \text{rank} \Phi_2 \leq r_m$. Consequently,
\[
\text{rank}_J (g \cdot \Phi) = \text{rank}_J [-\Phi_2 \cdot \Gamma, g \cdot \Phi_2] \leq r_m.
\]
It thus suffices to show that $\text{rank}_J \Phi = \text{rank}_J (g \cdot \Phi)$, but that is a consequence of Lemma 3.3.

**Remark 3.6.** Let $\Phi$ be as in Lemma 3.4(iv), and let $\pi_2 : A^p = A^{r_m} \oplus A^{p-r_m} \to A^{p-r_m}$ denote the canonical projection to the second direct summand. Then $$(\ker_J \Phi)(0) = \ker(\Phi_m) \text{ iff } \pi_2((\ker_J \Phi)(0)) = \pi_2(\ker(\Phi_m)),$$ where $J$ is an ideal in $R$. Indeed, since $(\ker_J \Phi)(0)$ is always contained in $\ker(\Phi_m)$ (cf. the proof of Lemma 3.2), it suffices to show that $\pi_2((\ker_J \Phi)(0)) \supset \pi_2(\ker(\Phi_m))$ implies $\ker(\Phi_m) \subset (\ker_J \Phi)(0)$. Let $\zeta = \zeta(x)$ be an element of $\ker(\Phi_m)$, and let $\xi \in \ker_J \Phi$ be such that $\pi_2(\zeta(0, x)) = \pi_2(\xi)$. It suffices to show that $\zeta(x) = \xi(0, x)$. Since $\eta(x) := \xi(0, x) - \zeta(x)$ belongs to $\ker \pi_2 \cap \ker \Phi_m$, it follows that $\eta = (\eta', 0) \in A^{r_m} \oplus A^{p-r_m}$ and $\alpha(0, x), \eta'(x) = 0$. Therefore $(\det \alpha)(0, x) \cdot \eta'(x) = 0$, and hence $\eta' = 0$, and $\eta = 0$, as required.

**Lemma 3.7.** Let $\Phi$ and $\pi_2 : A^p = A^{r_m} \oplus A^{p-r_m} \to A^{p-r_m}$ be as above, and let $J$ be an ideal in $R$. Then $\eta \in \pi_2(\ker_J \Phi)$ if and only if the following two conditions hold:
\[
(\alpha^\# \cdot \beta)(\eta) \in g \cdot A^{r_m} + J \cdot A^{r_m},
\]
\[
(g \cdot \delta - \gamma \cdot \alpha^\# \cdot \beta)(\eta) \in J \cdot A^{p-r_m},
\]
where $g$ denotes $\det \alpha$.

**Proof.** For the “only if” direction, let $(\xi, \eta)$ be an element of $\ker_J \Phi$. Then $\alpha \xi + \beta \eta \in J \cdot A^{r_m}$ and $\gamma \xi + \delta \eta \in J \cdot A^{p-r_m}$, and hence
\[
g \cdot \xi + (\alpha^\# \cdot \beta)(\eta) = \alpha^\# \cdot (\alpha \xi + \beta \eta) \equiv 0 \pmod{J \cdot A^{r_m}} \text{ and }
\]
\[
g \cdot \delta - \gamma \cdot \alpha^\# \cdot \beta)(\eta) = g \cdot (\gamma \xi + \delta \eta) - \gamma \cdot (g \cdot \xi + (\alpha^\# \cdot \beta)(\eta)) \equiv 0 \pmod{J \cdot A^{p-r_m}}.
\]
Now, for the “if” direction, let $\xi \in A^{r_m}$ be such that $g \cdot \xi + (\alpha^\# \cdot \beta)(\eta) \equiv 0$ modulo $J \cdot A^{r_m}$ and assume that $g \cdot \delta - \gamma \cdot \alpha^\# \cdot \beta)(\eta) \in J \cdot A^{p-r_m}$. Then
\[
g \cdot (\alpha \xi + \beta \eta) = \alpha \cdot (g \cdot \xi + (\alpha^\# \cdot \beta)(\eta)) \equiv 0 \pmod{J \cdot A^{r_m}} \text{ and }
\]
\[
g \cdot (\gamma \xi + \delta \eta) = (g \cdot \delta - \gamma \cdot \alpha^\# \cdot \beta)(\eta) + \gamma \cdot (g \cdot \xi + (\alpha^\# \cdot \beta)(\eta)) \equiv 0 \pmod{J \cdot A^{p-r_m}}.
\]
Therefore $g \cdot (\xi, \eta) \in \ker_J \Phi$; hence $(\xi, \eta) \in \ker_J \Phi$ by Lemma 3.3 and so $\eta \in \pi_2(\ker_J \Phi)$, as required. \qed


