HOMOLOGY-GENERICITY, HORIZONTAL DEHN SURGERIES AND UBIQUITY OF RATIONAL HOMOLOGY 3-SPHERES

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ABSTRACT. In this paper, we show that rational homology 3-spheres are ubiquitous from the viewpoint of Heegaard splitting. Let $M = H_+ \cup_F H_-$ be a genus g Heegaard splitting of a closed 3-manifold and c be a simple closed curve in F. Then there is a 3-manifold M_c which is obtained from M by horizontal Dehn surgery along c. We show that for c such that the homology class $[c]$ is generic in the set of curve-represented homology classes $\mathscr{H}_s \subset H_1(F)$, $rank(H_1(M_c, \mathbb{Q})) < max\{1, rank(H_1(M, \mathbb{Q}))\}$. As a corollary, for a set of curves $\{c_1, c_2, \ldots, c_m\}$, $m \geq g$, such that each $[c_i]$ is generic in $\mathscr{H}_s \subset H_1(F)$, $M_{(c_1, c_2, \ldots, c_m)}$ is a rational homology 3-sphere.

1. INTRODUCTION

Thurston conjectured [13] that any closed hyperbolic 3-manifold M admits a finite cover $\widetilde{M}$ which has positive first Betti number. There has been little progress on this conjecture for more than 20 years (see, for example, [1], [2] and [7]: a small list from the huge literature on this topic).

Why is this conjecture so difficult to confirm? One of the reasons is that in some sense, closed 3-manifolds with positive first Betti numbers are rare in the set of all closed 3-manifolds. For example, any closed 3-manifold M can be obtained from S^3 by a Dehn surgery along a link K, and the resulting manifold having positive first Betti number implies a strong restriction on the Dehn surgery coefficient; see [10]. Due to the lack of uniqueness of the surgery descriptions of 3-manifolds, the above argument just shows that "most" closed 3-manifolds have zero first Betti numbers in a philosophical sense, but not rigorously.

There are notions of random 3-manifolds using triangulations, Heegaard splittings and mapping class groups [2], and we can study the set of all 3-manifolds wholly from the probabilistic point of view. Since the set of all 3-manifolds is an infinite countable set, the notion of probability for some property of the set of 3-manifolds must be defined artificially, but naturally, for the sake of mathematical interest.

In this paper, we show that most closed 3-manifolds have zero first Betti numbers from the viewpoint of Heegaard splitting:

Theorem 1.1. *Let $M = H_+ \cup_F H_-$ be a genus $g \geq 2$ Heegaard splitting of a closed 3-manifold, for a simple closed curve c in F such that the homology class $[c]$ is generic in the set of curve-represented homology classes $\mathscr{H}_s \subset H_1(F)$. Then $rank(H_1(M_c, \mathbb{Q})) < max\{1, rank(H_1(M, \mathbb{Q})\}$, where M_c is the manifold obtained from M by horizontal Dehn surgery along c in the sense of Y. Moriah and J. Schultens.*

For precise definitions of the genericity, curve-represented homology classes and $\mathscr{H}_s$, see Definition 2.1.

Corollary 1.2. *Let $M = H_+ \cup_F H_-$ be a genus $g \geq 2$ Heegaard splitting of a closed 3-manifold. Then for a set of curves $\{c_1, c_2, \ldots, c_m\}$, $m \geq g$, such that each $[c_i]$ is generic in $\mathscr{H}_s \subset H_1(F)$, $M_{(c_1, c_2, \ldots, c_m)}$ is a rational homology 3-sphere.*

J. Johnson and T. Patel [4] introduced a set $\mathscr{B}^\infty$, called the generalized handlebody set, in the curve complex $\mathscr{C}(\partial H)$ for a handlebody H, and they proved that $M = H_+ \cup_F H_-$ has zero first Betti number if and only if the generalized handlebody sets $\mathscr{B}_+^\infty$ and $\mathscr{B}_-^\infty$ associated to H_+ and H_- have empty intersection. For any fixed copy of $\mathscr{B}^\infty$ in $\mathscr{C}(F)$ given by an identification $F = \partial H$ for some handlebody H, it seems difficult to detect whether or not $\mathscr{B}^\infty$ and $f(\mathscr{B}^\infty)$ intersect, for an automorphism f of F. However, we have the following:

Theorem 1.3. *Let $\mathscr{B}^\infty \subset \mathscr{C}(F)$ be given by an identification $F = \partial H$ for some handlebody H. For a set of curves $\{c_1, c_2, \ldots, c_m\}$, $m \geq g$, such that each $[c_i]$ is generic in $\mathscr{H}_s \subset H_1(F)$, the set $D_{c_1 c_2 \cdots c_m}(\mathscr{B}^\infty)$ is disjoint from $\mathscr{B}^\infty$, where $D_{c_1 c_2 \ldots c_m}$ is the composition of the Dehn twists along $c_1, c_2, \ldots, c_m$ iteratively.*

Remark 1.4. Our notion of the genericity in $H_1(F)$ is given with respect to a basis of $H_1(F)$ related to the Heegaard splitting $M = H_+ \cup_F H_-$, which has to satisfy some specific conditions (which are however natural in the given context). For an arbitrarily chosen basis of $H_1(F)$, whether or not results analogous to ours hold is unknown to the author.

Remark 1.5. Dunfield and Thurston proved a result similar to Corollary 1.2, in Corollary 8.5 of [2], using random walks in the mapping class group of a surface. Our results can be viewed as reconfirming the notion that rational homology 3-spheres are more common than the 3-manifolds that are not rational homology 3-spheres.

2. Preliminaries

For a surface F, M. Lustig and Y. Moriah introduced the notion of a generic curve in [8], which is based on the Lebesgue measure of $\mathscr{P}\mathscr{M}\mathscr{F}(F)$—the natural measure class on the projective measured foliation space induced by local charts, i.e., the measured foliations carried by bi-recurrent maximal train tracks. This notion is more natural than others.

For $M = H_+ \cup_F H_-$ a genus $g \geq 2$ Heegaard splitting of a closed 3-manifold, we can look at M as gluing H_+ and H_- by an orientation-reversing homeomorphism $\tau : \partial H_+ \longrightarrow \partial H_-$. Letting c be a simple closed curve in $F = \partial H_-$, we reglue H_+ and H_- by $D_c \circ \tau$, where D_c is the right Dehn twist along c, and we denote the resulting manifold by M_c. Note that M_c can also be looked at as obtained from M by 1/1 surgery along c, where the meridian-longitude coordinate is given by the

meridian on $\partial N(c)$ and the "horizontal" slope $F \cap \partial N(c)$, i.e., *the horizontal Dehn surgery* in the sense of Y. Moriah and J. Schultens; see [9].

We fix a canonical homology basis $\alpha = \{[a_1], [a_2], \ldots, [a_{2g}]\}$ of $H_1(F)$; i.e., $a_1, a_2, \ldots, a_{2g}$ are simple closed curves in F, and $F - \{a_1, a_2, \ldots, a_g\}$ is a $2g$-holed sphere, such that the geometric intersection number of a_i and a_{g+j} is δ_{ij}. Then an element of $H_1(F)$ can be written as $\sum_{i=1}^{2g} x_i [a_i]$, $x_i \in \mathbb{Z}$. We denote by $|x|_\alpha = \max\{|x_i|, 1 \le i \le 2g\}$, the α-*norm* of the homology class x. This homology class can be represented by a simple closed curve in F if and only if the greatest common denominator of $\{x_1, x_2, \ldots, x_{2g}\}$ is 1 (see [11]), and if this is the case, we say that the homology class is *curve-represented*. Note that whether or not a homology class is curve-represented is independent of the choice of the homology basis of $H_1(F)$. Let $\mathscr{H}_s$ be the collection of all homology classes in $H_1(F)$ which are curve-represented.

Now let c and d be two simple closed curves in F which represent the same homology class in $H_1(F)$. It is easy to see that $H_1(M_c) = H_1(M_d)$, for example, from the proof of Theorem 1.1. So the homology of the surgered manifold is a property preserved in the homology class. Thus, we introduce the notion of generic curve-represented homology classes:

Definition 2.1. For a property of points in $\mathscr{H}_s$, if the following holds:

$$(2.1) \qquad \lim_{N \to \infty} \frac{\sharp\{x \in \mathscr{H}_s \mid x \text{ satisfies the property } |x|_\alpha \le N\}}{\sharp\{x \in \mathscr{H}_s \mid |x|_\alpha \le N\}} = 1,$$

then we say that *the property is satisfied for generic curve-represented homology classes* with respect to the basis $\alpha = \{[a_1], [a_2], \ldots, [a_{2g}]\}$ of $H_1(F)$.

A family of homology classes $\{[c_i]\}_{i=1}^m$ is called *generic* (in $\mathscr{H}_s$) if each class $[c_i]$ in the family is *generic* (in $\mathscr{H}_s$), by which we mean precisely: $[c_i]$ belongs to a subset of $\mathscr{H}_s$ such that the property "belonging to this subset" is generic in the above-defined sense.

Definition 2.2 (see [4]). For a handlebody H of genus g, let $\mathcal{B}_i$ be the set of non-separating simple closed curves in ∂H that bound a genus i incompressible surface with one boundary in H. Let $\mathscr{B}^\infty = \bigcup_{i=0}^\infty \mathcal{B}_i$. Note that $\mathcal{B}_0$ is just the set of non-separating meridians in ∂H. The union $\mathscr{B}^\infty$ is called the *generalized handlebody set* or the *boundary set*.

Recall that the *curve complex* $\mathscr{C}(F)$ for a closed surface F of genus at least two is the graph whose vertices correspond to the isotopy classes of essential curves in F, and there is a length 1 edge between two vertices a, b whenever the isotopy classes a, b can be realized by disjoint curves in F.

J. Johnson and T. Patel [4] proved that $\mathscr{B}^\infty$ is 2-dense in $\mathscr{C}(\partial H)$ if H has genus at least three. By this we mean that for any essential simple closed curve c in ∂H, there is an essential simple closed curve b in $\mathscr{B}^\infty$ such that the distance between c and b in $\mathscr{C}(\partial H)$ is at most two. Lemma 5.10 of S. Schleimer [12] implies that $\mathscr{B}^\infty$ is 5-dense in $\mathscr{C}(\partial H)$ if H has genus two. Thus it is difficult to separate the boundary set from its f-image, for an automorphism f of F.

One of the potential approaches to show the ubiquity of rational homology 3-spheres is to show that the closure of $\mathscr{B}^\infty$ is small in $\mathscr{PMF}(F)$, where F is the boundary of a handlebody H. Recall that S. Kerckhoff [6] proved that the closure of $\mathcal{B}_0$ in $\mathscr{PMF}(F)$ has zero measure. But the following proposition blocks this approach:

Proposition 2.3. *For a handlebody H of genus $g \geq 2$, the closure of $\mathscr{B}^\infty$ in $\mathscr{PMF}(\partial H)$ is $\mathscr{PMF}(\partial H)$.*

Proof. We first assume that the genus of H is at least three. Let λ be a minimal uniquely ergodic foliation in ∂H. Then there is a sequence of simple closed curves $\{a_i\}_{i=1}^\infty$ in ∂H which converges to λ in $\mathscr{PMF}(\partial H)$. But by Klarreich's result on the boundary of the curve complex [5], we have that $\{a_i\}$ converges to λ in $\mathscr{C}(\partial H)$, where λ is understood as a point of the Gromov boundary of the curve complex. Since $\mathscr{B}^\infty$ is 2-dense in the curve complex, there is $b_i \in \mathscr{B}^\infty$ with $d_{\mathscr{C}(\partial H)}(a_i, b_i) \leq 2$. So $\{b_i\}$ converges to λ also; see Lemma 3.1 of [3]. Then, by Masur's result that almost every foliation is uniquely ergodic, the closure of $\mathscr{B}^\infty$ is $\mathscr{PMF}(\partial H)$.

The proof of the genus two case is similar, just replacing the 2-density of $\mathscr{B}^\infty$ in the curve complex by the 5-density. $\square$

The other potential approach is using the structure of the mapping class group $\mathcal{MCG}(F)$ and its dynamic properties. But note that the *Torelli group* $\mathscr{T}_g$, the subgroup of $\mathcal{MCG}(F)$ consisting of elements that act trivially on $H_1(F)$, has infinite index in $\mathcal{MCG}(F)$, and it is well known that the limit set of $\mathscr{T}_g$, say $\Lambda(\mathscr{T}_g)$, is the full space $\mathscr{PMF}(F)$ (for example, from Fact 3.2), so this approach is difficult, and it seems this approach always fails.

3. Proof of the theorems

In this section we prove our theorems.

Proof of Theorem 1.1. Let $M = H_+ \cup_F H_-$ be a genus g Heegaard splitting of a closed 3-manifold. Then $H_1(M, \mathbb{Z})$ has a presentation matrix X of size $g \times g$, and the first Betti number is the rank of $H_1(M, \mathbb{Q})$: take $a_1, a_2, \ldots, a_g$ to be a set of oriented simple closed curves in ∂H_+, each of which bounds a disk in H_+, such that these disks decompose H_+ into a 3-ball; we say that $\{a_1, a_2, \ldots, a_g\}$ is a *basis* of H_+. Similarly we define $b_1, b_2, \ldots, b_g$ in ∂H_-. Let $a_1^*, a_2^*, \ldots, a_g^*$ be a set of curves dual to $a_1, a_2, \ldots, a_g$ in ∂H_+: a_i^* is a simple closed curve such that the geometric intersection number of a_i^* and a_j is δ_{ij}. So $\{a_1^*, a_2^*, \ldots, a_g^*\}$ is a basis for $H_1(H_+)$, and $H_1(M, \mathbb{Z})$ is obtained from $H_1(H_+)$ by adding the relations $b_1, b_2, \ldots, b_g$. Then $H_1(M, \mathbb{Z})$ has the presentation matrix $X = ((a_i, b_j))_{g \times g}$, where $(.,.)$ is the algebraic intersection number of two curves. Note that the rank of X is just $g - rank(H_1(M, \mathbb{Q}))$. We denote by a_i also the curve $\tau(a_i)$ in $F = \partial H_-$.

Let c be a simple closed curve in F. Then in M_c the curve $\tilde{a}_i$ bounds a disk in H_+, where $\tilde{a}_i$ is the curve obtained from a_i by a right Dehn twist along c. Thus one has $(\tilde{a}_i, b_j) = (a_i, b_j) + (a_i, c)(c, b_j)$. If d is another simple closed curve in F such that $[c] = [d]$ in $H_1(F)$, then $(a_i, c)(c, b_j) = (a_i, d)(d, b_j)$, so $H_1(M_c)$ and $H_1(M_d)$ have the same presentation matrix. Thus M_c and M_d have the same homology.

Starting from X, we can transform it iteratively into $Y = diag\{\lambda_1, \lambda_2, \ldots \lambda_r, 0, \ldots, 0\}$, where $\lambda_i | \lambda_{i+1}$ are non-zero integers, by the following operations:

(1) interchange two rows or two columns,
(2) multiply a column (row) by -1,
(3) add to any column (row) an integer linear combination of other columns (rows).

Notice that the transformation from X to Y can be induced by the changes of the basis of H_+ and the basis of H_-: transformation (1) is induced by a relabeling of the
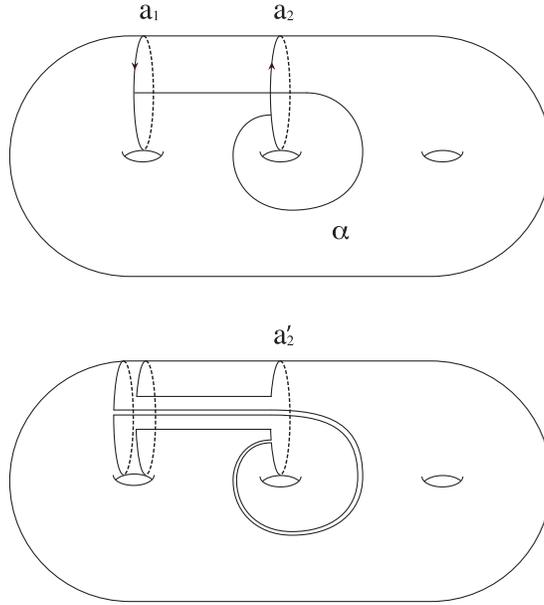
FIGURE 1

basis $\{a_i\}_{i=1}^g$ or $\{b_i\}_{i=1}^g$; transformation (2) is induced by reversing the orientation of one of a_i or b_i; transformation (3) is induced by band-sums of disks: for example, corresponding to adding k times the first column to the second column, we can take an oriented arc α from $\partial_1\alpha = \alpha \cap a_1$ to $\partial_2\alpha \in \alpha \cap a_2$, and $(a_1, \alpha) = 1$, $(\alpha, a_2) = k$ (note that k may be negative). We assume that $\alpha \cap a_2$ are $|k|$ points and we denote them by $p_1, p_2, \ldots, p_{|k|}$, where $p_{|k|} = \partial_2\alpha$ and we also denote $\partial_1\alpha$ by p_0. We can also assume $\alpha \cap a_i = \emptyset$ for $i \neq 1, 2$. Then performing the disk-sum of a_1 and a_2 along the sub-arc p_0p_1, we get a disk $a_1\sharp_{p_0p_1}a_2$, which is disjoint from a_i for $i \neq 2$, and α intersects $a_1\sharp_{p_0p_1}a_2$ in $p_2, p_3, \ldots, p_{|k|}$. Then we perform a disk-sum of a_1 and $a_1\sharp_{p_0p_1}a_2$ along p_0p_2. We also obtain a disk which is disjoint from a_i for $i \neq 2$. So performing disk-sums k times we get a disk a_2' such that $a_1, a_2', \ldots, a_g$ is a new basis of H_+. The intersection matrix of $a_1, a_2', \ldots, a_g$ and $b_1, b_2, \ldots, b_g$ corresponds to adding k times the first column to the second column. See Figure 1 for the case of adding twice the first column to the second column.

In other words, we can rechoose the basis $\{a_1, a_2, \ldots, a_g\}$ of H_+ and the basis $\{b_1, b_2, \ldots, b_g\}$ of H_-, such that $((a_i, b_j))_{g \times g} = diag\{\lambda_1, \lambda_2, \ldots \lambda_r, 0, \ldots, 0\}$ and $H_1(M) = \mathbb{Z}^{g-r} \oplus \mathbb{Z}_{\lambda_1} \oplus \mathbb{Z}_{\lambda_2} \oplus \cdots \oplus \mathbb{Z}_{\lambda_r}$.

We consider

$$\mathbf{S} = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

Now $\{[a_1], [a_2], \ldots, [a_g], [a_1^*], [a_2^*], \ldots, [a_g^*]\}$ is a standard homology basis of $H_1(F)$, and we assume $a_1 = (1, 0, 0 \ldots, 0)$, $a_2 = (0, 1, 0 \ldots, 0)$, $\ldots$, $a_g = (0, \ldots, 0, 1 \mid 0, 0 \ldots, 0)$ are $2g$-dimensional row vectors, and $a_1^* = (0, 0, 0 \ldots, 0 \mid 1, 0, 0 \ldots, 0)$, $a_2^* = (0, 0, 0 \ldots, 0 \mid 0, 1, \ldots, 0)$, $\ldots$, $a_g^* = (0, 0, 0 \ldots, 0 \mid 0, 0, \ldots, 1)$ are $2g$-dimensional row vectors.

For

$$\mathbf{A} = \left(\begin{array}{cc} I & 0 \end{array} \right)_{g \times 2g},$$

the vector a_i is the i-th row vector of A. Let B be a $g \times 2g$ matrix such that the i-th row vector of B is b_i in the coordinate given by $\{[a_1], [a_2], \ldots, [a_g], [a_1^*], [a_2^*], \ldots, [a_g^*]\}$. Then the intersection matrix $((a_i, b_j))_{g \times g}$ is just ASB^T, where B^T denotes the transposed matrix of B.

Now assume that $B = (B_1, B_2)$, where B_1 and B_2 are two $g \times g$ matrixes. Then we have $ASB^T = diag\{\lambda_1, \lambda_2, \ldots, \lambda_r, 0, \ldots, 0\}$. So from the above claim, we have $(B_2^T) = diag\{\lambda_1, \lambda_2, \ldots, \lambda_r, 0, \ldots, 0\} = B_2$. Denote by $\beta_1, \beta_2, \ldots, \beta_g$ the row vectors of B_1. Note that since $B = (B_1, B_2)$ is represented by g homology independent curves in F, the rank of $B = (B_1, B_2)$ is g. Hence the bottom submatrix of B_1 which consists of $\beta_{r+1}, \beta_{r+2}, \ldots, \beta_g$ has rank $g - r$. In particular, each β_j is non-zero, for $r + 1 \leq j \leq g$ if $r < g$.

Let $[c] = (x_1, x_2, \ldots, x_g, y_1, \ldots, y_g) = (x, y)$ be a homology class with respect to the coordinate $\{[a_1], [a_2], \ldots, [a_g], [a_1^*], [a_2^*], \ldots, [a_g^*]\}$ that can be represented by a simple closed curve c in F, where x and y are two g-dimensional row vectors.

Consider the presentation matrix $\widetilde{X}$ for $H_1(M_c, \mathbb{Z})$, which is $\widetilde{X} = ((\widetilde{a}_i, b_j))_{g \times g} = ((a_i, b_j) + (a_i, c)(c, b_j))_{g \times g}$. Note that $(a_i, c) = y_i$, and $(c, b_j) = x_j \lambda_j - \beta_j \cdot y$ for $1 \leq j \leq r$ and $(c, b_j) = -\beta_j \cdot y$ for $r + 1 \leq j \leq g$, where $\beta_j \cdot y$ is the inner product of two vectors β_j and y.

Now consider the left-top $(r + 1) \times (r + 1)$ submatrix of $\widetilde{X}$, which is:

$$\begin{pmatrix} \lambda_1 + y_1(x_1\lambda_1 - \beta_1 \cdot y) & \cdots & 0 + y_1(x_r\lambda_1 - \beta_r \cdot y) & 0 + y_1(-\beta_{r+1} \cdot y) \\ \vdots & \vdots & \vdots & \vdots \\ 0 + y_r(x_1\lambda_1 - \beta_1 \cdot y) & \cdots & \lambda_r + y_r(x_r\lambda_1 - \beta_r \cdot y) & 0 + y_r(-\beta_{r+1} \cdot y) \\ 0 + y_{r+1}(x_1\lambda_1 - \beta_1 \cdot y) & \cdots & 0 + y_{r+1}(x_r\lambda_1 - \beta_r \cdot y) & 0 + y_{r+1}(-\beta_{r+1} \cdot y) \end{pmatrix}.$$

Thus, if $-\beta_{r+1} \cdot y \neq 0$, using the last column, we can simplify the first r columns such that the determinant of it is $\lambda_1 \cdots \lambda_r y_{r+1}(-\beta_{r+1} \cdot y)$, which is non-zero if y_{r+1} is non-zero.

Note that the solution space of

$$(3.1) \qquad\qquad\qquad y_{r+1}(-\beta_{r+1} \cdot y) = 0$$

is just the union of two hyperplanes in $\mathbb{R}^g$, since β_{r+1} is a non-zero vector.

Thus, if one of $y_{r+1}(-\beta_{r+1} \cdot y), \ldots, y_g(-\beta_g \cdot y)$ is non-zero, then $\widetilde{X}$ has an $(r+1) \times (r+1)$ submatrix which has non-zero determinant, which implies $rank(\widetilde{X}) > rank(X)$. But note that $\beta_{r+1}, \beta_{r+2}, \ldots, \beta_g$ are linearly independent. The set of $(x_1, x_2, \ldots, x_g, y_1, \ldots, y_g) = (x, y) \in \mathbb{Z}^{2g}$ such that all $y_{r+1}(-\beta_{r+1} \cdot y), \ldots, y_g(-\beta_g \cdot y)$ are zeros has measure zero in $\mathbb{Z}^{2g}$. We are done.

If $r = g$, then $\widetilde{X}$ is given by

$$\begin{pmatrix} \lambda_1 + y_1(x_1\lambda_1 - \beta_1 \cdot y) & \cdots & y_1(x_{g-1}\lambda_1 - \beta_{g-1} \cdot y) & y_1(-\beta_g \cdot y) \\ \vdots & \vdots & \vdots & \vdots \\ y_{g-1}(x_1\lambda_1 - \beta_1 \cdot y) & \cdots & \lambda_{g-1} + y_{g-1}(x_{g-1}\lambda_1 - \beta_{g-1} \cdot y) & y_{g-1}(-\beta_g \cdot y) \\ y_g(x_1\lambda_1 - \beta_1 \cdot y) & \cdots & y_g(x_{g-1}\lambda_1 - \beta_{g-1} \cdot y) & \lambda_g + y_g(-\beta_g \cdot y) \end{pmatrix},$$

so that the determinant of it is a polynomial in $x_1, x_2, \ldots, x_g, y_1, \ldots, y_g$, with zero degree coefficient $\lambda_1\lambda_2 \cdots \lambda_g$. Hence the polynomial is non-zero and the set

of $\{x_1, x_2, \ldots, x_g, y_1, \ldots, y_g\} \in \mathbb{Z}^{2g}$ such that the determinant of $\widetilde{X}$ is zero has measure zero in $\mathbb{Z}^{2g}$. $\square$

Proof of Corollary 1.2. Now from Theorem 1.1, if $rank(H_1(M, \mathbb{Q})) = r \neq 0$, then for a homology-generic curve c, $rank(H_1(M_c, \mathbb{Q})) = rank(H_1(M, \mathbb{Q})) - 1$. Thus by performing iteratively r times horizontal Dehn surgeries along $c_1, c_2, \ldots, c_r$, we obtain that $M_{(c_1, c_2, \cdots c_r)}$ is a rational homology sphere, and for any homology-generic curve c_{r+1}, the manifold $M_{(c_1, c_2, \cdots c_{r+1})}$ is also a rational homology sphere by Theorem 1.1. $\square$

Remark 3.1. For two links K_1 and K_2 in S^3, it is difficult to show the relationship between the surgered manifolds along K_1 and K_2. For Heegaard splittings of manifolds as in Theorem 1.1, it can be shown that most different horizontal Dehn surgeries result in different manifolds: the main result of [8] by M. Lustig and Y. Moriah is that for any fixed number n, then for a generic curve c (this notion is defined in terms of the Lebesgue measure on $\mathscr{PMF}(F)$), the manifold M_c has Heegaard distance at least n. Hence for generic and homology-generic curves $\{c_{m+1}, \ldots, c_k\}$, the Heegaard distance of $M_{(c_1, c_2, \ldots, c_k)}$ is greater than the Heegaard distance of $M_{(c_1, c_2, \ldots c_m)}$ by Fact 3.2 below. Thus $M_{(c_1, c_2, \ldots c_k)}$ is not homeomorphic to $M_{(c_1, c_2, \ldots c_m)}$, for generic and homology-generic curves $c_{m+1}, \ldots, c_k$, if $k > m$.

Fact 3.2. For any x in $\mathscr{H}_s$, i.e., a homology class in $H_1(F)$ which can be realized by a simple closed curve in F, the closure of the set S_x of simple closed curves in F that represent x is the full space $\mathscr{PMF}(F)$.

Proof. This fact follows from the fact that the Torelli group $\mathscr{T}_g$ acts minimally on $\mathscr{PMF}(F)$. We include here a simple argument for completeness: take any c in S_x, and take a separating curve d which intersects c essentially. Then $D_d^n(c)$ is homologous to c in $H_1(F)$, and the limit of $D_d^n(c)$ in $\mathscr{PMF}(F)$ for $n \to \infty$ is d. Since separating curves are 1-dense in $\mathscr{C}(F)$, the claim follows from the arguments in Proposition 2.2. $\square$

Proof of Theorem 1.3. Fix a genus g handlebody H. The double of H is $N = \sharp^g S^2 \times S^1$, which has first Betti number g. For the genus g Heegaard splitting $N = H_+ \cup_F H_-$, we have $\mathscr{B}_+^\infty = \mathscr{B}_-^\infty$. We use Corollary 1.2 to obtain that $N_{(c_1, c_2, \ldots c_m)}$ is a rational homology sphere. Note that $N_{(c_1, c_2, \ldots c_m)}$ is obtained from H_+ by gluing the mirror copy H_- of it with the gluing map $D_{c_1 c_2 \ldots c_m}$. By [4], we have that $D_{c_1 c_2 \ldots c_m}(\mathscr{B}_+^\infty)$ is disjoint from $\mathscr{B}_-^\infty = \mathscr{B}_+^\infty$. $\square$

Acknowledgements

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