LINK BETWEEN NOETHERIANITY
AND THE WEIERSTRASS DIVISION THEOREM
ON SOME QUASIANALYTIC LOCAL RINGS

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Abstract. In the setting of well-behaved quasianalytic differentiable systems, we prove that the Weierstrass Division Theorem holds in such system if, and only if, the system is Noetherian.

1. Introduction

Let $C_k$, $k = 1, 2, \ldots$, be local quasianalytic rings of germs, at the origin in $\mathbb{R}^k$, of smooth functions. We suppose that the system $C = \{C_k, k \in \mathbb{N}\}$ satisfies some natural properties; see Section 2. We know by [3] that the Weierstrass Division Theorem never holds in such a system if $C_k$, $k = 1, 2, \ldots$, is not contained in the ring of germs of real analytic functions. Because of the lack of a Weierstrass Division Theorem, many problems remain open for such rings. For example, we do not know if the $C_k$ are Noetherian rings. The present study may be regarded as an inquiry as to what differences exist between a system $C = \{C_k, k \in \mathbb{N}\}$ in which we suppose that a Weierstrass Division Theorem holds and a system in which the $C_k$, $k = 1, 2, \ldots$, are Noetherian rings. This question is supported by the following conjecture, asked in the setting of quasianalytic Denjoy-Carleman classes (see [4, Ch. 7, An algebraic question]).

Conjecture. It may be that a Weierstrass Division Theorem holds in the system $C = \{C_k, k \in \mathbb{N}\}$ if and only if the $C_k$ are Noetherian rings.

We remark that in light of the result of [6], where this conjecture is established, we would know that some quasianalytic rings are not Noetherian.

It is clear that if a Weierstrass Division Theorem holds in the system $C = \{C_k, k \in \mathbb{N}\}$, then by repeating standard arguments, we show that the $C_k$ are Noetherian rings; see [12]. The aim of this paper is to prove the converse for some Noetherian local rings.

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2. DIFFERENTIABLE SYSTEM

**Definition 2.1.** A differentiable system is a sequence $\mathcal{C} = \{\mathcal{C}_n, n \in \mathbb{N}\}$ such that, for each $n \in \mathbb{N}$, $\mathcal{C}_n \subset \mathcal{E}_n$ is a local subring of the ring of germs, at the origin of $\mathbb{R}^n$, of $C^\infty$ functions. We suppose that, for each $n \in \mathbb{N}$, $\mathcal{C}_n$ is closed under taking derivatives, and the following hold:

$$(C_1) \quad \mathbb{R}[x_1, \ldots, x_n] \subset \mathcal{C}_n \subset \mathcal{E}_n,$$

for each $n \in \mathbb{N}$, where $\mathbb{R}[x_1, \ldots, x_n]$ is the ring of polynomials with coefficients in $\mathbb{R}$.

$$(C_2) \quad \text{The system } \mathcal{C} \text{ is closed under composition. This means that if } g \in \mathcal{C}_k \text{ and } f = (f_1, \ldots, f_k) \in (\mathcal{C}_n)^k \text{ with } f(0) = 0, \text{ then } g \circ f \in \mathcal{C}_n.$$ 

$$(C_3) \quad \text{For each } n \in \mathbb{N}, \mathcal{C}_n \text{ is closed under division by coordinates. This means that if } f \in \mathcal{C}_n \text{ and } f = (x_i - \alpha)g, \text{ where } g \in \mathcal{E}_n \text{ and } \alpha \in \mathbb{R}, \text{ then } g \in \mathcal{C}_n.$$ 

$$(C_4) \quad \text{The Implicit Function Theorem for } \mathcal{C}_n \text{ holds in the following sense: Suppose that } f = (f_1, \ldots, f_m) \in (\mathcal{C}_{n+m})^m \text{ with } f(0,0) = 0. \text{ Put } y = (y_1, \ldots, y_m) \text{ and suppose that }$$

$$det(\frac{\partial f_i}{\partial y_j}(0,0))_{i,j=1,\ldots,m} \neq 0.$$ 

Then there is a (unique) $g = (g_1, \ldots, g_m) \in (\mathcal{C}_m)^m$ with $g(0) = 0$ such that $f(x, g(x)) = 0$.

Call

$$\hat{\cdot} : \mathcal{C}_n \to \mathbb{R}[[x_1, \ldots, x_n]]$$

the map which associates to each $f \in \mathcal{C}_n$ its Taylor expansion at the origin. We consider the following conditions:

$$(C_5) \quad \hat{\cdot} \text{ is an injective homomorphism.}$$

$$(C_6) \quad \mathcal{C}_n \text{ is a Noetherian ring for each } n \in \mathbb{N}.$$ 

**Definition 2.2.** A differentiable system $\mathcal{C} = \{\mathcal{C}_n, n \in \mathbb{N}\}$ is called **quasianalytic** [resp. **Noetherian**] if condition $(C_5)$ holds [resp. if condition $(C_6)$ holds].

**Remark 2.3.** It is clear that every Noetherian differentiable system is a quasianalytic system.

**Example 2.4.**

i) If for each $n \in \mathbb{N}$, $\mathcal{C}_n$ is the ring of germs, at the origin in $\mathbb{R}^n$, of Nash functions, i.e. algebraic on the ring of polynomials $\mathbb{R}[x_1, \ldots, x_n]$, the system $\mathcal{C} = \{\mathcal{C}_n, n \in \mathbb{N}\}$ is a Noetherian differentiable system.

ii) If for each $n \in \mathbb{N}$, $\mathcal{C}_n$ is the ring of germs, at the origin in $\mathbb{R}^n$, of real analytic functions, the system $\mathcal{C} = \{\mathcal{C}_n, n \in \mathbb{N}\}$ is a Noetherian differentiable system.

iii) Let $\mathcal{R}$ be a polynomially bounded o-minimal structure which is an expansion of the ordered field of reals. For more details about an o-minimal structure over the field of reals, we refer the reader to [13]. We denote by $\mathcal{D}_n$ the ring of germs, at the origin in $\mathbb{R}^n$, of $C^\infty$ definable functions in a neighborhood of the origin in $\mathbb{R}^n$. By [3], the system $\mathcal{D} = \{\mathcal{D}_n, n \in \mathbb{N}\}$ is a quasianalytic differentiable system.

iv) Let $M = \{M_p\}_{p=0}^{\infty}$ be an increasing sequence of positive real numbers. We denote by $\mathcal{C}_n(M) \subset \mathcal{E}_n$ the subring of germs of $C^\infty$ functions in a neighborhood of the origin which are in the class $M$; see [4] and [10]. If we suppose that the class is quasianalytic, then the system $\mathcal{C}(M) = \{\mathcal{C}_n(M), n \in \mathbb{N}\}$ is a quasianalytic differentiable system.
In the following, for a differentiable quasianalytic system, we will not distinguish
notationally by : between the germ and its image, i.e. its Taylor expansion at the
origin.

In particular these conditions on a quasianalytic system imply that the maximal
ideal of $C_n$ is $m_n = \{ f \in C_n / f(0) = 0 \} = (x_1, \ldots, x_n)C_n$ and its completion with
the $m_n$-adic topology is the ring of formal series $R[[x_1, \ldots, x_n]]$.

3. Artin approximation property for a Noetherian system

Let $C = \{ C_n, n \in \mathbb{N} \}$ be a differentiable Noetherian system. For each $n \in \mathbb{N}$,
the completion of the ring $C_n$ with respect to the $m_n$-topology is the ring of formal
power series $R[[x_1, \ldots, x_n]]$, which is a regular ring. By [11, Proposition 24], $C_n$ is
also a regular ring; hence $C_n$ is an excellent ring [2, Theorem 24]. We see then that
the morphism

$$
C_n \rightarrow R[[x_1, \ldots, x_n]]
$$

is a regular homomorphism [12, Section 1]. The condition $(C_4)$ means that $C_n$ is
a Henselian ring. By [12, Theorem 2.4], the Artin approximation property holds
for the pair $(C_n, R[[x_1, \ldots, x_n]])$. This means that for every system of polynomial
equations $f = 0$, where $f = (f_1, \ldots, f_p)$ with $f_i \in C_n[Y]$ and $Y = (Y_1, \ldots, Y_n)$
a set of variables, for each $\nu \in \mathbb{N}$ and each formal solution $\hat{g} = (\hat{g}_1, \ldots, \hat{g}_N) \in
(R[[x_1, \ldots, x_n]])^N$, so that $f(\hat{g}) = 0$, we can find a solution $g = (g_1, \ldots, g_N) \in (C_n)^N
$ such that

$$
f(g) = 0 \quad \text{and} \quad g - \hat{g} \in m_n^\nu R[[x_1, \ldots, x_n]]^N.
$$

3.1. Monomialization lemma for Noetherian system. We recall a result
proved by Eakin-Harris [5, Lemma 5.1] for convergent power series. Here we give
its analogue for formal series; the proof is the same.

**Lemma 3.1.** Let $f \in R[[x_1, \ldots, x_n]]$. Then there exists $H = (x_1, M_2x_2, M_3x_3, \ldots,
M_nx_n)$ where for each $j = 2, \ldots, n$, $M_j$ is a monomial in only the variables
$x_1, \ldots, x_j$ such that

$$
f(x_1, M_2x_2, M_3x_3, \ldots, M_nx_n) = x_1^{\alpha_1} \ldots x_r^{\alpha_r} Q
$$

for some unit $Q \in R[[x_1, \ldots, x_n]]$, $i_1, \ldots, i_r \in \{ 1, \ldots, n \}$ and $\alpha_{i_1}, \ldots, \alpha_{i_r} \in \mathbb{N}$.

**Proposition 3.2.** Let $C = \{ C_n, n \in \mathbb{N} \}$ be a differentiable Noetherian system,
and let $\varphi \in C_n$. Then there exists $H = (x_1, M_2x_2, M_3x_3, \ldots, M_nx_n)$ where for each
$j = 2, \ldots, n$, $M_j$ is a monomial in only the variables $x_1, \ldots, x_j$ such that

$$
\varphi(x_1, M_2x_2, M_3x_3, \ldots, M_nx_n) = x_1^{\alpha_1} \ldots x_r^{\alpha_r} Q
$$

for some unit $Q \in C_n$, $i_1, \ldots, i_r \in \{ 1, \ldots, n \}$ and $\alpha_{i_1}, \ldots, \alpha_{i_r} \in \mathbb{N}$.

**Proof.** If $\varphi \in C_n$, by Lemma 3.1, there exists $H = (x_1, M_2x_2, M_3x_3, \ldots, M_nx_n)$
where for each $j = 2, \ldots, n$, $M_j$ is a monomial in only the variables $x_1, \ldots, x_j$ such that

$$
\varphi(x_1, M_2x_2, M_3x_3, \ldots, M_nx_n) = x_1^{\alpha_1} \ldots x_r^{\alpha_r} Q
$$

for some unit $Q \in R[[x_1, \ldots, x_n]]$, $i_1, \ldots, i_r \in \{ 1, \ldots, n \}$ and $\alpha_{i_1}, \ldots, \alpha_{i_r} \in \mathbb{N}$. We consider the equation $E(Z)$:

$$
x_1^{\alpha_1} \ldots x_r^{\alpha_r} Z - \varphi(x_1, M_2x_2, M_3x_3, \ldots, M_nx_n) = 0.
$$
We see that \( E(Z) \in \mathcal{C}_n[Z] \) and \( \hat{Q} \) is a formal solution of this equation. By the Artin Approximation Theorem, if \( \nu \in \mathbb{N}^* \), there exists \( Q \in \mathcal{C}_n \) that is a solution of the equation \( E(Z) \) such that \( Q - \hat{Q} \in m_n^\nu \mathbb{R}[[x_1, \ldots, x_n]] \), hence the result since \( \nu \geq 1 \).

4. Eakin-Harris property

In this section, we fix a quasianalytic differentiable system \( \mathcal{C} = \{ \mathcal{C}_n, n \in \mathbb{N} \} \). We put \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_k) \). We are concerned here with local homomorphisms \( \Phi : \mathcal{C}_n \to \mathcal{C}_k \), i.e. homomorphisms such that \( \Phi(m_n) \subset m_k \). The generic rank of \( \Phi \), denoted \( rk(\Phi) \), is the rank of the Jacobian matrix \( \frac{\partial \Phi(x_i)}{\partial y_j} \), considered as a matrix over the quotient field of \( \mathcal{C}_k \). Recall that \( \mathcal{C}_k \) is a domain by condition (\( C_5 \)). Since \( \Phi \) is a local homomorphism, we consider its natural extension to the completion \( \hat{\Phi} : \mathbb{R}[[x_1, \ldots, x_n]] \to \mathbb{R}[[y_1, \ldots, y_k]] \). It is well known that if \( rk(\Phi) = n \), then \( \Phi \) is injective. In [9], we can find an example (Osgood’s example) of an injective homomorphism \( \Phi \) for which \( rk(\Phi) < n \). Thus the condition is not necessary.

We let

\[
\Phi_* : \frac{\mathbb{R}[[x_1, \ldots, x_n]]}{\mathcal{C}_n} \to \frac{\mathbb{R}[[y_1, \ldots, y_k]]}{\mathcal{C}_k}
\]

be the homomorphism of groups induced by \( \Phi \) and \( \hat{\Phi} \) in the obvious manner.

**Definition 4.1.** We say that the homomorphism \( \Phi \) is strongly injective if the homomorphism \( \Phi_* \) is injective.

In the analytic setting, i.e. when each \( \mathcal{C}_n \) is the ring of germs of real analytic functions, Eakin and Harris [5] showed that \( \Phi \) is strongly injective if and only if \( rk(\Phi) = n \). Their result extends a result of Abhyankar and van der Put [1]. This result justifies the following definition:

**Definition 4.2.** We say that a local morphism \( \Phi : \mathcal{C}_n \to \mathcal{C}_k \) has the Eakin-Harris property if \( rk(\Phi) = n \) implies \( \Phi \) is strongly injective.

It is shown in [7] that if every morphism \( \Phi : \mathcal{C}_n \to \mathcal{C}_k \) has the Eakin-Harris property, then the system \( \mathcal{C} = \{ \mathcal{C}_n \} \) is contained in the analytic system.

**Lemma 4.3.** Let \( \Phi : \mathcal{C}_n \to \mathcal{C}_k \) and \( \Psi : \mathcal{C}_k \to \mathcal{C}_l \) be local homomorphisms. If \( \Psi \circ \Phi \) is strongly injective, then \( \Phi \) is strongly injective.

**Proof.** Follows from the definition. \( \square \)

**Remark 4.4.** Isomorphisms are strongly injective.

Recall that a local homomorphism \( u : A \to B \) between local rings is called finite if \( B \) is a finite module over the ring \( u(A) \).

**Lemma 4.5** ([1]). Let \( \mathcal{C} = \{ \mathcal{C}_n, n \in \mathbb{N} \} \) be a differentiable Noetherian system. If \( \Phi : \mathcal{C}_n \to \mathcal{C}_k \) is injective and finite, then \( \hat{\Phi} \) is injective and finite and \( \Phi \) is strongly injective.

**Proof.** Since each \( \mathcal{C}_k \) is a Zariski ring, the lemma follows from [13, Chapter VIII, Theorem 9].

In the following we identify local homomorphisms \( \Phi : \mathcal{C}_n \to \mathcal{C}_k \) defined by \( \Phi(f) = f(\varphi_1, \ldots, \varphi_n), \varphi_i \in \mathcal{C}_k, i = 1, \ldots, n \), by \((\varphi_1, \ldots, \varphi_n)\) and we write \( \Phi = (\varphi_1, \ldots, \varphi_n) \).
We consider the local homomorphism
\[ e = (y_1y_2, y_2, \ldots, y_k) : C_k \to C_k. \]
For \( d \in \mathbb{N}^+ \) we also consider the local homomorphism
\[ r_d = (y_1^d, y_2, \ldots, y_k) : C_k \to C_k. \]
It is clear that homomorphisms \( e, r_d \) \((d \in \mathbb{N})\) are injective and \( rk(r_d) = rk(e) = k \).

**Remark 4.6.** Let \( P = \sum_{\omega} a_{\omega} y_2^1 \omega_2^2 \cdots y_k^\omega_k \) be a polynomial. There exists \( N \in \mathbb{N} \) such that, for each \( \omega = (\omega_1, \ldots, \omega_k) \in \mathbb{N}^k \) with \( a_{\omega} \neq 0 \), \( N + \omega_2 \geq \omega_1 \). We have then \( y_2^N P = e(Q) \), where \( Q = \sum_{\omega} a_{\omega} y_2^\\omega_2 y_2^N+\omega_2-\omega_1 y_3^\omega_3 \cdots y_k^\omega_k \).

**Definition 4.7.** We say that the quasianalytic differentiable system \( C = \{C_n, n \in \mathbb{N}\} \) is well behaved if the homomorphisms \( e \) and \( r_d \) are strongly injective.

**Proposition 4.8.** The systems considered in i) and ii) of Example 2.4 are well behaved.

**Proof.**

(1) Analytic system.

Let \( \hat{f} = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} \), \( \alpha = (\alpha_1, \ldots, \alpha_n) \), be a formal power series such that \( e(\hat{f}) \) is a convergent series. Hence there exist \( r, M \in \mathbb{R}_+^k \), \( r < 1 \), such that for every \( \alpha \in \mathbb{N}^n \), \( |a_{\omega}| \leq M \), where \( |\alpha| = \alpha_1 + \cdots + \alpha_n \). We have then, for every \( \alpha \in \mathbb{N}^n \), \( |a_{\omega}| \leq M \), which proves that \( \hat{f} \) is a convergent power series; hence the homomorphism \( e \) is strongly injective. It is also an elementary calculation to show that the homomorphism \( r_d, d \in \mathbb{N}^+ \), is strongly injective. Hence the analytic system is well behaved.

(2) Nash system.

Let \( \hat{f} = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) be a formal power series such that \( e(\hat{f}) \) is algebraic on the ring of polynomials \( \mathbb{R}[x_1, \ldots, x_n] \). It is known already that \( e(\hat{f}) \) is a convergent series (the ring of convergent power series is algebraically closed in the ring of formal power series); hence \( \hat{f} \) is also a convergent power series. We have:

\[ P_q(e(\hat{f}))^q + \ldots + P_1 e(\hat{f}) + P_0 = 0, \tag{*} \]

with all the polynomials \( P_j \neq 0 \). By Remark 4.6, there exist \( m \in \mathbb{N} \) and polynomials \( Q_j, j = 1, \ldots, q \), such that \( x_2^m P_j = e(Q_j) \). By multiplying the equation \((*)\) by \( x_2^m \) and since \( e \) is injective, we have \( Q_j f^q + \ldots + Q_0 = 0 \); hence \( \hat{f} \) is algebraic on the ring of polynomials, which proves that the morphism \( e \) is strongly injective.

Now, let \( \hat{f} = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) be a formal power series such that \( r_d(\hat{f}) \) is algebraic on the ring of polynomials. Since the Weierstrass Division Theorem is true in the ring of Nash functions (see [3, 8.2.8]), we divide \( r_d(\hat{f}) \) by the polynomial \( x_1^d - T \), where \( T \) is an auxiliary variable,

\[ r_d(\hat{f}) = (x_1^d - T)Q(x, T) + \sum_{j=1}^d b_j(x_2, \ldots, x_n, T)x_1^{d-j}, \]

where \( Q(x, T), b_j(x_2, \ldots, x_n, T) \) are algebraic on the ring of polynomials, \( j = 1, \ldots, d \).
We also divide the formal series \( \hat{f} \) by the polynomial \( x_1 - T \) in the ring of formal series \( \mathbb{R}[[x_1, \ldots, x_n, T]] \).

\[
\hat{f} = (x_1 - T)Q_1(x, T) + W(x_2, x_3, \ldots, x_n, T),
\]

where \( Q_1(x, T) \in \mathbb{R}[[x_1, \ldots, x_n, T]], W(x_2, x_3, \ldots, T) \in \mathbb{R}[[x_2, \ldots, x_n, T]] \).

We have then

\[
\hat{f}(x_1^d, x_2, \ldots, x_n) = (x_1^d - T)Q_1(x_1^d, x_2, \ldots, x_n, T) + W(x_2, x_3, \ldots, x_n, T).
\]

Since the division is unique in \( \mathbb{R}[[x_1, \ldots, x_n, T]] \), we see that

\[
Q(x, T) = Q_1(x_1^d, x_2, \ldots, x_n, T)
\]

and

\[
\sum_{j=1}^{d} b_j(x_2, \ldots, x_n, T)x_1^{d-j} = W(x_2, x_3, \ldots, x_n, T).
\]

Hence \( b_1 = \ldots = b_{d-1} = 0 \), and \( W(x_2, x_3, \ldots, x_n, T) = b_d(x_2, \ldots, x_n, T) \), but, from (**), we have \( W(x_2, x_3, \ldots, x_n, x_1) = \hat{f}(x_1, \ldots, x_n) \), hence the result.

\[\square\]

**Remark 4.9.** For each \( l \in \{1, 2, \ldots, k - 1\} \), the local homomorphism \( H = (y_1, y_2, \ldots, y_l, y_{l+1}, \ldots, y_k) : \mathcal{C}_k \to \mathcal{C}_k \) is a finite compositions of \( e \) and permutations.

## 5. Eakin-Harris property for differentiable Noetherian system

We prove in this section a version of a theorem proved by Eakin and Harris \[5\]. This result is about local homomorphisms of rings in a given well-behaved differentiable Noetherian system. The proof is inspired by the proof of a similar result in the setting of an analytic system, proved by Eakin and Harris \[5\].

We put \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_k) \). If \( \mathcal{C} = \{\mathcal{C}_n, n \in \mathbb{N}\} \) is a quasianalytic differentiable system, let \( \Phi : \mathcal{C}_n \to \mathcal{C}_k \) be a local homomorphism.

**Definition 5.1.** An admissible modification of \( \Phi \) is a homomorphism \( \hat{\Phi} \) related to \( \Phi \) in one of the following ways:

(i) There is an isomorphism \( \Gamma : \mathcal{C}_n \to \mathcal{C}_n \) such that \( \Phi \circ \Gamma = \hat{\Phi} \).

(ii) There is a homomorphism \( \Psi : \mathcal{C}_k \to \mathcal{C}_k \) with \( rk(\Psi) = k \) such that \( \Psi \circ \Phi = \hat{\Phi} \).

(iii) There is a strongly injective homomorphism \( \omega : \mathcal{C}_n \to \mathcal{C}_n \) with \( rk(\omega) = n \) such that \( \Phi = \Phi \circ \omega \).

**Remark 5.2.** Let \( \hat{\Phi} \) be an admissible modification of \( \Phi \). Then

(i) \( rk(\hat{\Phi}) = rk(\Phi) \).

(ii) \( \Phi \) strongly injective \( \Rightarrow \) \( \Phi \) strongly injective.

From now on, we suppose that the system \( \mathcal{C} = \{\mathcal{C}_n, n \in \mathbb{N}\} \) is a well-behaved Noetherian differentiable system. Let \( \Phi : \mathcal{C}_n \to \mathcal{C}_k \), \( \Phi = (\varphi_1, \ldots, \varphi_n) \), be a local homomorphism. We suppose \( \varphi_1 \neq 0 \).

**Lemma 5.3.** There exist \( d \in \mathbb{N}^* \) and a homomorphism \( H : \mathcal{C}_k \to \mathcal{C}_k \) such that

\[
H(\varphi_1) = c_1y_1^d + c_2y_2^d + \ldots + c_ky_k^d + \psi
\]

where \( c_1, c_2, \ldots, c_k \in \mathbb{R}, c_1 \neq 0 \) and \( \psi \in \mathcal{C}_k \) such that \( \hat{\psi} = \{\text{mixed terms of degree } d\} + \sum_{|\omega| > d} a_\omega y_1^{\omega_1} \ldots y_k^{\omega_k} \).
Proof. \( \hat{\varphi}_1 = \sum_{\nu=0}^{\infty} P_{\nu}(y_1,\ldots,y_k) \), where \( P_{\nu}(y_1,\ldots,y_k) \) is a homogeneous polynomial of degree \( \nu, \nu = 0, 1, \ldots \). Let \( d \in \mathbb{N} \) be the least integer such that \( P_d \neq 0 \). We have then

\[
\hat{\varphi}_1 = P_d(y_1,\ldots,y_k) + \sum_{\nu > d} P_{\nu}(y_1,\ldots,y_k).
\]

Let \( (\lambda_{ij})_{i,j=1}^{k} \) be a nonsingular matrix of scalars such that \( P_d(\lambda_{11},\ldots,\lambda_{k1}) \neq 0 \). We define an isomorphism \( H : \mathcal{C}_k \to \mathcal{C}_k \) by

\[
H = \left( \sum_{l=1}^{k} \lambda_{1l}y_l, \sum_{l=2}^{k} \lambda_{2l}y_l, \ldots, \sum_{l=1}^{k} \lambda_{kl}y_l \right).
\]

We put \( \hat{\varphi}_1 := H(\varphi_1) \in \mathcal{C}_k \). We see then that

\[
\hat{\varphi}_1 = c_1 y_1^d + c_2 y_2^d + \ldots + c_k y_k^d + \{\text{mixed terms of degree } d\} + \sum_{|\omega| > d} b_{\omega} y_1^{\omega_1} \ldots y_k^{\omega_k},
\]

where \( c_1 = P_d(\lambda_{11},\ldots,\lambda_{k1}), c_2 = P_d(\lambda_{12},\ldots,\lambda_{k2}), \ldots, c_k = P_d(\lambda_{1k},\ldots,\lambda_{kk}) \).

If we put \( \gamma := \sum_{|\omega| > d} b_{\omega} y_1^{\omega_1} \ldots y_k^{\omega_k} \), we see that \( \gamma \) is a formal solution of the equation

\[
Z + c_1 y_1^d + c_2 y_2^d + \ldots + c_k y_k^d + \{\text{mixed terms of degree } d\} - \hat{\varphi}_1 = 0.
\]

By the Artin Approximation Theorem, if \( \nu \in \mathbb{N}, \nu > d \), there exists \( \psi \in \mathcal{C}_k \) that is a solution of the equation (2) such that \( \hat{\psi} - \gamma \in m^\nu \mathcal{R}[[x_1,\ldots,x_n]] \), which proves the lemma. \( \square \)

We consider \( H_1 = (y_1, y_1 y_2,\ldots,y_1 y_k) : \mathcal{C}_k \to \mathcal{C}_k \). We have

\[
H_1 H(\varphi_1)(y_1,\ldots,y_k) = y_1^d \psi_1,
\]

for some unit \( \psi_1 \in \mathcal{C}_k \).

Since \( \psi_1 \) is a unit, there is \( \psi_2 \in \mathcal{C}_k \) such that \( \psi_2^d \psi_1 = 1 \). Now we define \( H_2 := (y_1 \psi_2, y_2,\ldots,y_k) : \mathcal{C}_k \to \mathcal{C}_k \). We have \( H_2 H_1 H(\varphi_1)(y_1,\ldots,y_k) = y_1^d \).

Remark 5.4. We remark that the homomorphism

\[
(y_1^d, H_2 H_1 H(\varphi_1),\ldots,H_2 H_1 H(\varphi_n))
\]

is an admissible modification of \( \Phi \). Since our system is well behaved, we notice, also, that the homomorphism \( (y_1, H_2 H_1 H(\varphi_1),\ldots,H_2 H_1 H(\varphi_n)) \) is an admissible modification of \( (y_1^d, H_2 H_1 H(\varphi_1),\ldots,H_2 H_1 H(\varphi_n)) \).

Theorem 5.5. If \( rk(\Phi) = s \), then there is a finite sequence \( \Phi_1,\ldots,\Phi_l \) of homomorphisms \( \Phi_1 : \mathcal{C}_n \to \mathcal{C}_k \) such that \( \Phi_1 = \Phi, \Phi_{i+1} \) is related to \( \Phi_i \) by an admissible modification, \( i = 1,\ldots,l-1 \), and \( \Phi_l = (y_1,\ldots,y_s,0,0,\ldots,0) \).

Proof. By the above remark, we can suppose that \( \Phi = (y_1, \varphi_2,\ldots,\varphi_n) \). Assume we have found a sequence \( (\Phi_i)_{i=1}^{N} \) of admissible modifications such that \( \Phi_N = (y_1, y_2,\ldots,y_j, \varphi_{j+1},\ldots,\varphi_n) \) for some integer \( j \). Notice \( j \leq s \), since \( rk(\Phi_N) = s \). We consider two cases:

Case \( j < s \). We may assume that one of the germs, \( \varphi_{j+1},\ldots,\varphi_k \), depends on at least one of the remaining variables \( y_{j+1},\ldots,y_k \). If not, we consider the homomorphism \( \Gamma = (x_1,\ldots,x_j, x_{j+1} - \varphi_{j+1}, x_{j+2},\ldots,x_n) \). We see then that \( \Phi_N \circ \Gamma = (y_1,\ldots,y_j,0,\varphi_{j+2},\ldots,\varphi_n) \) is an admissible modification of \( \Phi_N \). We continue
with \( \varphi_{j+2} \). At the end we find an admissible modification of \( \Phi_N \) of the form 
\[
y_1, \ldots, y_j, 0, \ldots, 0, \]
which is a contradiction since \( rk(\Phi_N) = s \).

We can then suppose that \( \varphi_{j+1} \) depends on at least one of the variables \( y_{j+1}, \ldots, y_k \). We have

\[
\hat{\varphi}_{j+1} = P_d(y_{j+1}, \ldots, y_k) + \psi,
\]
where \( P_d \in \mathcal{C}_j[y_{j+1}, \ldots, y_k] \) is a homogeneous polynomial of degree \( d \), \( \psi \in (y_{j+1}, \ldots, y_k)^{d+1} \mathbb{R}[[y_1, \ldots, y_k]] \). We have then \( \hat{\psi} = \sum_{l=1}^q \lambda_l \hat{\beta}_l \), where \( \lambda_l \in (y_{j+1}, \ldots, y_k)^{d+1} \in \mathbb{R}[y_1, \ldots, y_k] \) and \( \hat{\beta}_l \in \mathbb{R}[[y_1, \ldots, y_k]] \). We put

\[
E(Z, Z_1, \ldots, Z_q) = (\varphi_{j+1} - P_d(y_{j+1}, \ldots, y_k) - Z - \sum_{l=1}^q \lambda_l Z_l),
\]

and \( E(Z, Z_1, \ldots, Z_q) \) is a formal solution of the equation

\[
E(Z, Z_1, \ldots, Z_q) = 0.
\]

By the Artin Approximation Theorem, there exists \((\psi, \beta_1, \ldots, \beta_q) \in (\mathcal{C}_k)^{q+1}\) that is a solution of the equation \( E(Z, Z_1, \ldots, Z_q) = 0 \). We have then that

\[
\varphi_{j+1} = P_d(y_{j+1}, \ldots, y_k) + \psi,
\]
with \( \psi \in (y_{j+1}, \ldots, y_k)^{d+1} \mathcal{C}_k \).

Let \((\lambda_{i,\nu})_{i,\nu=1}^k\) be a nonsingular matrix of scalars and define the homomorphism \( H : \mathcal{C}_k \to \mathcal{C}_k \) by

\[
H = (y_1, \ldots, y_j, \sum_{\nu=j+1}^k \lambda_{i,\nu} y_\nu, \ldots, \sum_{\nu=j+1}^k \lambda_{k,\nu} y_\nu).
\]

We see then that

\[
H(\varphi_{j+1}) = c_{j+1} y_{j+1}^d + \ldots + c_k y_k^d + \{\text{mixed terms of degree } d\} + H(\psi),
\]
where

\[
c_l = P_d(\lambda_{l,j+1}, \lambda_{l,j+2}, \ldots, \lambda_{l,k}), l = j+1, \ldots, k.
\]

We put

\[
Q = c_{j+2} y_{j+2}^d + \ldots + c_k y_k^d + \{\text{mixed terms of degree } d\}.
\]

Let us note that each term of \( Q \) is divisible by at least one of \( y_{j+2}, \ldots, y_k \).

We may choose the matrix \((\lambda_{i,\nu})_{i,\nu=1}^k\) so that \( P_d(\lambda_{j+1,j+1}, \lambda_{j+1,j+2}, \ldots, \lambda_{j+1,k}) \neq 0 \).

Define \( G : \mathcal{C}_k \to \mathcal{C}_k \) by

\[
G = (y_1, \ldots, y_{j+1}, y_{j+1} y_{j+2}, \ldots, y_{j+1} y_k).
\]

We see that \( y_{j+1}^d \) divides \( \Phi \circ H(\varphi_{j+1}) \). Since \( \Phi \circ H(\Phi_N) \) is an admissible modification of \( \Phi_N \), we can suppose that \( y_{j+1}^d \) divides \( \varphi_{j+1} \) and \( \varphi_{j+1} \) is still of the form (3).

On the other hand, by Proposition 3.2, there exists \( H_1 : \mathcal{C}_k \to \mathcal{C}_k \),

\[
H_1 = (y_1, y_1^{e_{1,1}} y_2^{e_{1,2}} y_3^{e_{1,3}}, \ldots, y_1^{e_{1,k}} y_2^{e_{2,k}} y_3^{e_{3,k}} \ldots y_{k-1}^{e_{k-1,k}} y_k),
\]

such that \( H_1(\varphi_{j+1}) = (\text{monomial}) \cdot Q_1 \) for some unit \( Q_1 \in \mathcal{C}_k \).

From (3), we have

\[
H_1(\varphi_{j+1}) = H_1(c_{j+1})(y_1^{e_{1,j+1}} y_2^{e_{2,j+1}} \ldots y_{j+1}^{e_{j+1,j+1}})^d + H_1(Q_1) + H_1(\psi).
\]
$H_1(c_{j+1}) \in \mathcal{C}_j$, $H_1(c_{j+1}) \neq 0$, and each term of $H_1(Q_1)$ is divisible by at least one of $y_{j+2}, \ldots, y_k$ and $H_1(\psi) \in (y_{j+1}, \ldots, y_k)^{d+1}\mathcal{C}_k$.

By (4), we see then that none of $y_{j+2}, \ldots, y_k$ divides $H_1(\varphi_{j+1})$. On the other hand, we know that $y^d_j$ divides $\varphi_{j+1}$, and hence $y^d_{j+1}$ divides $H_1(\varphi_{j+1})$. But we have $H_1(\varphi_{j+1}) = (\text{monomial}) \cdot Q_1$; hence

$$H_1(\varphi_{j+1}) = (y^d_{j+1} y^2_{j+1} \cdots y^d_{j+1,j+1} Q_1, \varphi_{j+2}, \ldots, \varphi_n).$$

Thus we have modified $\Phi_N$ to an $H_1(\Phi_N)$ of the form

$$H_1(\Phi_N) = (y_1, y^d_1, y^2_2, \ldots, y^d_{j+1}, \varphi_{j+1}, \varphi_{j+2} \cdots, \varphi_n).$$

Now we may absorb the unit $Q_1$ into $y_{j+1}$ as above. By successive applications of morphisms $e$ and $r_d$ we have modified $\Phi_N$ to the form

$$\Phi_N = (y_1, \ldots, y_{j+1}, \varphi_{j+2}, \varphi_n).$$

Case $j = s$. If $\Phi_N = (y_1, \ldots, y_s, \varphi_{s+1}, \ldots, \varphi_n)$, then for each $l = s + 1, \ldots, n$, $\varphi_l$ is independent of all $y_{s+1}, \ldots, y_k$. Otherwise we could modify $\Phi_N$ by the above procedure to $\Phi_N = (y_1, \ldots, y_s, y_{s+1}, \varphi_{s+2}, \ldots, \varphi_n)$, but $rk(\Phi_N) = s + 1 > s$.

Define $\Gamma : \mathcal{C}_n \to \mathcal{C}_n$ by

$$\Gamma = (x_1, \ldots, x_s, x_{s+1} - \varphi_{s+1}(x_1, \ldots, x_s), \ldots, x_n - \varphi_n(x_1, \ldots, x_s)).$$

We see then that $\Phi_n \circ \Gamma$ is defined by $(y_1, \ldots, y_s, 0, \ldots, 0)$, and hence the theorem is proved.

**Corollary 5.6.** Let $\Phi : \mathcal{C}_n \to \mathcal{C}_k$ be a local homomorphism. Suppose that $rk(\Phi) = n$. Then $\Phi$ is strongly injective.

**Proof.** By the above theorem we can modify $\Phi$ by successive admissible modifications to the form $(y_1, y_2, \ldots, y_n)$, which is trivially strongly injective. □

**Corollary 5.7.** The Weierstrass Division Theorem holds in the well-behaved Noetherian differentiable system $\mathcal{C} = \{\mathcal{C}_n, n \in \mathbb{N}\}$.

**Proof.** First, every Noetherian differentiable system is a quasianalytic differentiable system. By Corollary 5.6, every local homomorphism $\Phi : \mathcal{C}_n \to \mathcal{C}_k$ with $rk(\Phi) = n$ is strongly injective, hence the result by Theorem 4.1 of [7]. □

**Corollary 5.8.** Every well-behaved Noetherian differentiable system is continued in the analytic system.

**Proof.** By Corollary 5.7, the Weierstrass Division Theorem holds in this system. We deduce the result by [6]. □

### References


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