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NORMAL SUBGROUPS AND CLASS SIZES OF ELEMENTS OF PRIME POWER ORDER

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Abstract. If G is a finite group and N is a normal subgroup of G with two G-conjugacy class sizes of elements of prime power order, then we show that N is nilpotent.

1. Introduction

Let G be a finite group. Several researches have put forward that there exists a strong relation between the structure of a normal subgroup N of G and the sizes of the G-classes of elements in N, that is, the sizes of the conjugacy classes of G contained in N (see [1], [2]). The main theorem of [2], which uses the classification of nonsolvable CP-groups due to H. Heineken, establishes that a normal subgroup of G having exactly two G-class sizes is either abelian or is the direct product of a p-group by a central subgroup of G. This is an extension of the celebrated result of N. Itô, which asserts that if G has only two class sizes, 1 and m, then $m = p^a$ for some prime p and G is a direct product of a p-group with an abelian group.

In 1996, Li Shirong showed that if G has exactly two class sizes of elements of prime-power order, then G is solvable ([7]). It is remarkable how difficulties arise when one considers only the class sizes of prime-power order elements; Itô's result is quite elementary, while Li Shirong needs to appeal to the Classification of the Finite Simple Groups.

In this paper, our research goes further by investigating the influence of the G-class sizes of the elements of prime-power order of a normal subgroup on its structure by showing the following generalization of the main theorems of [2] and [7].

Theorem A. *Let G be a finite group and N a normal subgroup of G. Suppose that the G-class size of every element of prime-power order of N is 1 or m. Then N is nilpotent.*

We point out that the hypotheses of this theorem do not imply that the G-class sizes of all elements of N are 1 or m, and, in fact, the only information we can get on the G-class sizes of N is that all of them are 1 or divisible by m. In the particular case in which $N = G$, we want to remark that the approach in [7] (to prove that G is solvable) employs Feit-Thompson's Theorem so as to show that the Sylow 2-subgroups of $G/\mathbf{Z}(G)$ are elementary abelian. Then he finds a suitable quasisimple subnormal subgroup of G and makes a case-by-case analysis for the

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quasisimple groups having elementary abelian Sylow 2-subgroups in order to get a contradiction. Our approach in the proof of Theorem A, which is different and simpler, allows us to get more information on the structure of the normal subgroup. Furthermore, we will show with examples that m in Theorem A need not be a prime power as is the case of Itô's Theorem, and also that N may have all its Sylow subgroups nonabelian, differing from Itô's result and from the main theorem of [2] as well. However, in the particular case in which $N = G$, these two properties do not hold, and, in fact, we gain the following improvement of Shirong's result.

Corollary B. *Let G be a finite group and suppose that the class size of every element of prime-power order of G is 1 or m. Then G is nilpotent. More precisely, $m = p^n$ for some prime p, and $G = P \times A$ with A abelian and P a p-group.*

All groups are supposed to be finite.

2. Proofs

In order to prove our results we need two applications of the Classification of the Finite Simple Groups.

Theorem 1. *Let G be a transitive permutation group on a set Ω with $|\Omega| > 1$. Then there exist a prime p and an element $x \in G$ of order a power of p such that x acts without fixed points on Ω.*

Proof. This appears in [3]. □

Theorem 2. *If G is a nonabelian simple group, then there exists some prime dividing $|G|$ that does not divide the order of its Schur multiplier.*

Proof. We can examine (for instance in Chapter 5 of [5]) that the Schur multiplier of any simple group is a $\{2, 3\}$-group (including of course the trivial group) except at most for the groups $A_{n-1}(q)$ and ${}^2A_{n-1}(q)$, whose Schur multipliers have orders $(n, q-1)$ and $(n, q+1)$, respectively. However, the orders of $A_{n-1}(q)$ and ${}^2A_{n-1}(q)$ are always divisible by q. This shows that for every nonabelian simple group there always exists a prime satisfying the thesis of the theorem. □

The following property is elementary but useful for our purposes when dealing with the centralizers of elements of prime-power order.

Lemma 3. *Let G be a finite group. If the order of $x\mathbf{Z}(G) \in G/\mathbf{Z}(G)$ is a power of a prime p, then there exists some integer n such that the order of x^n in G is a power of p and $\mathbf{C}_G(x) = \mathbf{C}_G(x^n)$.*

Proof. Suppose that $o(x\mathbf{Z}(G)) = p^a$. Then we can write $x^{p^a} = z = z_p z_{p'}$, where z_p and $z_{p'}$ are the p and the p'-part of $z \in \mathbf{Z}(G)$. Assume that $o(z_p) = p^b$ and that $o(z_{p'}) = s$, where $(s, p) = 1$. Then there exist certain integers α, β such that $\alpha p^{a+b} + \beta s = 1$, and this implies that

$$(x^{p^{a+b}})^\alpha x^{s\beta} = x.$$

Notice that $x^{p^{a+b}} \in \mathbf{Z}(G)$, whence $\mathbf{C}_G(x) = \mathbf{C}_G(x^{s\beta})$. Moreover, since $x^{p^{a+b}s} = 1$, then $o(x^s)$ divides p^{a+b}, so in particular the order of $x^{s\beta}$ is a p-power too, and this is the required element of G. $\square$

Proof of Theorem A. We can assume that N is not nilpotent, so N is nontrivial, and we let N/K be a chief factor of G. Working by induction on $|N|$, we can assume that every normal subgroup of G properly contained in N is nilpotent and in particular K is nilpotent. Thus $K = \mathbf{F}(N)$ and in particular $\mathbf{Z}(N) \subseteq K$.

Assume first that $\mathbf{Z}(N)$ is properly contained in K. Then, there exists some prime p such that $\mathbf{Z}(N)_p < \mathbf{O}_p(K) = \mathbf{O}_p(N)$. Let us choose any prime $q \neq p$ dividing $|N|$ and take x to be a q-element of N such that $x \notin \mathbf{Z}(G)$. Such an element must exist; otherwise N has a central Sylow q-subgroup, so $N = N_q \times N_{q'}$ and N would be nilpotent by induction. We take P to be a Sylow p-subgroup of $\mathbf{C}_G(x)$ and consider the action of $P \times \langle x \rangle$ on $P_0 := \mathbf{O}_p(N)$. We claim that $\mathbf{C}_{P_0}(P) \subseteq \mathbf{C}_{P_0}(x)$. In fact, if $z \in \mathbf{C}_{P_0}(P)$ is noncentral in G, then $\langle P, z \rangle \leq \mathbf{C}_G(z) < G$. However, by hypothesis, $|\mathbf{C}_G(z)|_p = |\mathbf{C}_G(x)|_p = |P|$, so, in particular, $z \in P \cap P_0 \subseteq \mathbf{C}_{P_0}(x)$ as claimed. Then we can apply Thompson's $P \times Q$-Lemma (for instance 8.2.8 of [6]) to get that $x \in \mathbf{C}_N(P_0)$ and thus, we have shown that every Sylow q-subgroup of N lies in $\mathbf{C}_N(\mathbf{O}_p(N))$ for every prime $q \neq p$. This means that $|N : \mathbf{C}_N(\mathbf{O}_p(N)|$ is a p-number. Moreover, by induction, $\mathbf{C}_N(\mathbf{O}_p(N))$ is nilpotent, whence it is contained in K and thus N/K is a p-group. Then, for every prime $q \neq p$, we have $\mathbf{Z}(N)_q = \mathbf{O}_q(N)$; otherwise, we argue with q as with p, and we would get that N/K is a q-group, a contradiction. Therefore, $K = \mathbf{Z}(N)_{p'} \times \mathbf{O}_p(N)$, and since we have proved that N/K is a p-group, it follows that N is nilpotent, a contradiction.

Therefore, from now on we can assume that $\mathbf{Z}(N) = K$. Since N/K is a direct product of isomorphic simple groups and N/K cannot be abelian, then $N/\mathbf{Z}(N) = L_1/\mathbf{Z}(N) \times \ldots \times L_k/\mathbf{Z}(N)$, where the groups $L_i/\mathbf{Z}(N)$ are nonabelian simple and isomorphic. Furthermore, we know that the subgroups $L_i/\mathbf{Z}(N)$ are the only minimal normal subgroups of $N/\mathbf{Z}(N)$ and that the L_i's are G-conjugate.

Let L/K be one of the simple direct factors of N/K. Then $L''K = L$, so L/L'' is abelian, and hence L' is perfect. As $L/K \cong L'/(L' \cap K)$ is simple, then $L' \cap K$ is a homomorphic image of the Schur multiplier of this simple group. By Theorem 2, there is some prime divisor p of $|L/K|$ that does not divide $|L' \cap K|$. It follows that the group L' has no composition factor of order p. Since the G-conjugates of L' normalize each other, we have that their product also has no composition factor of order p. This product, however, is normal in G, contained in N and not nilpotent, and thus it must be equal to N. Then N has no composition factor of order p, and consequently p does not divide $|K|$.

Notice that the G-class size of every element in $N \setminus K$ is divisible by m, so we have $|N| = |K| + mt$ for some integer t. But, since p divides $|N|$ and does not divide $|K|$, we deduce that p does not divide m. Now, observe that $\mathbf{Z}(L) = K$. The simple group $L/\mathbf{Z}(L)$ acts transitively by conjugacy as a permutation group on its set of Sylow p-subgroups, and there are more than one of these. Then, by Theorem 1, there exists an element $\bar{x} = x\mathbf{Z}(L)$ of prime-power order acting without fixed points on the Sylow p-subgroups of $L/\mathbf{Z}(L)$. This implies that x does not normalize any Sylow p-subgroup P of L, and thus, in particular, $P \not\subseteq \mathbf{C}_L(x)$ for any such subgroup P. Hence, p divides $|L : \mathbf{C}_L(x)|$. Now, by Lemma 3, we may replace x by an appropriate power of x, say x^n, such that $o(x^n)$ is a prime power

and such that $\mathbf{C}_L(x) = \mathbf{C}_L(x^n)$. But L is subnormal in G, and hence $|L : \mathbf{C}_L(x)|$ divides $|G : \mathbf{C}_G(x)| = m$, which is a contradiction. □

Proof of Corollary B. By Theorem A, we have that G is nilpotent. Then, it is clear that the class size of every p-element of G is a p-power for every prime p. It follows that $m = p^n$ for a certain prime p and that all except one Sylow subgroup of G are abelian; the corollary is proved. □

Examples. As we have pointed out in the introduction, we are going to show that m appearing in Theorem A need not be in general a prime power, as occurs when $N = G$. Let

$$L = \langle x, y | x^3 = y^3 = 1, [x, y]^3 = 1, [x, [x, y]] = [y, [x, y]] = 1\rangle$$

be the extraspecial group of order 3^3 and exponent 3. Write $z = [x, y]$, so we have $\mathbf{Z}(L) = \langle z\rangle$. Let $\langle a\rangle$ be the automorphism of L defined by $x^a = x^2$ and $y^a = y^2$. Notice that the set of fixed points of a on L is exactly $\mathbf{Z}(L)$. On the other hand, let us consider an automorphism α of order 3 acting nontrivially on the quaternion group Q of order 8. Observe that α exactly fixes the elements in $\mathbf{Z}(Q)$. We form the group $G := Q\langle\alpha\rangle \times L\langle a\rangle$ and take the normal subgroup $N = Q \times L$. One can easily check that the G-class size of every element of prime-power order of N is exactly 1 or 6. However, we remark that the G-class sizes of all elements in N are $\{1, 6, 36\}$. Also, we want to notice that no Sylow subgroup of N is abelian, as happens in Itô's theorem.

Other easy examples can be constructed as follows. Let K be an abelian p'-group which admits a fixed point free automorphism α of order p for some prime p, and choose P to be a p-group such that its class sizes are $\mathrm{cs}(P)=\{1, p\}$ (this is equivalent to $|P'| = p$). We take $G := P \times K\langle\alpha\rangle$ and $N = P \times K$. Then every element of prime-power order of N has a G-class of size 1 or p, whilst the set of G-class sizes of all elements in N is $\{1, p, p^2\}$.

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