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CONES AND CONVEX BODIES WITH MODULAR FACE LATTICES

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Dedicated to Claus M. Ringel on the occasion of his 65th birthday

Abstract. If a convex body C in $\mathbb{R}^n$ has modular and irreducible face lattice and C is not strictly convex, there is a face-preserving homeomorphism from C to a set of positive-semidefinite Hermitian matrices of trace 1 over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, or C has dimension 8, 14 or 26.

1. Introduction

Let C be a convex body in $\mathbb{R}^n$. A subset F of C is a *face* of C if every open interval in C that contains a point of F is contained in F. An *extreme point* is a 1-point face. If S is any subset of C, the *face generated by* S is the minimal face of C containing S. The set $\mathcal{F}(C)$ of all faces of C ordered by inclusion is a lattice, which is always atomic, algebraic and complemented, as each face is generated by a finite number of extreme points and for every face F there exists a face G such that $F \wedge G = \emptyset$ and $F \vee G = C$. We want to consider convex bodies for which the lattice $\mathcal{F}(C)$ is modular. An atomic lattice is modular if and only if it is ranked (the lengths of all maximal chains between 2 elements are the same) and the rank (the length of a chain between 0 and x) satisfies $rk(x) + rk(y) = rk(x \vee y) + rk(x \wedge y)$ for all x and y [4]. The only examples of convex bodies with modular face lattice and a finite number of faces are the simplices [2], but beautiful examples with an infinite number of faces are given by the sets of positive-semidefinite Hermitian matrices of trace 1 over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ [3]. The main purpose of this paper is to show that in dimensions other than 8, 14 and 26, all convex bodies with modular and irreducible face lattices are deformations of these examples, in the sense that there is a face-preserving homeomorphism between them. The natural correspondence between convex bodies in $\mathbb{R}^n$ and closed cones in $\mathbb{R}^{n+1}$ gives analogous results for cones.

The paper is organized as follows. In section 2 we show that each convex body C with modular face lattice is a convex join of convex bodies whose face lattices are modular and irreducible. It is known that modular and irreducible lattices correspond to abstract projective spaces. We review their basic properties and describe the examples of convex bodies whose face lattices determine the classical projective spaces over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and an octonionic projective plane.

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In section 3 we prove that if the face lattice of a convex body C determines a projective space, it must be a topological projective space that is compact and connected. This is used to prove that, in most cases, the set of extreme points of C determines a classical projective space over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. In the remaining cases (corresponding to projective planes) the dimension of C is computed to be 5, 8, 14 or 26. We then show that an isomorphism between projective spaces formed by extreme points gives rise to a homeomorphism between the convex bodies.

In section 4 we consider convex bodies in $\mathbb{R}^n$ whose face lattices determine projective planes and show that the set of extreme points in a projective line is semi-algebraic in $\mathbb{R}^n$ and algebraic in case $n = 5$. This is used to prove that in $\mathbb{R}^5$ all convex bodies with modular and irreducible face lattice are projectively equivalent to the set of 3×3 real symmetric positive-semidefinite matrices of trace 1.

Finally, in section 5, we consider convex sets in $\mathbb{R}^n$ whose face lattices determine abstract affine spaces, and we prove a result analogous to the main theorem.

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2. Modular face lattices

If $\mathcal{L}_1$ and $\mathcal{L}_2$ are lattices, their *direct product* is given by $(\mathcal{L}_1 \times \mathcal{L}_2, \leq)$, where $(a, b) \leq (c, d)$ if and only if $a \leq c$ and $b \leq d$. It follows that the direct product of two lattices is modular if and only if the factors are modular. A lattice is called *irreducible* if it is not isomorphic to a direct product of two nontrivial lattices.

If $C_1 \subset \mathbb{R}^m$ and $C_2 \subset \mathbb{R}^n$ are convex bodies, define $C_1 * C_2 \subset \mathbb{R}^{m+n+1}$ as the convex hull of the union of a copy of C_1 and a copy of C_2 placed in such a way that their affine spans are disjoint and have no common directions. So $C_1 * C_2$ is well defined up to a linear transformation: it is the convex join of C_1 and C_2 of largest possible dimension. For example $C * \{pt\}$ is a pyramid with base C. Let us say that a convex body C is $*$-*decomposable* if $C = C_1 * C_2$ for two convex bodies C_1 and C_2.

The following lemma is equivalent to a result proved by Barker [2] for cones.

Lemma 1. *A convex body C is $*$-decomposable if and only if its lattice of faces $\mathcal{L}(C)$ is reducible.*

Proof. Let $C = C_1 * C_2$. Observe that each point x of $C_1 * C_2$ with $x \notin C_i$ lies in a unique line segment joining a point x_1 of C_1 and a point x_2: if x lies in x_1x_2 and $x_1'x_2'$, then the lines x_1x_1' and x_2x_2' are parallel or they intersect, contradicting the assumptions on the spans of C_1 and C_2. Moreover, if x moves along a straight line in $C_1 * C_2$, then the corresponding points x_1 and x_2 move along straight lines in C_1 and C_2: if p and q are points in C and $x \in pq$, then $x = tp + (1-t)q = t\lambda p_1 + t(1-\lambda)p_2 + (1-t)\mu q_1 + (1-t)(1-\mu)q_2$, which can be rewritten as a linear combination $\alpha x_1 + \beta x_2$ of the point $x_1 = \frac{t\lambda}{t\lambda+(1-t)\mu}p_1 + \frac{(1-t)\mu}{t\lambda+(1-t)\mu}q_1$ in p_1q_1 and the point $x_2 = \frac{t(1-\lambda)}{t(1-\lambda)+(1-t)(1-\mu)}p_2 + \frac{(1-t)(1-\mu)}{t(1-\lambda)+(1-t)(1-\mu)}q_2$ in p_2q_2, with $\alpha + \beta = 1$, so x lies in the line segment joining x_1 and x_2.

Now if C_i' is a face of C_i, then $C_1' * C_2'$ is a face of $C_1 * C_2$. For, if $x \in C_1' * C_2'$ and $x = \lambda p + (1-\lambda)q$ with $p, q \in C_1 * C_2$, then x_1 lies in p_1q_1 and x_2 lies in p_2q_2 so as C_i' is a face of C_i, p_i and q_i lie in C_i'; therefore p and q lie in $C_1' * C_2'$. Conversely, if C' is a face of $C_1 * C_2$ and $p \in C'$, then p_1 and p_2 lie in C', so $C' = (C' \cap C_1) * (C' \cap C_2)$. It remains to show that $C' \cap C_i$ is a face of C_i. If $x \in C' \cap C_1$ and $x = \lambda p + (1-\lambda)q$

with $p, q \in C_1 * C_2$, then as C' and $C_1 = C_1 * \emptyset$ are faces of $C_1 * C_2$, p and q lie in C' and also in C_1, so $C' \cap C_1$ is a face of C_1. Similarly $C' \cap C_2$ is a face of C_2. So $\mathcal{L}(C_1 * C_2) \simeq \mathcal{L}(C_1) \times \mathcal{L}(C_2)$.

If $\mathcal{L}(C) \approx \mathcal{L}_1 \times \mathcal{L}_2$, then $\mathcal{L}_1$ and $\mathcal{L}_2$ are isomorphic to sublattices of $\mathcal{L}(C)$, so $\mathcal{L}_i \approx \mathcal{L}(C_i)$ for two faces of C with $C_1 \wedge C_2 = \varnothing$ and $C_1 \vee C_2 = C$. To show that $C = C_1 * C_2$ we need to prove that $span(C_1)$ and $span(C_2)$ are disjoint and have no directions in common. Suppose that $x \in span(C_1) \cap span(C_2)$. Take $x_i \in Int(C_i)$. Then the line through x and x_i meets ∂C_i at two points a_i and b_i. As a_2 lies in a proper subface C_2' of C_2, the face generated by C_1 and a_2 lies in $C_1 \vee C_2'$, which is a proper subface of $C_1 \vee C_2$. But the points a_1, b_1, a_2, b_2 determine a plane quadrilateral whose side $a_i b_i$ lies in the interior of C_i, so its diagonals intersect at an interior point c of $C_1 \vee C_2$, so the face generated by C_1 and a_2 (which contains c) must be $C_1 \vee C_2$, a contradiction. Now suppose that $span(C_1)$ and $span(C_2)$ have a common direction v. Take $x_i \in Int(C_i)$. Then the line through x_i in the direction v meets ∂C_i at two points a_i and b. As before a_1, b_1, a_2, b_2 determine a plane quadrilateral whose diagonals intersect at an interior point c of $C_1 \vee C_2$, but c lies in the face generated by C_1 and a_2, which is a proper face of $C_1 \vee C_2$. $\square$

We recall now the relation between modular, irreducible lattices and projective spaces. A *projective space* consists of a set P (the points) and a set L (the lines) so that

(1) There are at least two lines; each line contains more than 2 points.
(2) Each pair of points is contained in a unique line.
(3) If a, b, c, d are distinct points and the lines ab and cd intersect, so do the lines ac and bd.

One also requires that all chains of subspaces (points, lines and projective spaces) ordered by inclusion have finite length. The maximum length of a chain starting with a point and ending with the projective space is its *rank*. The classic examples of projective spaces of rank n are provided by the set of linear subspaces of a vector space $\mathbb{K}^{n+1}$, with points corresponding to the 1-dimensional subspaces and lines to the 2-dimensional subspaces. Two projective spaces are *isomorphic* if there is a bijective map between their sets of points that preserves the lines.

It follows from the axioms that the lattice of subspaces of a projective space is atomic, algebraic, modular and irreducible. Conversely, any lattice with these properties is the lattice of subspaces of a projective space whose points are the atoms, and the lines are the joins of 2 atoms [6]. It is a classic result of Hilbert [9] that a projective space in which Desargues theorem holds is isomorphic to the projective space $\mathbb{A}\mathbb{P}^n$ determined by the linear subspaces of $\mathbb{A}^{n+1}$, for some division ring $\mathbb{A}$, and that $\mathbb{A}\mathbb{P}^n$ and $\mathbb{B}\mathbb{P}^m$ are isomorphic if and only if $\mathbb{A}$ and $\mathbb{B}$ are isomorphic and $m = n$. All projective spaces of rank larger than 2 are Desarguesian, but there are many non-Desarguesian projective planes.

Examples of convex bodies whose face lattices determine the projective spaces $\mathbb{R}\mathbb{P}^n$, $\mathbb{C}\mathbb{P}^n$, $\mathbb{H}\mathbb{P}^n$ and an octonionic projective plane arise from linear algebra, as cross-sections of some cones of matrices:

Lemma 2. *The set $C_n(\mathbb{F})$ of $n \times n$ positive-semidefinite Hermitian matrices with coefficients in $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ is a real cone whose face lattice is isomorphic to the lattice of linear subspaces of $\mathbb{F}^n$. Each cross-section of $C_n(\mathbb{F})$ is a convex body* *whose set of extreme points is homeomorphic to* $\mathbb{FP}^{n-1}$ *by a homeomorphism that identifies faces with subspaces.*

Proof. Recall that A is Hermitian if $\overline{A} = A^T$ and is positive-semidefinite if $\overline{v}Av^T \geq 0$ for all $v \in \mathbb{F}^n$. Consider the function from $C_n(\mathbb{F})$ to the set of subspaces of $\mathbb{F}^n$ that sends each matrix A to the orthogonal complement of its kernel, $(\ker A)^\perp$. This function is certainly surjective and not injective; however, for matrices of rank 1 only the scalar multiples of the same matrix have the same kernel, so there is a natural continuous bijection between matrices of rank 1 in a cross section of $C_n(\mathbb{F})$ and 1-dimensional linear subspaces of $\mathbb{F}^n$.

We claim that the matrices in $C_n(\mathbb{F})$ are arranged in the faces of the cone according to their kernels, with invertible matrices lying in the interior of the cone and matrices of minimal rank lying in minimal faces. To see this, let $A, B \in C_n(\mathbb{K})$, and let $F(B)$ be the face generated by B. Then

$A \in F(B) \iff$

$\exists \lambda > 0$ such that $B - \lambda A \in C_n(\mathbb{K}) \iff$

$\exists \lambda > 0$ such that $\overline{w}Bw^T \geq \lambda \overline{w}Aw^T \geq 0$ for all $w \in \mathbb{F}^n \iff$

$\overline{w}Bw^T = 0$ implies $\overline{w}Aw^T = 0$ for all $w \in \mathbb{F}^n \iff$

$\ker A \supseteq \ker B$, since for $A \in C_n(\mathbb{K})$, $\overline{w}Aw^T = 0$ if and only if $Aw^T = 0$.

Therefore $F(A) \to (\ker A)^\perp$ defines a bijection φ from the set of faces of $C_n(\mathbb{K})$ to the set of linear subspaces of $\mathbb{K}^n$. To prove that φ is an isomorphism of lattices observe that $\varphi(F(A) \vee F(B)) = \varphi(F(A+B)) = (\ker(A+B))^\perp = (\ker A \cap \ker B)^\perp = (\ker A)^\perp \vee (\ker B)^\perp) = \varphi F(A) \vee \varphi F(B)$, and on the other hand, if $F(A) \wedge F(B)$ is a nonempty face, then it is generated by a matrix C with $\ker C = \ker A \vee \ker B$, so $\varphi(F(A) \wedge F(B)) = (\ker C)^\perp = (\ker A)^\perp \cap (\ker B)^\perp = \varphi F(A) \wedge \varphi F(B)$. □

Lemma 3. *Let* $H_3(\mathbb{O})$ *be the set of* 3×3 *Hermitian matrices over the octonions. Then the subset* $C_3(\mathbb{O})$ *consisting of sums of squares of elements in* $H_3(\mathbb{O})$ *is a real cone whose face lattice determines an octonionic projective plane.*

Proof. We use the nontrivial fact that each matrix in $H_3(\mathbb{O})$ is diagonalizable by an automorphism of $H_3(\mathbb{O})$ that leaves the trace invariant; see [1]. Then:

(a) A matrix A in $H_3(\mathbb{O})$ lies in $C_3(\mathbb{O})$ if and only if it can be diagonalized to a matrix A' with nonnegative entries, because if A lies in $C_3(\mathbb{O})$, then A' is a sum of squares of matrices in $H_3(\mathbb{O})$, which have nonnegative diagonal entries.

(b) All the idempotent matrices in $H_3(\mathbb{O})$ lie in $C_3(\mathbb{O})$ as they are squares ($A = A^2$). The idempotent matrices of trace 1 correspond to the extreme rays of $C_3(\mathbb{O})$ since they cannot be written as nonnegative combinations of other idempotent matrices.

(c) Each face of $C_3(\mathbb{O})$ is generated by an idempotent matrix, because in any cone all the positive linear combinations of the same set of vectors generate the same face, so a diagonal matrix with nonnegative entries generates the same face as a matrix with only zeros and ones.

(d) Any two idempotent matrices of trace 1 lie in a face generated by an idempotent matrix of trace 2, because they can be put simultaneously in the form $\left[\begin{smallmatrix} a & x & 0 \\ \overline{x} & b & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$, and these lie in the face generated by $\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$.

(e) $A \in C_3(\mathbb{O})$ is an idempotent of trace 1 if and only if $I - A$ is an idempotent with trace 2. If A and B are idempotents of trace 1, then A lies in the face generated by $I - B$ if and only if B lies in the face generated by $I - A$. This duality and (d) show that any two faces generated by idempotent matrices of trace 2 meet in a face generated by an idempotent matrix of trace 1. □

3. Face lattices defining projective spaces

If C is a convex body in $\mathbb{R}^n$ whose face lattice is modular and irreducible and C is not strictly convex, the extreme points of C form a projective space with subspaces determined by the faces of C. The main result of the paper is the following:

Theorem 1. *Let C be a convex body whose face lattice defines a projective space of rank r. Then:*

If $r = 2$, C has dimension $5, 8, 14$ or 26.

If $r > 2$ or the projective space is Desarguesian, there is a face-preserving homeomorphism from C to a cross-section of a cone of positive-semidefinite Hermitian matrices over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

In order to prove the theorem, we need to show that among the infinitely many isomorphism classes of projective spaces of each rank, those defined by the extreme points of convex bodies are of a very special kind, namely, topological projective spaces which are compact and connected. A *topological projective space* is a projective space in which the sets of subspaces of each rank are given nontrivial topologies that make the join and meet operations $\vee$ and $\wedge$ continuous, when restricted to pairs of subspaces of fixed ranks whose join or meet have a fixed rank [8].

A natural topology for the set of faces of a convex body is given by the Hausdorff metric on $\mathbb{R}^n$. By the Blaschke selection theorem [7], the space of all compact, convex subsets of a convex body in $\mathbb{R}^n$, with the Hausdorff metric, is compact. The space of faces of a convex body, however, is not compact in general.

Lemma 4. *If C is a convex body whose face lattice $\mathcal{F}(C)$ is modular, then the set $\mathcal{F}_h(C)$ of faces of rank h is compact for each h.*

Proof. Let F_i be a sequence of faces of rank h. By Blaschke's theorem, F_i has a subsequence F_{i_j} that converges to a convex set K contained in ∂C, so K generates a proper face F. We claim that the rank of F is h and that K is all of F.

If the rank of F was less than h, there would be a face F^c of C of rank $n-h$ with $F^c \cap F = \phi$. As F and F^c are two disjoint compact sets in $\mathbb{R}^n$, there exists $\epsilon > 0$ such that the ε-neighborhoods of F and F^c in $\mathbb{R}^n$ are disjoint. But as $F_{i_j} \to K \subset F$ in the Hausdorff metric, then for sufficiently large j, F_{i_j} is contained in the ε-neighborhood of F; therefore $F_{i_j} \cap F^c = \varnothing$, but these 2 faces have ranks that add up to n, so they should meet, a contradiction.

If the rank of F is h and $K \neq F$, there is an extreme point $p \in F - K$. Let F' be a face of rank $n-h$ that meets F only at p ($F' = F^c \vee p$, where F^c is a complementary face of F) so $F' \cap K = \varnothing$ and the previous argument gives a contradiction.

To show that the rank of F cannot be larger than h, proceed inductively on $n-h$. As a limit of proper faces is contained in a proper face, the claim holds if $h = n-1$. Given a sequence F_i of faces of rank h, let F be a face generated by the limit of a convergent subsequence F_{i_j}. If p is an extreme point of C not in F, then for sufficiently large j, $p \notin F_{i_j}$ (otherwise p would be in F). Let G_{i_j} be a face of rank $h+1$ containing F_{i_j} and p. By induction we can assume that the limit of a convergent subsequence of G_{i_j} generates a face G of rank $h+1$. Then G contains F properly (because it contains p), so the rank of F is smaller than $h+1$. □

Lemma 5. *If C is a convex body whose face lattice is modular and irreducible, then C is strictly convex or the set of extreme points $\mathcal{E}(C)$ is a topological projective space which is compact and connected.*

Proof. The points of the projective space are the extreme points of C, and the points of a projective subspace are the extreme points of a face of C, so the projective lines are the boundaries of the strictly convex faces of C. With the natural topology for the set of subspaces given by the Hausdorff distance between the faces, the set of points $\mathcal{E}(\mathcal{C}) = \mathcal{F}_0(C)$ is compact (Lemma 4) and each line is a topological sphere of dimension at least 1. As every pair of points in $\mathcal{E}(C)$ is contained in one of these spheres, $\mathcal{E}(C)$ is connected.

It remains to show that $\vee$ and $\wedge$ are continuous on the preimages of each $\mathcal{F}_h(C)$. Suppose $A_i \to A$, $B_i \to B$, where all $A_i \wedge B_i$ and $A \wedge B$ are faces of the same rank h. We need to show that $A_i \wedge B_i \to A \wedge B$. By Lemma 4, $C_i = A_i \wedge B_i$ has convergent subsequences and the limit of a convergent subsequence C_{i_α} is a face C_α of rank h. As C_{i_α} is contained in A_{i_α} and B_{i_α}, C_α is contained in $A \wedge B$. But C_α and $A \wedge B$ are both faces of rank h, so $C_\alpha = A \wedge B$. Similarly, if $A_i \to A$, $B_i \to B$ and $A_i \vee B_i$, $A \vee B$ are faces of rank h, the limit of each convergent subsequence of $D_i = A_i \vee B_i$ is a face D of rank h. As D_i contains A_i and B_i, D contains $A \vee B$, and as both faces have rank h they must be equal. $\square$

Proof of Theorem 1. Assume first that the projective space determined by $\mathcal{E}(C)$ is Desarguesian. It is well known that every Desarguesian topological projective space of rank r is isomorphic to the space of linear subspaces of a vector space A^{r+1} for some *topological* division ring A, and this isomorphism is a homeomorphism [8, p. 1261]. A can be identified with a projective line minus a point, and in our case the projective lines are topological spheres. By a classic result of Pontryagin [8, p. 1263] the only locally compact, connected division rings are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, so the topological projective space determined by $\mathcal{E}(C)$ must be isomorphic to $\mathbb{RP}^r, \mathbb{CP}^r$ or $\mathbb{HP}^r$. The cones of Hermitian matrices described in Lemma 2 also determine these topological projective spaces, so for some cone section C', there exists a homeomorphism from $\mathcal{E}(C)$ to $\mathcal{E}(C')$ that is face-preserving. The following lemma shows that this homeomorphism can be extended to a face-preserving homeomorphism from C to C':

Lemma 6. *If C and C' are convex bodies with $\mathcal{F}(C)$ and $\mathcal{F}(C')$ compact, then any continuous "face-preserving" map $\varphi : \mathcal{E}(C) \to \mathcal{E}(C')$ extends naturally to a continuous map $\varphi : C \to C'$.*

We defer the proof of Lemma 6 to consider the remaining case of Theorem 1. Non-Desarguesian projective planes are not classified, but all the lines of a topological projective plane $\mathcal{P}$ are homeomorphic because if l is a line and p is a point not in l, then the projection $\phi : \mathcal{P} - p \to l$, $\phi(x) = (x \vee p) \wedge l$ is continuous and its restriction to each projective line not containing p is one-to-one. When the projective lines are topological spheres, as in our case, a famous result of Adams [8, p. 1278] shows that their dimension must be $d = 0, 1, 2, 4$ or 8.

To compute the dimension of C take 3 faces of rank 1, F_0, F_1 and F_2 so that F_1 and F_2 meet at a point p not in F_0. The proof of Lemma 6 shows that the projection $\phi : \mathcal{E}(C) - \{p\} \to \partial F_0$ extends to a continuous map ϕ from the union of the faces of rank 1 of C that do not contain p to F_0, whose restriction to each face is one-to-one. The union of the interiors of these faces is an open subset

U of ∂C and the function $\Phi : U \to F_0 \times (\partial F_1 - \{p\}) \times (\partial F_2 - \{p\})$ defined as $\Phi(x) = (\phi(x), \partial F(x) \wedge \partial F_1, \partial F(x) \wedge \partial F_2)$ is continuous and bijective, so U has the same dimension as $F \times \partial F \times \partial F$, which is $3d+1$; therefore C has dimension $3d+2$. Note that the discrepancy between the dimensions of the union of the boundaries of the faces ($2d$) and the union of the faces ($3d+1$) arises because the boundaries of the faces overlap (as the lines in a projective plane do) but the interiors of the faces are disjoint. When $r > 2$, there is a similar homeomorphism from an open subset of ∂C and a product $F_0 \times (\partial F_1 - \{p\}) \times (\partial F_2 - \{p\}) \times ... \times (\partial F_r - \{p\})$, where F_0 is a face of rank $r-1$ and $F_1, F_2, ..., F_r$ are faces of rank 1. So $\dim(C) = \dim(F_0) + rd + 1$, and it follows by induction that $\dim C = \frac{r(r-1)}{2}d + r - 1$. □

The proof of Lemma 6 relies on the following (probably known) result:

Lemma 7. *Let $\mathcal{C}$ be a convex body and $\mathcal{B}(C) \subset C$ be the set of baricenters of faces of C. If $\mathcal{F}(C)$ is compact, then the function $b : \mathcal{F}(C) \to \mathcal{B}(C)$ that maps each face to its baricenter is a homeomorphism. Moreover, a sequence of faces F_i converges to a face F if and only if $\mathcal{E}(F_i)$ converges to $\mathcal{E}(F)$.*

Proof. The function that assigns to each compact convex set in $\mathbb{R}^n$ its baricenter is continuous, so $b : \mathcal{F}(C) \to \mathcal{B}(C)$ is a continuous bijective map from a compact space to a Hausdorff space, so it is also a closed map. This proves the first part.

To prove the second part observe that the Hausdorff distance between two compact convex sets is bounded above by the Hausdorff distance between their sets of extreme points.

If F_i converges to F but $\mathcal{E}(F_i)$ does not converge to $\mathcal{E}(F)$, then there is a subsequence $\mathcal{E}(F_{i_j})$ that stays at a distance at least $\varepsilon > 0$ from $\mathcal{E}(F)$. For each i_j there is an extreme point $p_{i_j} \in F_{i_j}$ whose distance from $\mathcal{E}(F)$ is larger than ε, or an extreme point $q_i \in F$ whose distance from $\mathcal{E}(F_{i_j})$ is larger than ε. If there is a convergent subsequence $p_{i_k} \to p \in F$, then p is at a distance at least ε from $\mathcal{E}(F)$, so p can't be an extreme point of C.

If there is a convergent subsequence $q_{i_k} \to q \in F$, take $p'_{i_k} \in F_{i_k}$ with $p'_{i_k} \to q$. Eventually $\left|p'_{i_k} - q_{i_k}\right| < \frac{\varepsilon}{2}$, so the distance from p'_{i_k} to $\mathcal{E}(F_{i_k})$ is at least $\frac{\varepsilon}{2}$, so p'_{i_k} is the center of a straight interval I_{i_k} of length ε contained in F_{i_k}. A convergent subsequence of these intervals yields a straight interval centered at q and contained in F, so q can't be an extreme point of C, contradicting the compacity of $\mathcal{E}(C)$. □

Proof of Lemma 6. The map $\varphi : \mathcal{E}(C) \to \mathcal{E}(C')$ determines a function $\Psi : \mathcal{F}(C) \to \mathcal{F}(C')$. Ψ is continuous because by Lemma 7, $F_i \to F$ implies $\mathcal{E}(F_i) \to \mathcal{E}(F)$. Then uniform continuity of φ on $\mathcal{E}(C)$ implies that $\varphi(\mathcal{E}(F_i)) \to \varphi(\mathcal{E}(F))$, so by definition $\mathcal{E}(\Psi(F_i)) \to \mathcal{E}(\Psi(F))$ and so $\Psi(F_i) \to \Psi(F)$. So φ can be extended to a continuous function $\varphi : \mathcal{B}(C) \to \mathcal{B}(C')$ as $b \circ \Psi \circ b^{-1}$ (recall that $\mathcal{E}(C) \subset \mathcal{B}(C)$). Now we can extend φ to the interiors of the faces of C, starting with the faces of smallest dimension and using the function already defined on the boundaries of faces and their baricenters to extend it linearly on rays to their interiors. The delicate point is to prove that the extension is continuous, as C may have an infinite number of faces.

For each point $a \in C$, let $F(a)$ be the face of C generated by a and let $b(a)$ be the baricenter of $F(a)$. Although $F(a)$ and $b(a)$ are not continuous functions of a on all of C, they are continuous on the union of the interiors of the faces of dimension d, for each d. To see this suppose $a_i \to a$ is a converging sequence of points lying in the interiors of faces F_i and F of dimension d. As $\mathcal{F}(C)$ is compact, F_i has a subsequence converging to a face F' of dimension at most d. As a is contained in F' and a is contained in the interior of F, F is a face of F'. As the dimension of F' is at most that of F, so $F = F'$. This shows that $F(a)$ is continuous. The continuity of $b(a)$ follows.

Now if $a \neq b(a)$ let $p(a)$ be the projection of a to $\partial F(a)$ from $b(a)$ and let $\lambda(a) = \frac{|a-b(a)|}{|p(a)-b(a)|}$ (or 0 if $a = b(a)$) so $a = (1-\lambda(a))b(a) + \lambda(a)p(a)$. Define $\varphi(a) = (1-\lambda(a))\varphi(b(a)) + \lambda(a)\varphi(p(a))$.

Assume inductively that φ is continuous on the closed set formed by the union of $\mathcal{B}(C)$ and the faces of C of dimension less than d, and let us show that for each sequence of points a_i in the interiors of faces of dimension d, $a_i \to a$ implies $\varphi(a_i) \to \varphi(a)$. We may assume that the a_i are not baricenters, so $p(a_i)$ is well defined.

Case 1. If $F(a_i) \to F(a)$, then $b(a_i) \to b(a)$ by the continuity of b on faces.

If $b(a) \neq a$, then $p(a_i) \to p(a)$ and $\lambda(a_i) \to \lambda(a)$, so $\varphi(a_i) = (1-\lambda(a_i))\varphi(b(a_i)) + \lambda(a_i)\varphi(p(a_i)) \to (1-\lambda(a))\varphi(b(a)) + \lambda(a)\varphi(p(a)) = \varphi(a)$.

If $b(a) = a$, then $\lim b(a_i) = \lim a_i$, but $p(a_i)$ may not converge, so consider a convergent subsequence $p(a_{i_j})$: If $\lim p(a_{i_j}) \neq \lim b(a_{i_j}) = \lim a_{i_j}$, then $\lim \lambda(a_{i_j}) = 0$, so $\varphi(a_{i_j}) = \varphi(b(a_{i_j})) + \lambda(a_{i_j})\left[\varphi(p(a_{i_j})) - \varphi(b(a_{i_j}))\right] \to \varphi(b(a)) + 0 = \varphi(a)$. If $\lim p(a_{i_j}) = \lim b(a_{i_j})$, then $\lim \varphi(p(a_{i_j})) = \lim \varphi(b(a_{i_j}))$ (by continuity of φ in the baricenters and faces of lower dimension) and as $\varphi(a_{i_j})$ lies between them, $\lim \varphi(a_{i_j}) = \lim \varphi(b \circ F(a_{i_j})) = \varphi(b(a)) = \varphi(a)$.

Case 2. If $F(a_i) \nrightarrow F(a)$, then for any convergent subsequence $F(a_{i_j})$ with limit a face $F \neq F(a)$, a lies in F and so a must lie in ∂F, so $\left|a_{i_j} - p(a_{i_j})\right| \to 0$ and $\lambda(a_{i_j}) \to 1$, so $\lim \varphi(a_{i_j}) = \lim(1-\lambda(a_{i_j}))\varphi(b(a_{i_j})) + \lambda(a_{i_j})\varphi(p(a_{i_j})) = \lim \varphi(p(a_{i_j})) = \varphi(a)$ (by continuity of φ on the faces of lower dimension). $\square$

4. Projective planes and the case $n = 5$

Let's now consider more closely a convex body in $\mathbb{R}^n$ whose face lattice determines a projective plane, i.e., a convex body C different from a triangle in which every pair of extreme points lies in a face and every pair of faces with more than one point meet. By the proof of Theorem 1, $\mathcal{F}_0(C)$ and $\mathcal{F}_1(C)$ are closed manifolds of dimension $2d$ and C has dimension $n = 3d + 2$ for some $d \in \{1, 2, 4, 8\}$.

Lemma 8. *If the face lattice of a convex body C in $\mathbb{R}^n$ determines a projective plane, the boundaries of the faces of rank 1 of C are semi-algebraic sets. If $n = 5$, they are conic sections.*

The key to the proof of this lemma is the following:

Lemma 9. *If C is a convex body in $\mathbb{R}^n$ whose face lattice determines a projective plane, then each affine subspace of dimension $d+1$ of $\mathbb{R}^n$ that meets the affine spans of all faces in $\mathcal{F}_1(C)$ is the affine span of one of these faces.*

Proof. In the following, *span* will mean affine span. Let S be an affine subspace of dimension $d+1$ that intersects *span*(F) for every F in $\mathcal{F}_1(C)$. Then the set $\{F \in \mathcal{F}_1(C);\ \dim(S \cap \mathit{span}(F)) \geq i\}$ is closed in $\mathcal{F}_1(C)$, for each i.

Case 1. $d = 1$. We claim that if S is not the span of a face F, then S cannot intersect *span*(F) in more than one point. For, if $S \cap \mathit{span}(F)$ contains a line, then

$span(S \cup F)$ is 3-dimensional. Take an extreme point $p \notin span(S \cup F)$ and let F_1 and F_2 be 2 faces containing p, and meeting F at points p_1 and p_2 not in S. If p_1' and p_2' are points in $S \cap span(F_1)$ and $S \cap span(F_2)$ respectively, then p, p_i and p_i' are not collinear (otherwise p would be in $span(S \cup F)$). So $span(p \cup S \cup F) \supset span\{p, p_i, p_i'\}$ contains F, F_1 and F_2 and it contains each 2-dimensional face that meets F, F_1 and F_2 at 3 noncollinear points, but every face is a limit of such faces, so $span(p \cup S \cup F)$ contains all the faces of C. But $span(p \cup S \cup F)$ has dimension 4, contradicting the fact that C has dimension 5.

This shows that S intersects each $span(F)$ at exactly one point, and so S contains at most one extreme point p of C. The function $I : \mathcal{F}_1(C) \to S$ that maps each face F_i to the point of intersection of $span(F)$ with S is continuous, and as the spans of faces meet only at extreme points, I is injective outside the set of faces $\mathcal{F}_1^p(C)$ containing the extreme point p in S. As $d = 1$, $\mathcal{F}_1(C)$ is topologically a closed surface and $\mathcal{F}_1^p(C)$ is a closed curve, but there are no continuous maps from a closed surface to a plane that fail to be injective only along a curve.

Case 2. S does not contain extreme points of some face F in $\mathcal{F}_1(C)$. Choose F to minimize the dimension of the affine subspace $S \cap span(F)$ of S. Then for every face F' in a neighborhood of F, $S \cap span(F')$ is an affine subspace of minimal dimension of S that has no extreme points of F'. If S^0 is a maximal subspace of S meeting $span(F)$ in one point, then S^0 meets $span(F')$ in one point for all F' in a smaller neighborhood V of F. Then the function $I : V \to S^0$ that maps F' to $S^0 \cap span(F')$ is continuous, and it is injective as the spans of faces only meet at extreme points. But an injective map between manifolds can only exist when the domain has dimension no larger than the target, so $2d = \dim V \leq \dim S^0 \leq \dim S \leq d+1$, so $d = 1$ and we are in case 1.

Case 3. S contains extreme points of each face F. If S intersects C only in its boundary, then $S \cap C$ is contained in a face F_1 of C and so either $S \cap C = F_1$ (therefore $span(F_1) = S$) or there is an extreme point p of F_1 not contained in S, but then a face F_2 that meets F_1 at p does not meet $S \cap F_1 = S \cap C$ and so S does not contain extreme points of F_2, a contradiction.

If S meets the interior of C, then $S \cap \partial C = \partial_S(S \cap C)$. Consider the set $\mathcal{F}_1^p(C)$ of faces of rank 1 containing an extreme point p not in S. Choose F in $\mathcal{F}_1^p(C)$ so that $S \cap F$ has minimal dimension. Then for all F' in some neighborhood V of F in $\mathcal{F}_1^p(C)$, $S \cap F'$ has the same dimension and the map $I_V : V \to S \cap \partial C$ that sends F' to the baricenter of $S \cap F'$ is continuous and injective. As I_V is a map between manifolds, $d = \dim \mathcal{F}_1^p(C) = \dim V \leq \dim(S \cap \partial C) \leq d$ and so by invariance of domain the image of I_V is an open subset of $S \cap \partial C = \partial_S(S \cap C)$. As the baricenters of the faces of a convex body can fill an open set of its boundary only if the open set consists of extreme points, the image of I_V consists of extreme points of $S \cap C$ that (by the assumption that S contains extreme points of each F' in V) must be extreme points of C. So the line segment joining two extreme points in the image of I_V lies in $Int_S(S \cap C)$, and at the same time it lies in a face of C, so it must lie in $S \cap \partial C = \partial_S(S \cap C)$, a contradiction. □

Proof of Lemma 8. By the previous lemma, the set $\mathcal{S}$ of spans of faces of C is the same as the set of $(d+1)$-dimensional affine subspaces of $\mathbb{R}^n$ that intersect every element of $\mathcal{S}$. The set of all $(d+1)$-dimensional affine subspaces of $\mathbb{R}^n$ forms a real algebraic variety, and the condition that the subspaces meet a fixed subspace

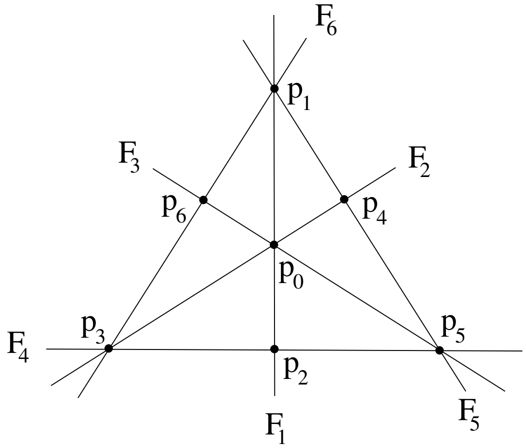

FIGURE 1

is algebraic, so (by the finite descending chain condition) there is a finite family of spans $S_1, S_2, ..., S_m \in \mathcal{S}$ such that $S \in \mathcal{S}$ if it intersects these S_i's (see [5]).

Now for $(x_1, x_2, ..., x_m) \in S_1 \times S_2 \times \cdots \times S_m$, the subspace $span(x_1, ..., x_m)$ has dimension at least $d+1$ (otherwise it would be contained in two subspaces of dimension $d+1$ that meet each S_i, so they would both be in $\mathcal{S}$, but two spans can only meet in 1 point). So $span(x_1, ..., x_m)$ lies in $\mathcal{S}$ if and only if its dimension is $d+1$, and this happens if and only if some determinants (given by polynomials in $x_1, ..., x_m$) vanish. Therefore the set $X = \{(x_1, x_2, ..., x_m) \in S_1 \times S_2 \times \cdots \times S_m \mid span(x_1, ..., x_m) \in \mathcal{S}\}$ is real algebraic, as is the set X^p formed by the elements of X that contain a fixed point p. If F_1 is the face in S_1 and p is an extreme point of C outside F_1, then ∂F_1 consists of the intersections of S_1 with the elements of $\mathcal{S}$ containing p. So ∂F_1 is the one-to-one projection of the algebraic set X^p to S_1, so ∂F_1 is at least semi-algebraic.

Now assume $n = 5$, so $d = 1$. Every projective plane contains 7 points and 6 lines so that each line contains 3 points as in Figure 1, so C has 7 extreme points and 6 faces intersecting in that way. The 7 points are in general position in $\mathbb{R}^5$ because as each face of C is spanned by 3 points, the span of any 6 of those points contains the span of 3 faces, which is all of $\mathbb{R}^5$. Therefore we may assume (by applying a projective transformation) that the 7 points are $p_0 = (0, 0, 0, 0, 0), p_1 = (1, 0, 0, 0, 0), p_2 = (0, 1, 0, 0, 0), ..., p_5 = (0, 0, 0, 0, 1), p_6 = (1, 1, 1, 1, 1)$. Let S_i be the plane spanned by the face F_i. A plane S that intersects S_1, S_2 and S_3 has a parametrization $(x, y, z, v, w) = r(a, b, 0, 0, 0) + s(0, 0, c, d, 0) + t(e, e, e, e, f)$ with $r + s + t = 1$. S intersects S_4, S_5 and S_6 only if three systems of linear equations in r, s, t represented by the following matrices have nontrivial solutions:

$$\begin{vmatrix} a & 0 & e \\ 0 & d & e \\ b-1 & c-1 & 2e+f-1 \end{vmatrix} \begin{vmatrix} b & 0 & e \\ 0 & c & e \\ a-1 & d-1 & 2e+f-1 \end{vmatrix} \begin{vmatrix} b & 0 & e-f \\ 0 & d & e-f \\ a-1 & c-1 & 2e-f-1 \end{vmatrix}.$$

As the determinants of these matrices are linear functions on the variables e and f, they vanish simultaneously if and only if the next matrix has determinant 0:

$$\det \begin{vmatrix} -ac+2ad-bd+a+d & ad & -ad \\ -ac+2bc-bd+b+c & bc & -bc \\ -ad-bc+2bd+b+d & ad+bc-bd-b-d & -bd \end{vmatrix} = 0.$$

This determinant factors as the product of a linear and a quadratic function of a and b (with coefficients in c and d). Since the boundary of the face F_1 is formed by the intersections of S_1 with the planes that meet all S_i's and go through a fixed point in the boundary of F_2 (this corresponds to fixing c and d), the boundary of F_1 is contained in the union of a line and a conic. As the boundary of F_1 is strictly convex, it must be the conic. □

Theorem 2. *Every* 5*-dimensional convex body with modular and irreducible face lattice is projectively equivalent to the set of* 3×3 *real symmetric positive-semidefinite matrices of trace* 1*.*

Proof. Let C and C' be two convex bodies in $\mathbb{R}^5 \subset \mathbb{RP}^5$ having modular and irreducible face lattices. The claim is that there is a projective transformation of $\mathbb{RP}^5$ taking C to C'. Take extreme points $p_0, p_1, ..., p_6$ and faces $F_1, ..., F_6$ of C as in Figure 1. Pick an extreme point p'_0 in C' and two faces F'_1 and F'_2 of C' intersecting at p'_0. Let S_i be the span of F_i. As the faces of C and C' are conics, there are linear transformations from S_1 to S'_1 taking F_1 to F'_1 and from P_2 to P'_2 taking F_2 to F'_2. Together, they define a linear transformation l from $span(F_1 \cup F_2)$ to $span(F'_1 \cup F'_2)$. Let $p'_i = l(p_i)$ for $i = 1, ..., 4$. The faces F_4, F_5, F_6 are generated by unique pairs of p_i's with $i \leq 4$. Let F'_4, F'_5, F'_6 be the faces generated by the corresponding pairs of p'_i's. Finally, let $p'_5 = S'_4 \cap S'_5$, let F'_3 be the face generated by p'_0 and p'_5 and let $p'_6 = S'_3 \cap S'_6$. The linear transformation l can be extended to a projective transformation ρ in $\mathbb{RP}^5$ that takes p_5 to p'_5 and p_6 to p'_6. As ρ sends each p_i to p'_i, it sends each S_i to S'_i, so it sends each plane in $\mathbb{R}^5$ intersecting every S_i to a plane intersecting every S'_i. Since by construction ρ takes those planes that meet ∂F_1 and ∂F_2 to planes that meet $\partial F'_1$ and $\partial F'_2$, Lemma 9 implies that ρ maps spans of faces of C to spans of faces of C', and therefore it maps faces to faces. □

5. Face lattices defining affine spaces

Projective spaces are closely related to affine spaces, whose axioms capture the basic properties of parallelism in $\mathbb{R}^n$. Every affine space A can be embedded in a projective space by attaching to A a point at infinity for each parallelism class of affine lines, and conversely, in any projective space P the complement of a maximal projective subspace is an affine space. So it seems natural to look at convex sets whose faces meet as the subspaces of an affine space. For this, one needs to consider convex sets in $\mathbb{R}^n$ that are closed but not necessarily compact. Observe that if a closed convex set C in $\mathbb{R}^n$ is noncompact, it contains a ray (half of a Euclidean line); and if C contains a ray, then it contains all the parallel rays starting at points of C (we say that C contains an infinite direction). So if C contains a line, C is the product of that line and a closed convex set C' of lower dimension and the face lattices of C and C' are isomorphic. So from now on we will assume that C does not contain lines.

It is easy to see that a convex body with a finite number of faces cannot determine an affine space: the faces of rank i would have dimension i, two parallel faces of rank 1 generate a face of rank 2 with at least 4 vertices, but the sides of a polygon do not define an affine plane. On the other hand, if C is a cone in $\mathbb{R}^{n+1}$ whose face lattice determines a projective space, the intersection of C with a hyperplane parallel to one that touches C in a maximal face is a closed (noncompact) convex set in $\mathbb{R}^n$ whose face lattice determines an affine space. In particular, there are closed convex sets whose faces determine real, complex and quaternionic affine spaces and an octonionic affine plane.

We want to prove a result analogous to the main theorem for closed convex sets $\mathbb{R}^n$ whose face lattices determine affine spaces. For this we will show that these sets have compactifications in $\mathbb{RP}^n$ that are convex bodies whose face lattices define projective spaces. Think of $\mathbb{RP}^n$ as the result of attaching to $\mathbb{R}^n$ a "point at infinity" for each class of parallel lines, and also as the space of lines through the origin in $\mathbb{R}^{n+1}$. Identifying $\mathbb{R}^n$ with a hyperplane of $\mathbb{R}^{n+1}$ that does not contain the origin gives an embedding of $\mathbb{R}^n$ as a dense open subset of $\mathbb{RP}^n$. The remaining points of $\mathbb{RP}^n$ correspond to lines through the origin in $\mathbb{R}^{n+1}$ that do not meet the hyperplane, and these correspond to unoriented directions in $\mathbb{R}^n$. Different identifications of $\mathbb{R}^n$ with a hyperplane give different embeddings of $\mathbb{R}^n$ into $\mathbb{RP}^n$. Define a set in $\mathbb{RP}^n$ to be *convex* if it is the image of a convex set in $\mathbb{R}^n$ under one of these embeddings. So convex sets in $\mathbb{RP}^n$ correspond to convex cones based at the origin of $\mathbb{R}^{n+1}$ (by definition a cone does not contain lines). If C is a closed convex set in $\mathbb{R}^n$, its closure in $\mathbb{RP}^n$, denoted by $\overline{C}$, is obtained by attaching to C a point at infinity for each infinite direction of C. If C does not contain lines, the closure of the cone over C in $\mathbb{R}^{n+1}$ is a cone, so $\overline{C}$ is convex in $\mathbb{RP}^n$ and the faces of $\overline{C}$ are the closures of the faces of C in $\mathbb{RP}^n$ and the intersections of these faces with the set of points at infinity.

Lemma 10. *If the faces of a closed convex set C in $\mathbb{R}^n$ define an affine space, each face representing an affine line contains a unique infinite direction, which is the same for faces representing parallel lines.*

Proof. We are assuming that C does not contain lines, so the points of the affine space are the extreme points of C and the affine lines are the boundaries of strictly convex faces of C of dimension at least two.

We claim that a face A representing an affine line cannot be compact. Otherwise, let P be a face that contains A and represents an affine plane. Let p and q be two extreme points of A and let q_i be a sequence of extreme points of P, not in A, that converge to q. If A_i is the face generated by p and q_i, then for sufficiently large i, A_i lies in an ε-neighborhood U_ε of A. If not, take a subsequence A_{i_j} having extreme points r_{j_j} lying outside U_ε. As A is compact we may assume that the sequence r_{i_j} is bounded and therefore has a subsequence converging to a point r outside U_ε. As the triangles $pq_{i_j}r_{i_j}$ lie in ∂P, the triangle pqr is contained in ∂P and therefore is contained in a proper face of P. This is impossible because the only proper face of P that contains p and q is A, and r is not in A.

Now take a face A' in P that does not meet A (i.e., A and A' represent parallel affine lines). As A is compact and A' is closed in $\mathbb{R}^n$, there is an ε-neighborhood of A that does not intersect A'. By the previous argument there is a point q_i not in A so that the face A_i generated by p and q_i is contained in the ε-neighborhood of A. So A_i does not meet A', but A was supposed to be the only face containing p and disjoint from A', a contradiction This proves that a face A representing an affine line is not compact, so it contains rays.

The rays in A passing through a point p form a convex cone, and each ray l_+ in the boundary of this cone is the limit of a sequence of intervals pp_i in A joining p with extreme points of A. If q is an extreme point of C outside A, q and A generate a face P representing an affine plane. The sequence of intervals qp_i lie in ∂P and converge to a ray m_+ parallel to l_+ that contains q, so (as P is closed) m_+ is contained in a face A' of P. As two faces that contain parallel rays cannot meet at a single point, A' does not meet A, so A' represents the affine line parallel to A through p. This shows that two faces A and A' representing parallel affine lines have the same cones of infinite directions. If the cones have more than one infinite direction, a line going across a cone for A meets A in a closed interval, and the same is true for a parallel line going across a cone for A'.

So there are 4 extreme points a_1, a_2 and a_1', a_2' of A and A' respectively that are coplanar in $\mathbb{R}^n$. They determine a convex quadrilateral whose diagonals meet at an interior point, but this is impossible because the diagonals are contained in faces representing different affine lines, and these can only meet at an extreme point $\square$

Theorem 3. *If C is a closed convex set in $\mathbb{R}^n$ whose faces determine an affine space, then there is a face-preserving homeomorphism from C to a "parabolic section" of a cone of positive-semidefinite Hermitian matrices over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, or C has dimension $8, 14$ or 26.*

Proof. Consider $C \subset \mathbb{R}^n \subset \mathbb{RP}^n$. By the previous lemma the faces of C representing affine lines share an infinite direction in $\mathbb{R}^n$ if and only if they represent parallel affine lines. The closure of C in $\mathbb{RP}^n$ contains one point at infinity for each infinite direction in C, so $\overline{C}$ contains an extreme point at infinity for each class of faces of C representing parallel lines, and this corresponds precisely with the definition of the projective completion of an affine space (we are not assuming any topologies in the affine or the projective space).

Therefore $\overline{C}$ is a convex body in $\mathbb{RP}^n$ whose face lattice determines a projective space, and we can apply Theorems 1 and 2 to show that there is a face-preserving homeomorphism φ from $\overline{C}$ to an "elliptic" section of a cone of Hermitian matrices, or $\overline{C}$ (and therefore C) has dimension 8, 14 or 26. As C is obtained from $\overline{C}$ by removing a maximal face, φ maps C to the set of points of the elliptic section lying outside a maximal face, and there is a natural bijection between these points and the points in a "parabolic section" of the cone obtained by intersecting the cone with a hyperplane parallel to one that touches the cone in the maximal face. $\square$

6. Questions

The results above suggest some questions:

Can two convex bodies of the same dimension define nonisomorphic projective planes (so they are not related by a face-preserving homeomorphism)? In dimensions 8 and 14 this is equivalent to asking if these projective planes are always Desarguesian. In dimension 26 there might be many nonisomorphic examples.

Are all the convex bodies whose face lattices determine classical projective spaces projectively equivalent to sections of cones of Hermitian matrices?

Are there examples of convex sets with face lattice isomorphic to the lattice of subspaces of a hyperbolic space?

What can be said about convex bodies whose face lattice is lower semimodular?

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