AUSLANDER-REITEN COMPONENTS DETERMINED BY THEIR COMPOSITION FACTORS

ALICJA JAWORSKA, PIOTR MALICKI, AND ANDRZEJ SKOWROŃSKI

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Abstract. We provide sufficient conditions for a component of the Auslander-Reiten quiver of an Artin algebra to be determined by the composition factors of its indecomposable modules.

1. Introduction and main results

Let $A$ be an Artin algebra over a commutative Artin ring $R$. We denote by $\text{mod} A$ the category of finitely generated right $A$-modules, by $K_0(A)$ the Grothendieck group of $A$, and by $[M]$ the image of a module $M$ from $\text{mod} A$ in $K_0(A)$. Thus, for modules $M$ and $N$ in $\text{mod} A$, $[M] = [N]$ if and only if $M$ and $N$ have the same composition factors including the multiplicities. An interesting open problem is to find handy criteria for two indecomposable modules $M$ and $N$ in $\text{mod} A$ with the same composition factors to be isomorphic. It was shown in [16] that this is the case when $M$ does not lie on a short cycle $M \to X \to M$ of non-zero non-isomorphisms in $\text{mod} A$ with $X$ an indecomposable module, generalizing earlier results about directing modules proved in [6], [8]. In fact, it follows from [7] and [16] that an indecomposable module $M$ in $\text{mod} A$ lies on a short cycle $M \to X \to M$ in $\text{mod} A$ if and only if $M$ is the middle term of a chain $Y \to M \to D\text{Tr} Y$ of non-zero homomorphisms in $\text{mod} A$ with $Y$ a non-projective indecomposable module. Hence the above result from [16] gives in fact another interpretation of a result from [3]. An important combinatorial and homological invariant of the module category $\text{mod} A$ of an Artin algebra $A$ is its Auslander-Reiten quiver $\Gamma_A$ [4]. Sometimes, we may recover the algebra $A$ and the category $\text{mod} A$ from the shape of the components $\mathcal{C}$ of $\Gamma_A$ and their behaviour in the category $\text{mod} A$. By a component of $\Gamma_A$ we mean a connected component of the translation quiver $\Gamma_A$.

In this article we are concerned with the problem of finding handy criteria for a component $\mathcal{C}$ of the Auslander-Reiten quiver $\Gamma_A$ of an Artin algebra $A$ to be uniquely determined in $\Gamma_A$ by the composition factors of its indecomposable modules. We say that two components $\mathcal{C}$ and $\mathcal{D}$ of $\Gamma_A$ have the same composition factors if, for any element $x \in K_0(A)$, $x = [M]$ for an indecomposable module $M$ in $\mathcal{C}$ if and only if $x = [N]$ for an indecomposable module $N$ in $\mathcal{D}$.

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In order to state the main results, we recall some concepts. For an Artin algebra $A$, we denote by $\text{rad} A$ the Jacobson radical of $\text{mod} A$, generated by all non-isomorphisms between indecomposable modules in $\text{mod} A$, and by $\text{rad}_A^n$ the infinite radical of $\text{mod} A$, which is the intersection of all powers $\text{rad}_A^i$, $i \geq 1$, of $\text{rad} A$. Recall that, by a result of M. Auslander [2], $\text{rad}_A^n = 0$ if and only if $A$ is of finite representation type, that is, there are in $\text{mod} A$ only finitely many indecomposable modules up to isomorphism. Following [24], a component quiver $\Sigma_A$ of $A$ is the quiver whose vertices are the components $\mathcal{C}$ of $\Gamma_A$, and two components $\mathcal{C}$ and $\mathcal{D}$ of $\Gamma_A$ are linked in $\Sigma_A$ by an arrow $\mathcal{C} \to \mathcal{D}$ provided $\text{rad}^A(X,Y) \neq 0$ for some modules $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. We note that a component $\mathcal{C}$ of $\Gamma_A$ is generalized standard in the sense of [22] if and only if $\Sigma_A$ has no loop at $\mathcal{C}$. By a short cycle in $\Sigma_A$ we mean a cycle $\mathcal{C} \to \mathcal{D} \to \mathcal{C}$, where possibly $\mathcal{C} = \mathcal{D}$. We also mention that a component $\mathcal{C}$ of $\Gamma_A$ lies on a short cycle $\mathcal{C} \to \mathcal{D} \to \mathcal{C}$ in $\Sigma_A$ with $\mathcal{C} \neq \mathcal{D}$ if and only if $\mathcal{C}$ has an external short path $X \to Y \to Z$ with $X$ and $Z$ in $\mathcal{C}$ and $Y$ in $\mathcal{D}$ [15]. Recall also that a translation quiver of the form $\mathbb{Z}A_\infty/(r^n)$, $r \geq 1$, is called a stable tube of rank $r$. We note that every regular component (without projective modules and injective modules) of the Auslander-Reiten quiver $\Gamma_A$ of an Artin algebra $A$ is either a stable tube or is acyclic (without oriented cycles) of the form $\mathbb{Z}A^\infty$ for an acyclic locally finite connected valued quiver $\Delta$ (see [13], [27]).

The following theorem is the first main result of this article.

**Theorem 1.** Let $A$ be an Artin algebra and $\mathcal{C}$ and $\mathcal{D}$ be two components of $\Gamma_A$ with the same composition factors. Assume that $\mathcal{C}$ is not a stable tube of rank one and does not lie on a short cycle in $\Sigma_A$. Then $\mathcal{C} = \mathcal{D}$.

The above theorem says that a generalized standard Auslander-Reiten component $\mathcal{C}$ of an Artin algebra $A$ without external short paths, different from a stable tube of rank one, is uniquely determined in $\Gamma_A$ by the composition factors of its indecomposable modules. We point out that the assumption on $\mathcal{C}$ not being a stable tube of rank one is essential for the validity of the above theorem. For example, if $H$ is the path algebra $K\Delta$ of a Euclidean quiver $\Delta$ over an algebraically closed field $K$, then the component quiver $\Sigma_H$ of $H$ is acyclic and the Auslander-Reiten quiver $\Gamma_H$ of $H$ contains infinitely many pairwise different stable tubes of rank one having the same composition factors (see [17], [20]).

The second main result of the article clarifies the situation in general.

**Theorem 2.** Let $A$ be an Artin algebra, $\mathcal{C}$ a stable tube of rank one in $\Gamma_A$ which does not lie on a short cycle in $\Sigma_A$, and $\mathcal{D}$ a component of $\Gamma_A$ different from $\mathcal{C}$ and having the same composition factors as $\mathcal{C}$. Then there is a quotient algebra $B$ of $A$ such that the following statements hold:

(a) $B$ is a concealed canonical algebra.
(b) $\mathcal{C}$ and $\mathcal{D}$ are stable tubes of a separating family of stable tubes of $\Gamma_B$.
(c) $\mathcal{D}$ is a stable tube of rank one.

Recall that a concealed canonical algebra is an algebra of the form $B = \text{End}_A(T)$, where $\Lambda$ is a canonical algebra in the sense of C. M. Ringel [19] (see also [17]) and $T$ is a multiplicity-free tilting module in the additive category $\text{add}(\mathcal{P}^\Lambda)$, for the canonical decomposition $\Gamma_A = \mathcal{P}^\Lambda \vee \mathcal{T}^\Lambda \vee Q^\Lambda$ of $\Gamma_A$, with $\mathcal{T}^\Lambda$ the canonical infinite separating family of stable tubes of $\Gamma_A$. Then $\Gamma_B$ admits a decomposition $\Gamma_B = \mathcal{P}^B \vee \mathcal{T}^B \vee Q^B$, where the image $\mathcal{T}^B = \text{Hom}_A(T, \mathcal{T}^\Lambda)$ of the family $\mathcal{T}^\Lambda$ via the functor $\text{Hom}_A(T, -) : \text{mod} \Lambda \to \text{mod} B$ is an infinite separating family of stable
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2. Proofs of Theorems 1 and 2

Let \( A \) be an Artin algebra over a commutative Artin ring \( R \). We denote by \( \tau_A \) and \( \tau_A^\perp \) the Auslander-Reiten translations \( D \tau \) and \( \tau D \), respectively. For a module \( V \) in \( \text{mod} \ R \), we denote by \( |V| \) its length over \( R \). In the proofs a crucial role will be played by the following formulas from [23, Proposition 4.1], their being consequences of [3, (1.4)] (see also [4, Corollary IV.4.3]).

For indecomposable modules \( M, N \) and \( X \) in \( \text{mod} \ A \) with \( |M| = |N| \) the following equalities hold:

\[
\begin{align*}
(i) & \quad |\text{Hom}_A(X, M)| = |\text{Hom}_A(M, \tau_A X)| = |\text{Hom}_A(X, N)| = |\text{Hom}_A(N, \tau_A X)|, \\
(ii) & \quad |\text{Hom}_A(M, X)| = |\text{Hom}_A(\tau_A^\perp X, M)| = |\text{Hom}_A(N, X)| = |\text{Hom}_A(\tau_A^\perp X, N)|.
\end{align*}
\]

Let \( \mathcal{C} \) and \( \mathcal{D} \) be components of \( \Gamma_A \) with the same composition factors such that \( \mathcal{C} \) does not lie on a short cycle in \( \Sigma_A \). We assume that \( \mathcal{C} \neq \mathcal{D} \) and show in several steps that \( \mathcal{C} \) and \( \mathcal{D} \) are stable tubes of rank one of a separating family of stable tubes in the Auslander-Reiten quiver \( \Gamma_B \) of a concealed canonical algebra \( B \).

(1) \( \mathcal{C} \) is a semi-regular component of \( \Gamma_A \) (\( \mathcal{C} \) does not contain both a projective module and an injective module). Assume \( \mathcal{C} \) contains a projective module \( P \) and an injective module \( I \). Since \( \mathcal{C} \) and \( \mathcal{D} \) have the same composition factors, there exist modules \( M \) and \( N \) in \( \mathcal{D} \) such that \( |P| = |M| \) and \( |I| = |N| \). Then we have \( \text{Hom}_A(P, M) \neq 0 \) and \( \text{Hom}_A(N, I) \neq 0 \), because the top of \( P \) is a composition factor of \( M \), and the socle of \( I \) is a composition factor of \( N \). Hence, we have in \( \Sigma_A \) the short cycle \( \mathcal{C} \to \mathcal{D} \to \mathcal{C} \), because \( \text{Hom}_A(P, M) = \text{rad}_A^\infty(P, M) \) and \( \text{Hom}_A(N, I) = \text{rad}_A^\infty(N, I) \), a contradiction. Therefore, \( \mathcal{C} \) is a semi-regular component of \( \Gamma_A \).

(2) \( \mathcal{C} \) is a cyclic component of \( \Gamma_A \) (every module in \( \mathcal{C} \) lies on an oriented cycle in \( \mathcal{C} \)). Take a module \( X \) in \( \mathcal{C} \). It follows from our assumption that \( |X| = |Y| \) for some module \( Y \) in \( \mathcal{D} \), and so \( X \) is not uniquely determined by \( |X| \), because \( \mathcal{C} \neq \mathcal{D} \). Applying [16, Corollary 2.2], we conclude that we have in \( \text{mod} \ A \) a short cycle \( X \to Z \to X \). Observe that then \( Z \) belongs to \( \mathcal{C} \), because \( \mathcal{C} \) does not lie on a short cycle in \( \Sigma_A \). Moreover, since there is no loop at \( \mathcal{C} \) in \( \Sigma_A \), \( \mathcal{C} \) is a generalized standard component of \( \Gamma_A \), and hence \( \text{rad}_A^\infty(X, Z) = 0 \) and \( \text{rad}_A^\infty(Z, X) = 0 \). Then \( \text{Hom}_A(X, Z) \neq 0 \) and \( \text{Hom}_A(Z, X) \neq 0 \) imply that there exist paths of irreducible homomorphisms in \( \text{mod} \ A \) from \( X \) to \( Z \) and from \( Z \) to \( X \) (see [3, Proposition V.7.5]), and consequently there exists an oriented cycle in \( \mathcal{C} \) passing through \( X \) and \( Z \). Hence, \( \mathcal{C} \) is a cyclic component.

(3) \( \mathcal{C} \) is a ray tube (obtained from a stable tube by a finite number (possibly empty) of ray insertions) or a coray tube (obtained from a stable tube by a finite
number (possibly empty) of coray insertions) in the sense of [17] (4.5) (see also [21, XV.2]). This is a direct consequence of [14] (2.6), since by (1) and (2) $C$ is semi-regular with oriented cycles.

(4) We may assume (without loss of generality) that $C$ is a ray tube, hence without injective modules. Let $\text{ann}_A(C)$ be the annihilator of $C$ in $A$, that is, the intersection of the annihilators $\text{ann}_A(X) = \{a \in A \mid Xa = 0\}$ of all modules $X$ in $C$, and $B = A/\text{ann}_A(C)$. Then $C$ is a faithful component of $\Gamma_B$. Since $C$ does not lie on a short cycle in $\Sigma_A$, we conclude that $C$ is without external short paths [15]; that is, there are no paths $U \to V \to W$ in mod $A$ with $U$ and $W$ in $C$ but $V$ not in $C$. Then it follows from [9, Theorem 2] that $B$ is an almost concealed canonical algebra and $C$ is a faithful ray tube of a separating family $\mathcal{T}^B$ of ray tubes of $\Gamma_B$. Recall that then there exists a canonical algebra $\Lambda$ (in the sense of C. M. Ringel [17], [19]) such that $B = \text{End}_A(T)$ for a tilting module $T$ in the additive category $\text{add}(\mathcal{P}^A \cup T^A)$ of $\mathcal{P}^A \cup T^A$, for the canonical decomposition $\Gamma_A = \mathcal{P}^A \vee \mathcal{T}^A \vee \mathcal{Q}^A$ of $\Gamma_A$ with $\mathcal{T}^A$ the canonical separating family of stable tubes. By general theory (see [11], [12], [17], [19], [25]), $\Gamma_B$ admits a decomposition

$$\Gamma_B = \mathcal{P}^B \vee \mathcal{T}^B \vee \mathcal{Q}^B,$$

where $\mathcal{T}^B$ is a family of ray tubes separating $\mathcal{P}^B$ from $\mathcal{Q}^B$ (in the sense of [19]). In particular, $\mathcal{T}^B$ is an infinite family of pairwise orthogonal generalized standard ray tubes, $\text{Hom}_B(\mathcal{T}^B, \mathcal{P}^B) = 0$, $\text{Hom}_B(\mathcal{Q}^B, \mathcal{T}^B) = 0$, and $\text{Hom}_B(\mathcal{Q}^B, \mathcal{P}^B) = 0$. In fact, since $C$ is a faithful ray tube of $\mathcal{T}^B$, all ray tubes of $\mathcal{T}^B$ except $C$ are stable tubes. Moreover, the separation property of $\mathcal{T}^B$ implies that $\text{Hom}_B(\mathcal{X}, C) = 0$ for any component $\mathcal{X}$ from $\mathcal{P}^B$ and $\text{Hom}_B(C, \mathcal{Y}) = 0$ for any component $\mathcal{Y}$ from $\mathcal{Q}^B$. Moreover, we note that $\mathcal{Q}^B$ contains all indecomposable injective $B$-modules.

(5) $\mathcal{D}$ is a component of $\Gamma_B$. Write $A = P' \oplus P''$, where the simple summands of $P'/\text{rad} P'$ are exactly the simple composition factors of modules in $C$. Denote by $t_{p''}(A)$ the ideal of $A$ generated by the images of all homomorphisms in mod $A$ from $P''$ to $A$. Since $C$ is a semi-regular component of $\Gamma_A$ without external short paths, it follows from arguments in [16, Section 1] that $\text{End}_A(P') \cong A/t_{p''}(A)$ and $\text{End}_A(P'') \cong \text{ann}_A(C)$. Observe that $1_A = e + f$ for orthogonal idempotents $e$ and $f$ in $A$ with $P' = eA$ and $P'' = fA$, and consequently $\text{End}_A(P') \cong eAe$ and $t_{p''}(A) = AfA$. Clearly, then $B = A/\text{ann}_A(C) \cong eAe$. On the other hand, since $\mathcal{D}$ has the same composition factors as $C$, we have $Nf = \text{Hom}_A(fA, N) = \text{Hom}_A(P'', N) = 0$, and consequently $\text{ann}_A(C) = N(AfA) = (Nf)A = 0$, for any module $N$ in $\mathcal{D}$. This shows that $\mathcal{D}$ is a component of $\Gamma_B$.

(6) $\mathcal{D}$ is a component of $\mathcal{T}^B$. Assume $\mathcal{D} \notin \mathcal{T}^B$. Fix a stable tube $\mathcal{T}^* \subseteq \mathcal{T}^B$ of rank one, which is different from $C$. By general theory ([11], [12], [19]), $B$ is a tubular (branch) extension of a concealed canonical algebra $C$ such that $\Gamma_C = \mathcal{P}^C \vee \mathcal{T}^C \vee \mathcal{Q}^C$, where $\mathcal{T}^C$ is a separating family of stable tubes, $\mathcal{P}^C = \mathcal{P}^C$, $\mathcal{Q}^C$ is obtained from a stable tube $T$ of $\mathcal{T}^C$ by a finite number (possibly empty) of ray insertions and the remaining tubes of $\mathcal{T}^C$ and $\mathcal{T}^B$ coincide ($\mathcal{T}^C \setminus \mathcal{T} = \mathcal{T}^B \setminus C$). Clearly, $C$ is a quotient algebra of $B$.

Let $M$ be a module in $\mathcal{C}$ which lies in $\mathcal{T}$. In particular, the composition factors of $M$ are $C$-modules. Take a module $N \in \mathcal{D}$ such that $[M] = [N]$. Assume $\mathcal{D} \in \mathcal{Q}^B$. Since $[M] = [N]$ there exists a projective module $P \in \mathcal{P}^B = \mathcal{P}^C$ such that $\text{Hom}_B(P, N) \neq 0$. By the separation property of $\mathcal{T}^B$ we have $\text{Hom}_B(X, N) \neq 0$
for some module $X \in \mathcal{T}^*$. Then, applying formula (i), we obtain
\[
0 = |\text{Hom}_A(X, M)| - |\text{Hom}_A(M, \tau_A X)| = |\text{Hom}_A(X, N)| - |\text{Hom}_A(N, \tau_A X)|
\]
\[
= |\text{Hom}_A(X, N)| > 0,
\]
since $M$ and $X$ belong to orthogonal tubes of $\mathcal{T}^B$ and $\text{Hom}_B(Q^B, \mathcal{T}^B) = 0$. Dually, if $\mathcal{D} \in \mathcal{P}^B$, then there exists an injective module $I$ in $Q^B$ such that $\text{Hom}_B(N, I) \neq 0$.

By the separation property of $\mathcal{T}^B$, we have $\text{Hom}_B(N, Y) \neq 0$ for some module $Y \in \mathcal{T}^*$. Then, by formula (ii), we have
\[
0 = |\text{Hom}_A(M, Y)| - |\text{Hom}_A(\tau_A^{-1} Y, M)| = |\text{Hom}_A(N, Y)| - |\text{Hom}_A(\tau_A^{-1} Y, N)|
\]
\[
= |\text{Hom}_A(N, Y)| > 0,
\]
since $M$ and $Y$ belong to orthogonal tubes of $\mathcal{T}^B$ and $\text{Hom}_B(\mathcal{T}^B, \mathcal{P}^B) = 0$. The above contradictions show that $\mathcal{D} \in \mathcal{T}^B$.

(7) $\mathcal{T}^B$ is a family of stable tubes. Assume $\mathcal{C}$ contains a projective module $P$. Take an indecomposable module $Y$ in $\mathcal{D}$ with $[P] = [Y]$. Then the top of $P$ is a composition factor of $Y$ and hence $\text{Hom}_B(P, Y) \neq 0$. Therefore, $\text{Hom}_B(\mathcal{C}, \mathcal{D}) \neq 0$, which contradicts the fact that $\mathcal{C}$ and $\mathcal{D}$ are orthogonal. We conclude that $\mathcal{C}$ is a stable tube of $\mathcal{T}^B$. Clearly, then $\mathcal{T}^B$ is a separating family of stable tubes of $\Gamma_B$, and consequently $B$ is a concealed canonical algebra, by [11].

(8) $\mathcal{C}$ and $\mathcal{D}$ are stable tubes of rank one. Since $\mathcal{C}$ and $\mathcal{D}$ belong to the separating family $\mathcal{T}^B$ of stable tubes of $\Gamma_B$, we know that $\mathcal{C}$ and $\mathcal{D}$ are orthogonal, generalized standard, and without external short paths. In particular, $\mathcal{C}$ and $\mathcal{D}$ do not lie on short cycles in $\Sigma_B$. Then, applying [23] Lemmas 3.1 and 3.3], we conclude that $\mathcal{C}$ and $\mathcal{D}$ consist of modules which do not lie on infinite short cycles in mod $B$. Assume $\mathcal{C}$ is of rank $r \geq 2$. Take a module $X$ lying on the mouth of $\mathcal{C}$ ($X$ has one immediate predecessor and one immediate successor in $\mathcal{C}$). Then, by [23] Corollary 4.4), $X$ is uniquely determined by $[X]$, which contradicts the fact that $[X] = [Y]$ for some module $Y$ in $\mathcal{D}$ and $\mathcal{C} \neq \mathcal{D}$. Therefore, $\mathcal{C}$ is of rank one. Applying the same arguments, we conclude that $\mathcal{D}$ is also of rank one.

Summing up, the proofs of Theorems 1 and 2 are provided.

3. Examples

Let $K$ be an algebraically closed field and $Q$ be a finite quiver. For any arrow $\alpha \in Q$, by $s(\alpha)$ and $t(\alpha)$ we mean the source and the target of $\alpha$, respectively. By $KQ$ we denote the path algebra of $Q$. Recall that, if the quiver $Q$ is acyclic, then $KQ$ is a hereditary algebra [1]. For a finite-dimensional algebra $H$ over $K$, we denote by $T(H)$ the trivial extension algebra of $H$ by its duality $H$-$H$-bimodule $D(H) = \text{Hom}_K(H, K)$. Recall that $T(H) = H \oplus D(H)$ as a $K$-vector space and the multiplication in $T(H)$ is given by $(a, f)(b, g) = (ab, ag + fb)$ for $a, b \in H$ and $f, g \in D(H)$. Then $T(H)$ is a symmetric algebra and $H$ is the quotient algebra of $T(H)$ by the ideal $D(H)$.

For a natural number $n \geq 4$, $Q_n$ will be the quiver of the following form:

```
1 ---- 2 ---- 3 ---- 4 ---- \cdots ---- n-2 ---- n-1 ---- n + 1
```

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Each arrow in $Q_n$ will be named either by $\alpha$ or by $\beta$ in such a way that an arrow which starts in the vertex 3 and terminates in the vertex 1 is $\alpha$, and $s(\alpha) = t(\beta)$, $t(\alpha) = s(\beta)$, for all arrows $\alpha$ and $\beta$. Let $I_n$ be the admissible ideal in the path algebra $KQ_n$ generated by all paths $\alpha\beta, \beta\alpha$ such that $s(\alpha\beta) \neq t(\alpha\beta), s(\beta\alpha) \neq t(\beta\alpha)$, and all commutativity relations $\omega_1 - \omega_2$, where $\omega_1, \omega_2$ are all paths of length 2 in $Q_n$ such that their source and target coincide with the vertex $i$, for all $i \in \{3, \ldots, n - 1\}$. Then by $\Lambda_n$ we denote the quotient algebra $KQ_n/I_n$.

We consider now the quiver $\Delta_n$ of Euclidean type $D_n$, for any $n \geq 4$, defined in the following way. If $n$ is an odd number, then $\Delta_n$ is of the form:

\[
\begin{array}{cccccccc}
1 & \longrightarrow & 3 & \leftarrow & 4 & \leftarrow & \cdots & \leftarrow & n - 2 & \leftarrow & n - 1 & \leftarrow & n & \leftarrow & n + 1
\end{array}
\]

and similarly, for an even number $n$, the quiver $\Delta_n$ is of the form:

\[
\begin{array}{cccccccc}
1 & \longrightarrow & 3 & \leftarrow & 4 & \leftarrow & \cdots & \leftarrow & n - 2 & \leftarrow & n - 1 & \leftarrow & n & \leftarrow & n + 1
\end{array}
\]

(in particular, all maximal subquivers of type $A_{n-1}$ of $Q_n$ have an alternate orientation of arrows).

Let $H_n$ be the path algebra $K\Delta_n$ and $H_n^*$ the path algebra $K\Delta_n^*$, where $\Delta_n^*$ is the opposite quiver of $\Delta_n$. Note that $\Delta_n$ is a subquiver of $Q_n$ given by the arrows $\alpha$ and $\Delta_n^*$ is a subquiver of $Q_n$ given by the arrows $\beta$. Moreover, observe that $\Lambda_n$ is the trivial extension algebra $T(H_n)$ of $H_n$ and the trivial extension algebra $T(H_n^*)$ of $H_n^*$. In particular, $H_n$ and $H_n^*$ are quotient algebras of $\Lambda_n$.

Assume now that $n > 4$ is an odd number. For each arrow $\alpha$ in $\Delta_n$ such that $s(\alpha) = i$ and $t(\alpha) \in \{i - 1, i + 1\}$, for some $i \in \{3, \ldots, n\}$, we put $\alpha_l$ instead of $\alpha$, where $l$ is given by the formula:

\[
l = \begin{cases}
\frac{i - 1}{2}, & \alpha : i \rightarrow i + 1, \\
\frac{n - i + 1}{2}, & \alpha : i \rightarrow i - 1.
\end{cases}
\]

Observe that $l \in \{1, \ldots, n - 2\}$. We define the family of indecomposable representations $F_{\alpha_1}, \ldots, F_{\alpha_{n-2}}$ of $H_n$ over $K$:

- $F_{\alpha_l}$ for $l \notin \{\frac{n - 1}{2}, n - 2\}$:

\[
\begin{array}{cccccccc}
0 & \longrightarrow & \cdots & K & \longrightarrow & 0 & \longrightarrow & \cdots & K & \longrightarrow & 0 & \longrightarrow & \cdots & 0
\end{array}
\]

where $K$ stands in for the vertices $s(\alpha_l), t(\alpha_l)$, zero space elsewhere (here by $\longrightarrow$ we mean $\longrightarrow$ or $\leftarrow$).
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\[ F_{\alpha_{n-1}} : \]

\[
\begin{array}{c}
\downarrow 1 \\
K \\
\downarrow 1 \\
0 \\
\end{array}
\begin{array}{c}
\downarrow 1 \\
0 \\
\downarrow 0 \\
\end{array}
\begin{array}{c}
\leftarrow \cdots \leftarrow 0 \\
\end{array}
\begin{array}{c}
\leftarrow \cdots \leftarrow 0 \\
\end{array}
\begin{array}{c}
\downarrow 1 \\
K \\
\downarrow 1 \\
0 \\
\end{array}
\]

where \( K \) stands in for the vertices \( n - 1, n, n + 1 \), zero space elsewhere;

\[ F_{\alpha_{n-2}} : \]

\[
\begin{array}{c}
\downarrow 1 \\
K \\
\downarrow 1 \\
0 \\
\end{array}
\begin{array}{c}
\downarrow 1 \\
K \\
\downarrow 1 \\
0 \\
\end{array}
\begin{array}{c}
\leftarrow \cdots \leftarrow 0 \\
\end{array}
\begin{array}{c}
\leftarrow \cdots \leftarrow 0 \\
\end{array}
\begin{array}{c}
\downarrow 1 \\
K \\
\downarrow 1 \\
0 \\
\end{array}
\]

where \( K \) stands in for the vertices \( 1, 2, 3 \), zero space elsewhere.

Let \( E_l = F_{\alpha_l} \) for \( l \in \{1, \ldots, n - 2\} \). Obviously \( E_1, \ldots, E_{n-2} \) are pairwise orthogonal bricks. Direct calculation shows that \( \tau_{H_n} E_{l+1} = E_l \) if \( l \in \{1, \ldots, n-3\} \) and \( \tau_{H_n} E_1 = E_{n-2} \). Moreover, \( \Ext_{H_n}^2(E_r, E_p) = 0 \) for any \( r, p \in \{1, \ldots, n-2\} \), because \( H_n \) is a hereditary algebra. It allows us to state that \( E_1, \ldots, E_{n-2} \) form the mouth of a standard stable tube \( T \) of rank \( n - 2 \) in \( \Gamma_{H_n} \) (see [17], [20]). Since \( \text{pd}_{H_n} X \leq 1 \) for any \( H_n \)-module \( X \) in \( T \), it follows from [26] Proposition 1.1] that \( T \) is also a component of the Auslander-Reiten quiver \( \Gamma_{\Lambda_n} \).

Analogously, let \( E_1^*, E_2^*, \ldots, E_{n-2}^* \) be the indecomposable \( H_n^* \)-modules, where the indices \( l \) are given in such a way that, for any \( l \in \{1, \ldots, n-2\} \), \( E_l \) and \( E_l^* \) have the same composition factors in \( \text{mod} \Lambda_n \) including the multiplicities. It is easy to see that these modules form the mouth of a stable tube \( T^* \) of rank \( n - 2 \) in \( \Gamma_{H_n^*} \) such that \( \tau_{H_n^*} E_l^* = E_l^* \) for \( l \in \{1, \ldots, n-3\} \) and \( \tau_{H_n^*} E_{n-2}^* = E_1^* \). Using once more [26] Proposition 1.1] we get that \( T^* \) is also a component of the Auslander-Reiten quiver \( \Gamma_{\Lambda_n} \). Note that \( \text{top}(E_l^*) = \text{soc}(E_l) \) and \( \text{top}(E_l) = \text{soc}(E_l^*) \) in \( \text{mod} \Lambda_n \), for any \( l \in \{1, \ldots, n-2\} \). Therefore, \( T \) has an external short path \( E_l \to E_l^* \to E_l \) in \( \text{mod} \Lambda_n \), which implies existence of a short cycle \( T \to T^* \to T \) in \( \Sigma_{\Lambda_n} \). Observe also that \( T \) and \( T^* \) have the same composition factors since \( [E_l] = [E_l^*] \) for all \( l \in \{1, \ldots, n-2\} \). Moreover, \( T \) and \( T^* \) are generalized standard stable tubes in \( \Gamma_{\Lambda_n} \), since they are generalized standard in \( \Gamma_{H_n} \) and \( \Gamma_{H_n^*} \), respectively (see for example [20] Chapter X).

Assume \( n \geq 4 \) is an even number. For each arrow \( \alpha \) in \( H_n \) such that \( s(\alpha) = i \) and \( t(\alpha) \in \{i-1, i+1\} \), for some \( i \in \{3, \ldots, n-1\} \), we define the index \( l \) in the previous way. Similarly, we define the indecomposable representations \( F_{\alpha_1}, \ldots, F_{\alpha_{n-2}} \) of \( H_n \) over \( K \):

\[ F_{\alpha_l} \quad \text{for} \quad l \notin \{\frac{n-2}{2}, n-2\} : \]

\[
\begin{array}{c}
\downarrow 1 \\
0 \\
\downarrow 0 \\
\end{array}
\begin{array}{c}
\leftarrow \cdots \leftarrow K \\
\end{array}
\begin{array}{c}
\leftarrow \cdots \leftarrow 0 \\
\end{array}
\begin{array}{c}
\downarrow 1 \\
0 \\
\downarrow 0 \\
\end{array}
\]

where \( K \) stands in for the vertices \( n - 1, n, n + 1 \), zero space elsewhere;
where \( K \) stands in for the vertices \( s(\alpha_1), t(\alpha_1) \), zero space elsewhere
(here by \( \rightarrow \rightarrow \) we mean \( \rightarrow \rightarrow \) or \( \rightarrow \rightarrow \));

- \( F_{\alpha_{n-2}} : \)

\[
\begin{array}{c}
0 \\
\rightarrow \\
0 & 0 & \cdots & \cdots & \cdots & 0 \\
\leftarrow \\
0
\end{array}
\]

where \( K \) stands in for the vertices \( n-1, n, n+1 \), zero space elsewhere;

- \( F_{\alpha_{n-2}} : \)

\[
\begin{array}{c}
K \\
\rightarrow \\
0 & \cdots & \cdots & 0 & 0 \\
\leftarrow \\
K
\end{array}
\]

where \( K \) stands in for the vertices 1, 2, 3, zero space elsewhere.

As before the modules \( E_l = F_{\alpha_l}, l \in \{1, \ldots, n-2\} \), form the mouth of a stable tube \( T \) of rank \( n-2 \) in \( \Gamma_{H_n} \) in such a way that \( \tau_{H_n} E_{l+1} = E_l \) for \( l \in \{1, \ldots, n-3\} \) and \( \tau_{H_n} E_1 = E_{n-2} \). Similarly, let \( T^* \) be the stable tube of rank \( n-2 \) in \( \Gamma_{H_n} \) whose mouth consists of the modules \( E_1^*, E_2^*, \ldots, E_{n-2}^* \), where the indices \( l \) are given in such a way that, for any \( l \in \{1, \ldots, n-2\} \), \( E_l \) have the same composition factors and \( \top(E_l^*) = \soc(E_l), \top(E_l^*) = \soc(E_l^*) \) in mod \( \Lambda_n \). Therefore, there is a short cycle \( T \to T^* \to T \) in \( \Sigma_{\Lambda_n} \). Moreover, \( T \) and \( T^* \) are generalized standard components in \( \Gamma_{\Lambda_n} \).

Summing up, we have proved that for an arbitrary \( m \geq 2 \), the Auslander-Reiten quiver \( \Gamma_{\Lambda_{m+2}} \) of \( \Lambda_{m+2} \) contains a generalized standard stable tube of rank \( m \) which is not uniquely determined by its composition factors.

We end this section with comments concerning acyclic generalized standard Auslander-Reiten components. It has been proved in [22] Corollaries 2.4 and 3.3] that every acyclic generalized standard component \( C \) of the Auslander-Reiten quiver \( \Gamma_A \) of an Artin algebra \( A \) is of the form \( \mathbb{Z} \Delta \) for a finite acyclic connected valued quiver \( \Delta \) with at least three vertices, \( B = A/\ann_A(\mathcal{C}) \) is a tilted algebra of the form \( \End_H(T) \), for some wild hereditary Artin algebra \( H \) and a regular tilting \( H \)-module, and \( \mathcal{C} \) is the connecting component \( \mathcal{C}_T \) of \( \Gamma_B \) determined by \( T \). Moreover, C. M. Ringel proved in [18] that, for any connected wild hereditary Artin algebra \( H \) whose ordinary valued quiver has at least three vertices, there exists a multiplicity-free regular tilting module \( T \) in mod \( H \), and consequently the connecting component \( \mathcal{C}_T \) of the Auslander-Reiten quiver \( \Gamma_B \) of the associated tilted algebra \( B = \End_H(T) \) is an acyclic generalized standard faithful regular component of \( \Gamma_B \). We refer also to [10] for constructions of tilted algebras having regular connecting components with arbitrary large composition factors.

Let \( K \) be an algebraically closed field, \( Q \) an arbitrary connected acyclic wild quiver with at least three vertices, and \( H = KQ \). Then it follows from [11] Corollary 4 that there are infinitely many pairwise non-isomorphic tilted algebras \( B = \End_H(T) \), for multiplicity-free regular tilting modules \( T \) in mod \( H \), such that the connecting component \( \mathcal{C}_T \) determined by \( T \) is regular and without simple
modules. Take such a tilted algebra $B = \text{End}_H(T)$ and consider the trivial extension algebra $\Lambda = T(B)$ of $B$ by the $B$-$B$-bimodule $D(B) = \text{Hom}_K(B,K)$. Then it follows from [5] Section 5] that the Auslander-Reiten quiver $\Gamma_\Lambda$ of $\Lambda$ consists of two acyclic generalized standard regular sincere components $C = C_T$ and $D$, having sections of type $\Delta = Q^{op}$, and infinitely many components whose stable parts are of the form $\mathbb{Z}A_\infty$. However, it is not clear if $C$ and $D$ may have the same composition factors. It would be interesting to know if such a situation may occur.

References


Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland
E-mail address: jaworska@mat.uni.torun.pl

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland
E-mail address: pmalicki@mat.uni.torun.pl

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland
E-mail address: skowron@mat.uni.torun.pl