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SCHRÖDINGER OPERATORS WITH A COMPLEX VALUED POTENTIAL

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Abstract. If M_n is a compact Riemannian manifold for which $H^1(M_n,\mathbb{Z}) = 0$, V is a continuous complex valued function whose imaginary part is of constant sign, and $-\Delta\psi + V\psi = 0$ for some C^2 complex valued function ψ on M_n, then either ψ vanishes somewhere or there is a constant c and an everywhere positive function F such that $\psi = cF$.

Let M_n be a smooth compact oriented connected Riemannian manifold for which $H^1(M_n,\mathbb{Z}) = 0$. Let Λ be the $n-$form on M_n giving rise to the standard measure there, and let V be a continuous complex valued function on M_n such that either the imaginary part of V is greater than or equal to zero everywhere or the imaginary part is less than or equal to zero everywhere.

Theorem 1. *If ψ is a complex valued function of class C^2 on M_n such that $-\Delta\psi + V\psi = 0$, then either $\psi(x) = 0$ for some x or there is a constant c and an everywhere positive function F such that $\psi = cF$.*

Proof. Suppose ψ is different from 0 everywhere. Since $H^1(M_n,\mathbb{Z}) = 0$, it follows that there exist real valued continuous functions k and u such that $\psi = e^{(k+iu)}$. Because M_n is compact, there is at least one point p at which u assumes a maximum value.

It is known that one can always find a coordinate patch about p in which $\Lambda = dx_1 \wedge \cdots \wedge dx_n$. In [2] it is proven that if $D_{\nabla u}$ denotes Lie differentiation with respect to the vector field ∇u, then the imaginary part of $(\Delta\psi/\psi)e^{2k}\Lambda$ equals $D_{\nabla u}(e^{2k}\Lambda)$. A computation in [2] shows that in our special coordinate system,

$$D_{\nabla u}(e^{2k}\Lambda) = \left(\left[2g^{ij}\frac{\partial k}{\partial x_i} + \frac{\partial g_{ij}}{\partial x_i}\right]\frac{\partial u}{\partial x_j} + g^{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}\right)e^{2k}dx_1 \wedge \cdots \wedge dx_n.$$

Since $\Delta\psi/\psi = V$, if $\mathrm{Im}(V) \geq 0$ everywhere, then the same thing holds for the above expression. However, a theorem of E. Hopf [1] tells us that when u satisfies an inequality of the form

$$h^j\frac{\partial u}{\partial x_j} + g^{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} \geq 0$$

in a connected region and assumes a maximum at some point in the region, then u must be constant throughout the region.

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The function u that satisfies $\psi = e^{k+iu}$ must therefore be constant because the set of points at which u assumes its maximum must be both open and closed and M_n is connected. Hence ψ equals a constant times a real valued function. Since ψ vanishes nowhere, this real valued function must be of constant sign. It follows that ψ is a constant times a positive function.

There is a parallel statement when

$$h^i \frac{\partial u}{\partial x_i} + g^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \leq 0$$

everywhere, so a similar argument shows that this assumption yields the same conclusion in this case.

If one assumes V is real valued, then our present result yields the theorem at the end of [2] in the case where the manifold has no boundary.

Actually, the case where there is a boundary is more interesting than our present theorem, even though it only applies when V is real valued, because the case where there is a boundary includes the situation in which M_n is the closure of the interior of an $(n-1)-$sphere that is smoothly imbedded in $\mathbb{R}^n$.

It should be mentioned that the principal part of [2] is devoted to proving a theorem of the present type in the case in which M_n is complete rather than compact. To compensate for the absence of compactness, one assumes that our real valued potential V is bounded below. $\square$

References

[1] Bochner, S. and Yano, K. *Curvature and Betti Numbers*, Princeton University Press, 1953 (p. 26). MR0062505 (15:989f)

[2] Sol Schwartzman, *Schrödinger Operators and the Zeroes of Their Eigenfunctions*, Communications in Mathematical Physics **306** (2011), 187–191. MR2819423 (2012e:81079)

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