KAZHDAN’S PROPERTY (T) WITH RESPECT TO NON-COMMUTATIVE $L_p$-SPACES

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Abstract. We show that a group with Kazhdan’s property (T) has property $(TB)$ for $B$ the Haagerup non-commutative $L_p(M)$-space associated with a von Neumann algebra $M$, $1 < p < \infty$. We deduce that higher rank groups have property $F_{L_p(M)}$.

1. Introduction

Kazhdan’s property (T) of a topological group $G$ is an important rigidity property, defined in terms of the unitary representations of $G$ on Hilbert spaces. We recall the precise definition:

Definition 1.1. A pair $(G, H)$ of topological groups, where $H$ is a closed subgroup of $G$, is said to have relative property (T) if there exist a compact subset $Q$ of $G$ and $\epsilon > 0$ such that whenever a unitary representation $\pi$ of $G$ on a Hilbert space $H$ has a $(Q, \epsilon)$-invariant vector, that is, a vector $\xi \in H$ such that

$$\sup_{g \in Q} ||\pi(g)\xi - \xi|| < \epsilon ||\xi||,$$

then $\pi$ has a non-zero $\pi(H)$-invariant vector. The pair $(Q, \epsilon)$ is called a Kazhdan pair.

A topological group $G$ is said to have property (T) if the pair $(G, G)$ has relative property (T).

For more details on property (T), see the monograph [2].

The following variant of this property for Banach spaces was recently introduced by Bader, Furman, Gelander and Monod in [1]. Let $B$ be a Banach space and $O(B)$ the orthogonal group of $B$, that is, the group of linear bijective isometries of $B$.

Recall that an orthogonal representation of a topological group $G$ on a Banach space $B$ is a homomorphism $\rho : G \to O(B)$ such that the map $g \mapsto \rho(g)x$ is continuous for every $x \in B$. If $\rho : G \to O(B)$ is an orthogonal representation of a group $G$, we denote the subspace of $\rho(G)$-invariant vectors by

$$B^{\rho(G)} = \{ x \in B \mid \rho(g)x = x \text{ for all } g \in G \}.$$ 

Observe that $B^{\rho(G)}$ is invariant under $G$. The representation $\rho$ is said to almost have invariant vectors if it has a $(Q, \epsilon)$-invariant vector for every compact subset $Q$ of $G$ and $\epsilon > 0$.
Definition 1.2. Let $G$ be a topological group and $H$ be a closed normal subgroup of $G$. The pair $(G, H)$ has relative property $(T_B)$ for a Banach space $B$ if, for any orthogonal representation $\rho : G \to O(B)$, the quotient representation $\rho' : G \to O(B/B^{\rho(H)})$ does not almost have $\rho'(G)$-invariant vectors.

A topological group $G$ has property $(T_B)$ if the pair $(G, G)$ has relative property $(T_B)$.

The authors of [1] studied the case where $B$ is a superreflexive Banach space, and among other things, they showed that a group which has property $(T_B)$ has property $(T_{L^p(\mu)})$ for $\mu$ a $\sigma$-finite measure on a standard Borel space $(X, \mathcal{B})$ and $1 < p < \infty$. We will extend this result to the non-commutative setting.

Non-commutative $L_p$-spaces were introduced by Dixmier [3] and studied by various authors, among them Yeadon [13] and Haagerup [4] (for a survey on these spaces, see Pisier and Xu [6]). Apart from the standard $L^p(\mu)$-spaces, common examples are the $p$-Schatten ideals $S_p = \{x \in \mathcal{B}(\mathcal{H}) \mid \text{tr}(|x|^p) < \infty\}$, where $\mathcal{H}$ is a separable Hilbert space.

We review below (in Section 2) Haagerup’s definition of these non-commutative $L_p$-spaces. Here is our main result:

**Theorem 1.3.** Let $G$ be a topological group and $H$ a closed normal subgroup of $G$. Assume that the pair $(G, H)$ has relative property $(T)$ for every von Neumann algebra $\mathcal{M}$, the pair $(G, H)$ has relative property $(T_{L^p(\mathcal{M})})$ for $1 < p < \infty$.

In particular, if $G$ has property $(T)$, then $G$ has property $(T_{L^p(\mathcal{M})})$ for $1 < p < \infty$. Property $(T_B)$ has a stronger version which is a fixed point property for affine actions.

Definition 1.4. Let $B$ be a Banach space. A topological group $G$ has property $(F_B)$ if every continuous action of $G$ by affine isometries on $B$ has a $G$-fixed point.

The authors of [1] showed that higher rank groups and their lattices have property $(F_{L^p(\mu)})$.

Definition 1.5. For $1 \leq i \leq m$, let $k_i$ be local fields and $G_i(k_i)$ be the $k_i$-points of connected simple $k_i$-algebraic groups $G_i$. Assume that each simple factor $G_i$ has $k_i$-rank $\geq 2$. The group $G = \prod_{i=1}^m G_i(k_i)$ is called a higher rank group.

Our next result shows that Theorem B in [1] remains true for non-commutative $L_p$-spaces.

**Theorem 1.6.** Let $G$ be a higher rank group and $\mathcal{M}$ a von Neumann algebra. Then $G$, as well as every lattice in $G$, has property $(F_{L^p(\mathcal{M})})$ for $1 < p < \infty$.

Theorem 1.6 was proved by Puschnigg in [7] in the case $L^p(\mathcal{M}) = S_p$. The strategy of the proof of Theorem [1,3] (as in [7]) follows the one from [1]. To achieve the result, we will need some results on the Mazur map and the description of the surjective isometries of $L_p(\mathcal{M})$ given by Sherman in [9].

The paper is organized as follows. In Section 2, useful properties of the Mazur map are established. Group representations on $L_p(\mathcal{M})$ are studied in Section 3. The proof of Theorem [1,3] is given in Section 4. In Section 5, we show how Theorem 1.6 can be obtained from a variant of Theorem [1,3].
2. SOME PROPERTIES OF THE MAZUR MAP

Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and equipped with a normal semi-finite weight $\varphi_0$. Let $t \mapsto \sigma_t^{\varphi_0}$ be the one-parameter group of modular automorphisms of $\mathcal{M}$ with respect to $\varphi_0$. We denote by $\mathcal{N}_{\varphi_0} = \mathcal{M} \rtimes_{\varphi_0} \mathbb{R}$ the crossed product von Neumann algebra, which is a von Neumann algebra acting on $L^2(\mathbb{R}, \mathcal{H})$ and generated by the operators $\pi_{\varphi_0}(x)$, $x \in \mathcal{M}$, and $\lambda_s$, $s \in \mathbb{R}$, defined by

$$\pi_{\varphi_0}(x)(\xi)(t) = \sigma_t^{\varphi_0}(x)\xi(t),$$

$$\lambda_s(\xi)(t) = \xi(t-s) \quad \text{for any } \xi \in L^2(\mathbb{R}, \mathcal{H}) \text{ and } t \in \mathbb{R}.$$ 

There is a dual action $s \mapsto \theta_s$ of $\mathbb{R}$ on $\mathcal{N}_{\varphi_0}$. Then let $\tau_{\varphi_0}$ be the semi-finite normal trace on $\mathcal{N}_{\varphi_0}$ satisfying

$$\tau_{\varphi_0} \circ \theta_s = e^{-s}\tau_{\varphi_0} \text{ for all } s \in \mathbb{R}.$$ 

We denote by $L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$ the $*$-algebra of $\tau_{\varphi_0}$-measurable operators affiliated with $\mathcal{N}_{\varphi_0}$. For $1 \leq p \leq \infty$, the Haagerup non-commutative $L_p$-space associated with $\mathcal{M}$ is defined by

$$L_p(\mathcal{M}) = \{x \in L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0}) \mid \theta_s(\xi)(x) = e^{-s/p}x \text{ for all } s \in \mathbb{R}\}.$$ 

It is known that this space is independent of the weight $\varphi_0$ up to isomorphism. The space $L_1(\mathcal{M})$ is isomorphic to $\mathcal{M}_+$. The identification goes as follows: there exists a normal faithful semi-finite operator-valued weight from $\mathcal{N}_{\varphi_0}$ to $\mathcal{M}$ defined by

$$\Phi_{\varphi_0}(x) = \pi_{\varphi_0}^{-1}(\int_{\mathbb{R}} \theta_s(x)ds), \quad \text{for } x \in \mathcal{N}_{\varphi_0}.$$ 

Now, if $\varphi \in \mathcal{M}_+^*$ and $\hat{\varphi}$ denotes the extension of $\varphi$ to a a normal weight on $\hat{\mathcal{M}}^+$, the extended positive part of $\mathcal{M}$, we then put

$$\hat{\varphi}^{\varphi_0} = \hat{\varphi} \circ \Phi_{\varphi_0}.$$ 

We associate to $\varphi$ the Radon-Nikodym derivative $d\hat{\varphi}^{\varphi_0}/d\tau_{\varphi_0}$ of $\hat{\varphi}^{\varphi_0}$ with respect to the trace $\tau_{\varphi_0}$. This isomorphism between $\mathcal{M}_+^*$ and $L_1(\mathcal{M})^+$ extends to the whole spaces by linearity.

If $x \in L_1(\mathcal{M})$, and $\varphi_x$ is the element of $\mathcal{M}_+^*$ associated to $x$, we define a linear functional $\text{Tr}$ by

$$\text{Tr}(x) = \varphi_x(1)$$

and we have, $p'$ being the conjugate exponent of $p$,

$$\text{Tr}(xy) = \text{Tr}(yx) \text{ for } x \in L_p(\mathcal{M}), \ y \in L_{p'}(\mathcal{M}).$$ 

For $1 \leq p < \infty$, if $x = u|x|$ is the polar decomposition of $x \in L_p(\mathcal{M})$, we define

$$||x||_p = \text{Tr}(|x|^p)^{1/p}.$$ 

Equipped with $||.||_p$, $L_p(\mathcal{M})$ is a Banach space. For $1 < p < \infty$, the dual space of $L_p(\mathcal{M})$ is $L_{p'}(\mathcal{M})$ and $L_p(\mathcal{M}, \tau)$ is known to be superreflexive.

We now introduce the Mazur map and establish some of its properties.
Definition 2.1. Let $1 \leq p, q < \infty$. For an operator $a$, let $\alpha |a|$ be its polar decomposition. The map

$$M_{p,q} : L_0(N_{\varphi_0, \tau_\varphi_0}) \to L_0(N_{\varphi_0, \tau_\varphi_0}),$$

$$x = \alpha |a| \mapsto \alpha |a|^{\frac{p}{q}}$$

is called the Mazur map.

We will need the following lemma.

Lemma 2.2. Let $1 \leq p, q, r < \infty$. Then $M_{r,q} \circ M_{p,r} = M_{p,q}$.

Proof. Let $\alpha |x|$ be the polar decomposition of $x \in L_0(N_{\varphi_0, \tau_\varphi_0})$. Let $\beta > 0$, and set $y = \alpha |x|^{\beta}$. We claim that the polar decomposition of $y$ is given by $\alpha$ and $|x|^{\beta}$. To show this, it suffices to prove that $\text{Im}(|x|^{\beta}) = \text{Im}(|x|)$.

By taking orthogonals, we have to show that $\text{Ker}(|x|) = \text{Ker}(|x|^{\beta})$ for all $\beta > 0$. Recall that the domain $D(|x|^{\beta})$ of $|x|^{\beta}$ is

$$D(|x|^{\beta}) = \{ \xi | \int_0^\infty \lambda^{2\beta} d\mu_\xi(\lambda) < \infty \}.$$

If $\xi \in \text{Ker}(|x|)$, we have for all $\eta \in L^2(\mathbb{R}, \mathcal{H})$,

$$\langle |x|\xi, \eta \rangle = \int_0^\infty \lambda d\mu_\xi,\eta(\lambda) = 0.$$

In particular, $\mu_\xi([0, \infty]) = 0$. So $\xi \in D(|x|^{\beta})$ and $\xi \in \text{Ker}(|x|^{\beta})$ thanks to

$$\langle |x|^{\beta} \xi, \eta \rangle = \int_0^\infty \lambda^\beta d\mu_\xi,\eta(\lambda) = 0.$$

By exchanging the role of $|x|$ and $|x|^{\beta}$, we get the equality.

Let $1 \leq p, q, r < \infty$, and $\beta = p/r$; then $M_{p,r}(x) = \alpha |x|^{\beta}$. It follows from what we have just seen that $M_{r,q}(M_{p,r}(x)) = \alpha |x|^{\frac{p}{q}} = M_{p,q}(x)$. \hfill $\square$

Proposition 2.3. Let $1 \leq p, q < \infty$, and $a \in L_p(M)$. Then

$$||M_{p,q}(a)||_q^q = ||a||_p^p.$$  

Proof. We denote again by $\alpha |a|$ the polar decomposition of $a$. We already seen that $|M_{p,q}(a)| = |a|^{\frac{p}{q}}$. So we have

$$\text{Tr}(|M_{p,q}(a)|^q) = \text{Tr}(|a|^p).$$ \hfill $\square$

Proposition 2.4. Let $p, q \in ]1, \infty[$ be conjugate. The map

$$L_p(M) \to L_q(M),$$

$$x \mapsto M_{p,q}(x)^*$$

is the duality map from $L_p(M)$ to $L_q(M)$.

Proof. We first notice that $M_{p,q}$ sends $L_p(M)$ into $L_q(M)$. Let $x = \alpha |x| \in L_p(M)$ and $s \in \mathbb{R}$. By uniqueness in the polar decomposition, we have $\theta_s(\alpha) = \alpha$ and $\theta_s(|x|) = e^{-s/p}|x|$, and then

$$\theta_s(M_{p,q}(x)) = \theta_s(\alpha) \theta_s(|x|^{\frac{p}{q}})$$

$$= \alpha (\theta_s(|x|))^{\frac{p}{q}}$$

$$= e^{-s/q} M_{p,q}(x).$$
Thanks to the uniqueness of the duality map in superreflexive spaces, we just have to check that \( \text{Tr}(M_{p,q}(a^*a)) = 1 \) for \( a \) in the unit sphere \( S(L_p(M)) \) of \( L_p(M) \).

Let \( a = \alpha|a| \in S(L_p(M)) \); then \( M_{p,q}(a) = \alpha|a|^{\frac{p}{p}} \). Since \( \alpha^*\alpha|a| = |a| \), it follows that
\[
\text{Tr}(|a|^{\frac{p}{p}} \alpha^*\alpha|a|) = \text{Tr}(|a|^{\frac{p}{p}}|a|) = \text{Tr}(|a|^p) = 1.
\]

\begin{proposition}
If \( a, b \in L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0}) \) and if \( e, f \) are two central projections in \( \mathcal{N}_{\varphi_0} \) such that \( ef = 0 \), then \( M_{p,q}(ae + bf) = M_{p,q}(ae) + M_{p,q}(bf) \).
\end{proposition}

\begin{proof}
As is easily checked, we have
\[
|ae + bf| = |ae + bf|.
\]

Let \( \gamma \) be the partial isometry occurring in the polar decomposition of \( ae + bf \), and let \( a = \alpha|a|, b = \beta|b| \) be the polar decompositions of \( a \) and \( b \). We claim that \( \gamma = ae + \beta f \). Indeed, we have
\[
ae + bf = \gamma|ae + bf|,
\]
and
\[
ae + bf = (ae)(|a|e) + (\beta f)(|b|f) = (ae + \beta f)|ae + bf|.
\]

Since \( ae \) is zero on \( \text{Ker}(|ae|) \) and \( \beta f \) is zero on \( \text{Ker}(|bf|) \), \( ae + \beta f \) is zero on \( \text{Im}(|ae + bf|) = \text{Ker}(|ae + bf|) \) (\( ef = 0 \)).

Using again the fact that \( ef = 0 \) and that \( e, f \) are central elements, we deduce that
\[
M_{p,q}(ae + bf) = (ae + \beta f)|ae + bf|^{\frac{p}{p}}
= (ae + \beta f)(|a|^p + f|b|^p)
= M_{p,q}(ae) + M_{p,q}(bf).
\]

\end{proof}

\begin{proposition}
Let \( J \) be a Jordan isomorphism of \( \mathcal{N}_{\varphi_0} \), and let \( 1 \leq p, q < \infty \). Then we have
\[
J(x) = M_{p,q} \circ J \circ M_{q,p}(x) \quad \text{for all} \quad x \in \mathcal{N}_{\varphi_0}.
\]

\end{proposition}

\begin{proof}
By Lemma 3.2 in [10], we have a decomposition \( J = J_1 + J_2 \) with the following properties: \( J_1 \) is a \(*\)-homomorphism, \( J_2 \) is a \(*\)-anti-homomorphism and \( J_1(x) = J(x)e, J_2(x) = J(x)f \) for all \( x \in M \), with \( e, f \) two orthogonal and central projections such that \( e + f = I \).

Observe first that, for \( a \in \mathcal{N}_{\varphi_0} \) with \( a \geq 0 \) and a positive real number \( r \), we have
\[
J_1(a^r) = J_1(a)^r
\]
and the same is true for \( J_2 \).

If \( \alpha \) is a partial isometry, then \( J_1(\alpha) \) and \( J_2(\alpha) \) are partial isometries with initial supports \( J_1(\alpha^*\alpha) \) and \( J_2(\alpha^*\alpha) \), and final supports \( J_1(\alpha^*\alpha) \) and \( J_2(\alpha^*\alpha) \) respectively.

Let \( x = \alpha|x| \in \mathcal{N}_{\varphi_0} \). Since the supports of \( J_1 \) and \( J_2 \) are orthogonal, it follows from Proposition 2.5 that
\[
M_{p,q} \circ J \circ M_{q,p}(x) = M_{p,q}(J_1(M_{q,p}(x)) + J_q(M_{q,p}(x)))
= M_{p,q}(J_1(M_{q,p}(x))) + M_{p,q}(J_2(M_{q,p}(x)));
\]

\end{proof}
Moreover, we have
\[ M_{p,q}(J_1(M_{q,p}(x))) = M_{p,q}(J_1(\alpha|x|^\frac{p}{2})) = M_{p,q}(J_1(\alpha)J_1(|x|^\frac{p}{2})) = J_1(x) \]
and
\[ M_{p,q}(J_2(M_{q,p}(x))) = M_{p,q}(J_2(\alpha|x|^\frac{p}{2}\alpha^*)\alpha) = M_{p,q}(J_2(\alpha)J_2(\alpha|x|^\frac{p}{2}\alpha^*)) = M_{p,q}(J_2(\alpha)J_2((\alpha|x|^\alpha^*)^\frac{p}{2})) = M_{p,q}(J_2(\alpha)J_2(\alpha|x|^\alpha^*)^\frac{p}{2}) = J_2(x). \]

An essential tool for the proof of Theorem 1.3 is the following result about the local uniform continuity of \( M_{p,q} \), which is proved in Lemma 3.2 of [8] (for an independent proof in the case \( L_p(M,\tau) = S_p \), see [7]).

**Proposition 2.7** ([8]). For \( 1 \leq p, q < \infty \), the Mazur map \( M_{p,q} \) is uniformly continuous on the unit sphere \( S(L_p(M)) \).

### 3. Group representations on \( L_p(M) \)

Sherman’s description of the surjective isometries of \( L_p(M) \) in [9] is a crucial tool in the following result (non-surjective isometries in the semi-finite case, and 2-isometries in the general case are described in [13] and [5] respectively). This will allow us to transfer a representation of a group \( G \) on \( L_p(M) \) to a representation of \( G \) on \( L_2(M) \).

**Proposition 3.1.** For \( p > 2 \), and \( U \in O(L_p(M)) \), the map \( V = M_{p,2} \circ U \circ M_{2,p} \) belongs to \( O(L_2(M)) \).

**Proof.** The fact that \( ||V(x)||_2 = ||x||_2 \) for all \( x \in L_2(M) \) follows from Proposition 2.3 and \( V \) is bijective by Lemma 2.2. We have to prove that \( V \) is linear on \( L_2(M) \).

By Theorem 1.2 in [9], there exist a Jordan isomorphism \( J \) of \( M \) and a unitary \( w \in M \) such that
\[ U(\varphi^{1/p}) = w(\varphi \circ J^{-1})^{1/p} \text{ for all } \varphi \in M^+_* \.
\]
It was shown in [12] that \( J \) extends to a Jordan *-isomorphism \( \widetilde{J} \) between \( L_0(N_{\varphi_0}, \tau_{\varphi_0}) \) and \( L_0(N_{\varphi_0\circ J^{-1}}, \tau_{\varphi_0\circ J^{-1}}) \); moreover, \( \widetilde{J} \) is an extension of an isomorphism between \( N_{\varphi_0} \) and \( N_{\varphi_0\circ J^{-1}} \) as well as a homeomorphism for the measure topology on \( L_0(N_{\varphi_0}, \tau_{\varphi_0}) \) and \( L_0(N_{\varphi_0\circ J^{-1}}, \tau_{\varphi_0\circ J^{-1}}) \). The isomorphism \( \widetilde{J} \) satisfies the relations
\[ \tau_{\varphi_0} \circ \widetilde{J}^{-1} = \tau_{\varphi_0\circ J^{-1}}, \]
\[ J^{-1} \circ \Phi_{\varphi_0\circ J^{-1}} = \Phi_{\varphi_0} \circ \widetilde{J}^{-1}. \]

**Lemma 3.2.** For \( \varphi \in M^+_* \), we have
\[ \frac{d(\varphi^\tau_{\varphi_0})}{d\tau_{\varphi_0}} = \widetilde{J}^{-1}(\frac{d\varphi \circ J^{-1} \varphi_0 \circ J^{-1}}{d\tau_{\varphi_0 \circ J^{-1}}}). \]
Proof. For all \( \varphi \in \mathcal{M}^+_+ \), we have

\[
\tau_{\varphi_0}(\frac{d\varphi_0}{d\tau_{\varphi_0}}) = \varphi \circ \Phi_{\varphi_0} \\
= \varphi \circ J^{-1} \circ \Phi_{\varphi_0} \circ J^{-1} \circ \tilde{J} \\
= \tau_{\varphi_0} \circ J^{-1}(\frac{d\varphi \circ J^{-1} \varphi_0 \circ J^{-1}}{d\tau_{\varphi_0} \circ J^{-1}}) \\
= \tau_{\varphi_0} \circ J^{-1}(\frac{d\varphi \circ J^{-1} \varphi_0 \circ J^{-1}}{d\tau_{\varphi_0} \circ J^{-1}}) \\
= \tau_{\varphi_0}(\tilde{J}^{-1}(\frac{d\varphi \circ J^{-1} \varphi_0 \circ J^{-1}}{d\tau_{\varphi_0} \circ J^{-1}})) ,
\]

where in the last equality we used the fact that \( \tilde{J} \) is a Jordan homomorphism. \( \square \)

In Lemma 2.1 in [11], it is shown that there exists a topological *-isomorphism \( \tilde{K} \) between \( L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0}) \) and \( L_0(\mathcal{N}_{\varphi_0} J^{-1}, \tau_{\varphi_0} J^{-1}) \) which satisfies the following relation on the Radon-Nikodým derivatives:

\[
\tilde{K}(\frac{d\varphi_0}{d\tau_{\varphi_0}}) = \frac{d\varphi_0 \circ J^{-1}}{d\tau_{\varphi_0} \circ J^{-1}} \text{ for all } \varphi \in \mathcal{M}^+_+ .
\]

From Lemma 3.2 we obtain

\[
\frac{d\varphi \circ J^{-1} \varphi_0}{d\tau_{\varphi_0}} = \tilde{K}^{-1} \circ \tilde{J}(\frac{d\varphi_0}{d\tau_{\varphi_0}}) \text{ for all } \varphi \in \mathcal{M}^+_+ .
\]

As a consequence, the linear and bijective isometry \( U \) of \( L_p(\mathcal{M}) \) is given by the following relation on positive elements:

\[
U(x) = w(\tilde{K}^{-1} \circ \tilde{J}(x)) \text{ for all } x \in L_p(\mathcal{M})^+. 
\]

This relation extends by linearity to the whole \( L_p(\mathcal{M}) \).

Now notice that \( \tilde{K}^{-1} \circ \tilde{J} \) is a Jordan isomorphism on \( \mathcal{N}_{\varphi_0} \) and a topological isomorphism (for the measure topology) on \( L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0}) \). By Proposition 2.6 for \( x \in \mathcal{N}_{\varphi_0} \), we have

\[
V(x) = M_{p,2} \circ U \circ M_{2,p}(x) \\
= w(M_{p,2} \circ \tilde{K}^{-1} \circ \tilde{J} \circ M_{2,p}(x)) \\
= w(\tilde{K}^{-1} \circ \tilde{J}(x)).
\]

Recall from [8] that the Mazur map is continuous for the measure topology on \( L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0}) \). So by density of \( \mathcal{N}_{\varphi_0} \) in \( L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0}) \) for the measure topology, we have

\[
V(x) = w(\tilde{K}^{-1} \circ \tilde{J}(x)) \text{ for all } x \in L_2(\mathcal{M}),
\]

which gives the linearity of \( V \) on \( L_2(\mathcal{M}) \). \( \square \)

Remark 3.3. The proof of the linearity of the map \( V \) in Proposition 3.1 is simpler in the case where \( \mathcal{M} \) is a von Neumann algebra equipped with a faithful semi-finite normal trace \( \tau \). Indeed, by Theorem 2 in [14], there exist a Jordan isomorphism...
Let $J$, a positive operator $B$ commuting with $J(\mathcal{M})$, and a partial isometry $W$ in $\mathcal{M}$ with the property that $W^* W$ is the support of $B$, such that

\[ U(x) = WBJ(x) \text{ for all } x \in \mathcal{M} \cap L_p(\mathcal{M}, \tau). \]

Using the fact that $B$ commutes with $J(\mathcal{M})$, and as in the proof of Proposition 2.6 for all $x = \alpha|x| \in \mathcal{M} \cap L_p(\mathcal{M}, \tau)$, we have

\[
V(x) = WM_{p,2}(BJ_1(\alpha|x|^\frac{1}{2}) + BJ_2(\alpha|x|^\frac{1}{2})) \\
= WM_{p,2}(BJ_1(\alpha|x|^\frac{1}{2}) + WM_{p,2}(BJ_2(\alpha|x|^\frac{1}{2})) \\
= WJ_1(\alpha)B_{x}^2J_1(|x|) + WJ_2(\alpha)B_{x}^2J_2(\alpha|x|\alpha^*) \\
= WB_{x}^2 J(x).
\]

The linearity on the whole $L_p(\mathcal{M}, \tau)$ follows from the density of $\mathcal{M} \cap L_p(\mathcal{M}, \tau)$ in $L_p(\mathcal{M}, \tau)$.

**Corollary 3.4.** Let $G$ be a topological group, $p \geq 2$, and $U : G \rightarrow O(L_p(\mathcal{M}))$ be a representation on $L_p(\mathcal{M})$. For $g \in G$, define $V(g) : L_2(\mathcal{M}) \rightarrow L_2(\mathcal{M})$ by

\[ V(g) = M_{p,2} \circ U(g) \circ M_{2,p}. \]

Then $V$ is a representation of $G$ on $L_2(\mathcal{M})$.

**Proof.** By the previous proposition, $V(g) \in O(L_2(\mathcal{M}))$ for every $g$ in $G$. Moreover, the map $g \mapsto V(g)x$ is continuous, since $g \mapsto U(g)M_{2,p}(x)$ is continuous and since $M_{p,2} : L_p(\mathcal{M}) \rightarrow L_2(\mathcal{M})$ is continuous.

It remains to check that $V$ is a homomorphism. For this, let $g_1, g_2 \in G$. Then, by Lemma 2.2

\[
V(g_1)V(g_2) = M_{p,2} \circ U(g_1) \circ M_{2,p} \circ M_{p,2} \circ U(g_2) \circ M_{2,p} \\
= M_{p,2} \circ U(g_1) \circ U(g_2) \circ M_{2,p} \\
= M_{p,2} \circ U(g_1g_2) \circ M_{2,p} \\
= V(g_1g_2).
\]

Let $U$ be a representation of a topological group $G$ on $L_p(\mathcal{M})$ and let

\[ L_p(\mathcal{M})^{U(G)} = \{x \in L_p(\mathcal{M}) \mid U(g)x = x \text{ for all } g \in G\} \]

be the space of $U(G)$-invariant vectors in $L_p(\mathcal{M})$. Let $p'$ be the conjugate of $p$ and $U^*$ the contragredient representation of $U$ on the dual space $L_{p'}(\mathcal{M})$ of $L_p(\mathcal{M})$. Since $L_p(\mathcal{M})$ is superreflexive, there exists a complement $L_p(\mathcal{M})'$ for $L_p(\mathcal{M})^{U(G)}$ (see Proposition 2.6 in [1]), and we have

\[ L_p(\mathcal{M})' = \{v \in L_p(\mathcal{M}) \mid \operatorname{Tr}(vc) = 0 \text{ for all } c \in L_{p'}(\mathcal{M})^{U^*(G)}\}. \]

**Proposition 3.5.** Let $v \in S(L_p(\mathcal{M})')$. Then

\[ d(v, L_p(\mathcal{M})^{U(G)}) \geq \frac{1}{2}. \]

**Proof.** Assume, by contradiction, that there exists $b \in L_p(\mathcal{M})^{U(G)}$ such that

\[ ||v - b||_p < \frac{1}{2}. \]

Then $\frac{1}{2} \leq ||b||_p \leq \frac{3}{2}$. Setting $c = \frac{b}{||b||_p}$, we have $||b - c||_p \leq \frac{1}{2}$. 

Since \( c \in L_p(\mathcal{M})^{U(G)} \), it is easily checked that \( M_{p,p'}(c) \ast \in L_{p'}(\mathcal{M})^{U'(G)} \); hence
\[
\text{Tr}((c - v)M_{p,p'}(c) \ast) = \text{Tr}(cM_{p,p'}(c) \ast) = ||c||_p^p = 1.
\]
On the other hand, using Hölder’s inequality, we have
\[
1 = \text{Tr}((c - v)M_{p,p'}(c) \ast) \\
\leq ||c - v||_p ||M_{p,p'}(c) \ast||_{p'} \\
= ||c - v||_p ||c||_{p'}^{p'} \\
= ||c - v||_p.
\]
This implies that
\[
||v - b||_p \geq ||v - c||_p - ||c - b||_p \\
\geq \frac{1}{2},
\]
and this is a contradiction.

\[\square\]

4. Proof of Theorem \[\underline{1.3}\]

We follow the strategy of the proof of Theorem A in [4,1]. Let \( p \in ]1, \infty[ \), and let \( U \) be a representation on \( L_p(\mathcal{M}) \) of a group \( G \). Let \( H \) be a closed subgroup of \( G \) such that the pair \( (G, H) \) has property \( (T) \). We claim that the representation \( U' \) of \( G \) on the complement \( L_p(\mathcal{M})' \) of \( L_p(\mathcal{M})^{U(H)} \) has no almost \( U'(G) \)-invariant vectors. This will prove Theorem \[\underline{1.3}\].

Let \( Q \) be a compact subset in \( G \), and take \( \epsilon > 0 \). Assume by contradiction that there exist almost \( U(G) \)-invariant vectors in \( L_p(\mathcal{M})' \). Then, we can find, for every \( n \), a unit vector \( v_n \) such that
\[
\sup_{g \in Q} ||U(g)v_n - v_n||_p < \frac{1}{n}.
\]
By Corollary \[\underline{3.4}\] \( V = M_{p,2} \circ U \circ M_{2,p} \) defines a representation of \( G \) on \( L_2(\mathcal{M}) \). Let \( w_n \) be the orthogonal projection of \( M_{p,2}(v_n) \) on the orthogonal complement \( L_2(\mathcal{M})' \) of \( L_2(\mathcal{M})^{V(H)} \). We claim that \( w_n \) is \((Q, \epsilon)\)-invariant for \( V \) for \( n \) sufficiently large. This will contradict property \( (T) \) for the pair \( (G, H) \).

We first show that there exists \( \delta' > 0 \) such that
\[
d(M_{p,2}(v_n), L_2(\mathcal{M})^{V(H)}) \geq \delta' \quad \text{for all } n.
\]
Indeed, otherwise for some \( n \), there exists \( a_k \in L_2(\mathcal{M})^{V(H)} \) such that
\[
||M_{p,2}(v_n) - a_k||_2 \xrightarrow{k \to \infty} 0.
\]
By Proposition \[\underline{2.3}\] we have
\[
||M_{p,2}(v_n)||_2 = ||v_n||_{p}^{p} = 1.
\]
Since \( ||a_k||_2 \xrightarrow{k \to \infty} ||M_{p,2}(v_n)||_2 = 1 \), we can assume that \( ||a_k||_2 = 1 \). Notice that
\[
M_{2,p}(L_2(\mathcal{M})^{V(H)}) = L_p(\mathcal{M})^{U(H)}.
\]
Hence, \( M_{2,p}(a_k) \) belongs to \( L_p(\mathcal{M})^{U(H)} \) for every \( k \). Moreover
\[
||v_n - M_{2,p}(a_k)||_p \xrightarrow{k \to \infty} 0
\]
by the uniform continuity of $M_{2,p}$ on the unit sphere (see Proposition 3.5). This is a contradiction to Proposition 2.4.

In particular, we have

$$||w_n||_2 = d(M_{p,2}(v_n), L_2(M)^{V(H)}) \geq \delta'.$$

For $g \in Q$, we have

$$||V(g)w_n - w_n||_2 \leq ||V(g)M_{p,2}(v_n) - M_{p,2}(v_n)||_2 = ||M_{p,2}(U(g)v_n) - M_{p,2}(v_n)||_2.$$

Recall that $||v_n||_p^p = 1$ and that

$$\sup_{g \in Q} ||U(g)v_n - v_n||_p < \frac{1}{n}.$$  

Hence, by the uniform continuity of $M_{p,2}$ on $S(L_2(M))$, there exists an integer $N$ (depending only on $(Q, \epsilon)$) such that

$$\sup_{g \in Q} ||V(g)w_n - w_n||_2 < \epsilon\delta'$$  

for $n \geq N$.

Since $||w_n||_2 \geq \delta'$, it follows that

$$\sup_{g \in Q} ||V(g)w_n - w_n||_2 < \epsilon||w_n||_2$$  

for $n \geq N$.

This shows that $w_n$ is $(Q, \epsilon)$-invariant for $U$ when $n \geq N$. This finishes the proof of Theorem 1.3.

5. Property $(\mathcal{F}L_p(M))$ for higher rank groups

Let $H$ be a closed normal subgroup of $G$ and let $L$ be a closed group of $G$. Assume that $G = L \rtimes H$. The following strong relative property $(T_B)$ was considered in [1]:

**Definition 5.1.** A pair $(L \rtimes H, H)$ has property $(T_B)$ if, for any orthogonal representation $\rho : L \rtimes H \to O(B)$, the quotient representation $\rho' : L \to O(B/B^\rho(H))$ does not almost have $\rho'(L)$-invariant vectors.

A straightforward modification of our proof of Theorem 1.3 shows that we also have the following result:

**Theorem 5.2.** Let $(L \rtimes H, H)$ be a pair with strong relative property $(T)$. Then $(L \rtimes H, H)$ has strong relative property $(T_{L_p(M)})$ for $1 < p < \infty$.

Let $G$ be a higher rank group as defined in the introduction. Using an analogue of Howe-Moore’s theorem on the vanishing of matrix coefficients, the authors of [1] showed that $G$ has property $(F_B)$ whenever $B$ is a superreflexive Banach space and a certain pair $(L \rtimes H, H)$ of subgroups, which has property $(T)$, has also $(T_B)$. The property $(F_{L_p(M)})$ for higher rank groups in Theorem 1.6 is then a consequence of Theorem 5.2. Moreover, the result for lattices in higher rank groups is obtained by an induction process exactly as in Proposition 8.8 of [1].

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REFERENCES


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