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ON PREHOMOGENEITY OF A RANK VARIETY

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Abstract. If a linear algebraic group G acts on $M(m,n)$, then it also acts on a rank variety $M^{(r)}(m,n) = \{X \in M(m,n) | \operatorname{rank} X = r\}$. In this paper, we give the necessary and sufficient condition that this variety has a Zariski-dense G-orbit. We consider everything over the complex number field $\mathbb{C}$.

Introduction

Let G be a linear algebraic group. Let $\rho : G \to GL(m)$ and $\sigma : G \to GL(n)$ be its rational representations over $\mathbb{C}$. Then G acts on $M(m,n)$ by $\rho \otimes \sigma$, i.e., $X \mapsto \rho(g)X^t\sigma(g)$ $(X \in M(m,n),\ g \in G)$. By this action, G also acts on a rank variety $M^{(r)}(m,n) = \{X \in M(m,n) |\ \operatorname{rank} X = r\}$. On the other hand, if a rational representation $\tau : H \to GL(V)$ of a linear algebraic group H on a finite-dimensional vector space V has a Zariski-dense H-orbit, we call a triplet (H, τ, V) a prehomogeneous vector space (abbrev. PV). A point of the Zariski-dense H-orbit is called a generic point, and the isotropy subgroup at a generic point is called a generic isotropy subgroup. For the basic facts of PVs, see [K]. In this paper, we shall prove the following theorem.

Theorem 0.1. *The following assertions are equivalent.*

(1) *$M^{(r)}(m,n)$ has a Zariski-dense G-orbit by the action $\rho \otimes \sigma$.*
(2) *$(G \times GL(r),\ \rho \otimes \Lambda_1 + \sigma \otimes \Lambda_1^*,\ M(m,r) \oplus M(n,r))$ is a PV.*

Here the action of (2) is given by $(X,Y) \mapsto (\rho(g)X^tA, \sigma(g)YA^{-1})$ for $(X,Y) \in M(m,r) \oplus M(n,r))$ and $(g,A) \in G \times GL(r)$. Note that we have $m \geq r$ and $n \geq r$ in (2).

1. Proof of theorem

The following lemma is the key for our proof.

Lemma 1.1 (M. Sato)**.** *Assume that an algebraic group G acts on both of two irreducible algebraic varieties W and W'. Let $\varphi : W \to W'$ be a morphism satisfying*

(I) $\varphi(gw) = g\varphi(w) \quad (g \in G, w \in W)$,
(II) $\overline{\varphi(W)} = W'$.

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Then the following assertions are equivalent:

(1) $W = \overline{G \cdot w}$ *for some* $w \in W$*; that is,* W *is* G*-prehomogeneous.*
(2) (a) $W' = \overline{G \cdot w'}$ *for some* $w' \in W'$.
(b) *For the above point* $w' \in W'$ *in* (a)*, there exists a point* $w \in \varphi^{-1}(w')$ *such that* $\varphi^{-1}(w') = \overline{G_{w'} \cdot w}$*, where* $G_{w'} = \{g \in G : gw' = w'\}$ *is the isotropy subgroup of* G *at* w'.

Proof. For the proof, see Lemmas 7.2 and 7.6 in [K]. □

Now we shall prove Theorem 0.1. Put $W = \{(X, Y) \in M(m, r) \oplus M(n, r) |\ \operatorname{rank} X = \operatorname{rank} Y = r\}$. Then clearly $(G \times GL(r),\ \rho \otimes \Lambda_1 + \sigma \otimes \Lambda_1^*,\ M(m, r) \oplus M(n, r))$ is a PV if and only if W is $G \times GL(r)$-prehomogeneous. Note that $\operatorname{rank} X^t Y = r$ for any $(X, Y) \in W$. Hence we have a $(G \times GL(r))$-equivariant map $\varphi : W \to M^{(r)}(m, n)$ by $(X, Y) \mapsto X^t Y$. Note that $GL(m) \times GL(n)$ also acts on both W and $M^{(r)}(m, n)$. Since φ is also $(GL(m) \times GL(n))$-equivariant and $M^{(r)}(m, n)$ is a single $(GL(m) \times GL(n))$-orbit, we see that φ is surjective. For $Z_0 = \varphi(X_0, Y_0) \in M^{(r)}(m, n)$, we shall show that the fiber $\varphi^{-1}(Z_0)$ is a principal $GL(r)$-orbit. Assume that $(X, Y) \in \varphi^{-1}(Z_0)$. If $X = (x_1 | \cdots | x_r) \in M(m, r)$, let $\langle X \rangle$ be an r-dimensional subspace of $\mathbb{C}^m$ generated by column vectors $x_1, \ldots, x_r$. Since we have $X^t Y = (\sum_{j=1}^r y_{1j} x_j | \cdots | \sum_{j=1}^r y_{nj} x_j)$ for $Y = (y_{ij})$ and $\operatorname{rank} X = \operatorname{rank} X^t Y (= r)$, we have $\langle X \rangle = \langle X^t Y \rangle = \langle Z_0 \rangle = \langle X_0^t Y_0 \rangle = \langle X_0 \rangle$, and hence there exists uniquely $A \in GL(r)$ satisfying $X = X_0\ {}^t A$. Then we have $X_0\ {}^t Y_0 = X^t Y = X_0\ {}^t A^t Y$ and since $\operatorname{rank} X_0 = r$, we obtain ${}^t Y_0 = {}^t A^t Y$, i.e., $Y = Y_0 A^{-1}$. This shows that the fiber $\varphi^{-1}(Z_0)$ is a principal $GL(r)$-orbit. Since $GL(r)$ is contained in the isotropy subgroup of Z_0, by Lemma 1.1, we obtain Theorem 0.1.

Note that (2) in Theorem 0.1 is equivalent to:

(3) $(G \times GL(r),\ \rho \otimes \Lambda_1, M(m, r))$ is a PV with a generic isotropy subgroup H, and $(H, \sigma \otimes \Lambda_1^*,\ M(n, r))$ is a PV.

Example 1.2. Let $\rho : Spin(10) \to GL(16)$ be a half-spin representation of the spin group $Spin(10)$. Then $Spin(10) \times GL(14)$ acts on $M(16, 14)$ and $M^{(r)}(16, 14)$ has the Zariski-dense $(Spin(10) \times GL(14))$-orbit if and only if $r = 1, 2, 3, 13, 14$. By (3), $(Spin(10) \times GL(14) \times GL(r)$, a half-spin rep.$\otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^*, M(16, r) \oplus M(14, r))$ is a PV if and only if $(Spin(10) \times GL(r)$, a half-spin rep. $\otimes \Lambda_1, M(16, r))$ is a PV since the latter is a trivial PV (see [K]). So by [SK], we have our result. Note that the case $r = 13$ (resp. $r = 14$) is a castling transform of the case $r = 3$ (resp. $r = 2$).

Remark 1.3. Professors Shigefumi Mori and Yasuo Teranishi proved that (2) in Theorem 0.1 is equivalent to (4) : $U = \{(\langle X \rangle, \langle Y \rangle, X^t Y) \in \operatorname{Grass}_r(V(m)) \oplus \operatorname{Grass}_r(V(n)) \oplus M(m, n) |\ (X, Y) \in W\}$ is G-prehomogeneous, and later Professor Masanobu Taguchi proved that it is equivalent to (5) : $U' = \{(\langle X \rangle, X^t Y) \in \operatorname{Grass}_r(V(m)) \oplus M(m, n) |\ (X, Y) \in W\}$ is G-prehomogeneous (see [KKTI, Remark 1.38]).

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