PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 141, Number 1, January 2013, Pages 205–216
S 0002-9939(2012)11321-8
Article electronically published on May 24, 2012

OBSERVABILITY IN INVARIANT THEORY

LEX E. RENNER

(Communicated by Harm Derksen)

Abstract. We consider actions $G \times X \to X$ of the affine, algebraic group G on the affine, algebraic variety X. We say that $G \times X \to X$ is *observable in codimension one* if for any height-one, G-invariant, prime ideal $\mathfrak{p} \subset k[X]$, $\mathfrak{p}^G \neq (0)$. Many familiar actions are observable in codimension one. We characterize such actions geometrically and indicate how they fit into the general framework of invariant theory. We look at what happens if we impose further restrictions, such as G being reductive or X being factorial. We indicate how Grosshans subgroups are involved.

1. Introduction

Let $G \times X \to X$ be an action of the connected affine algebraic group on the irreducible affine variety X. The fundamental theme of invariant theory is the relationship between the orbit structure of X and the invariant ring $k[X]^G$. At one extreme we end up with $k[X]^G = k$. In such cases there is little or no useful relationship between the orbit structure and the invariant ring. In [8] we considered observable actions. An action $G \times X \to X$ is *observable* if for any nonzero G-stable ideal $\mathfrak{a}$ of $k[X]$, $\mathfrak{a}^G \neq \{0\}$. In the case of a reductive group this is the same as saying that the action is stable. An action is *stable* if there is a nonempty open subset U of X such that the G-orbit, Gx, is closed in X for any $x \in U$. For unipotent groups any action is observable. Observable actions have the desirable property that $[k[X]^G] = [k[X]]^G$, where [...] denotes the "quotient field". We call an action *visible* if $[k[X]^G] = [k[X]]^G$. The main result of [8] is the following characterization of observable actions.

Theorem 1.1 ([8]). *The following are equivalent for $G \times X \to X$.*

(1) *The action is observable.*
(2) *The action is stable and visible.*

If G is reductive, then any stable action is observable.

There is an important difference between reductive groups and most non-reductive groups. First of all, stability is a local condition for any group action. If the group G is reductive and it acts on the affine variety X, then the quotient is also affine. If the orbit structure is nice this will be reflected in the ring of invariants. However, if the group is not reductive, then there is no a priori reason to expect the action to be visible, even if every orbit is closed. This is not necessarily pathological, but rather a subtle combination of local and global factors. In many

Received by the editors March 8, 2011 and, in revised form, May 31, 2011 and June 22, 2011.
2010 *Mathematics Subject Classification.* Primary 13A50, 14L30.

important examples the action is not visible, but there is still a very nice quotient $X \to X/G$ with X/G nonaffine.

With this as background we continue the investigation of actions $G \times X \to X$. We say that an action

$$G \times X \to X$$

is **observable in codimension one** if for any G-invariant, height-one, prime ideal $\mathfrak{p} \subset k[X]$, $\mathfrak{p}^G \neq \{0\}$. See Definition 2.1 and the subsequent discussion below.

The purpose of this paper is to identify the study of such actions as an important part of invariant theory. Many familiar actions are observable in codimension one (Corollary 3.11). We characterize such actions under mild assumptions (Theorem 2.5). We look at what happens if G is reductive (Theorem 3.13), or if X is factorial (Theorem 3.8), or if the typical orbit is affine (Theorem 2.9). We also indicate how Grosshans subgroups are involved (Corollaries 2.6 and 2.8).

The results of this paper can be thought of as a further development of the investigation that began in [8]. Some of the key ideas that go into our investigation have been previously well established. The idea of *stability* was originally studied by Popov in [6]. It also becomes clear that *Grosshans subgroups* are involved here in a fundamental way. This idea had been developed extensively by Grosshans [2].

We now assemble some of the terminology and background that is used in the paper. Let A be an integral domain. We denote by $[A]$ the quotient field of A. If $G \times X \to X$ is an action of G on the affine variety X we define

$$k[X]^G = \{f \in k[X] \mid g \cdot f = f\},$$

where $g \cdot f(x) = f(g^{-1}x)$. We let

$$X/G = Spec(k[X]^G).$$

This is not the customary terminology. One usually reserves the notation X/G for situations where the canonical morphism, $X \to X/G$, is a geometric quotient. It is well known that, in general, $k[X]^G$ may not be finitely generated, and consequently X/G will not be an algebraic variety. Nor can we always expect X/G to be a reasonable approximation of the orbit space for the action of G on X. Indeed, the canonical morphism $\pi : X \to X/G$ may not even be surjective, and it does not always separate the orbits. But X/G is the primary scheme-theoretic object that arises from the given group action. We observe, however, that by a theorem of Rosenlicht (see [9, 10] or §13.5 of [1]), for any G-action $G \times X \to X$, there exists a G-invariant, open set $X_0 \subset X$, an algebraic variety Y (with trivial G-action) and a separable, dominant G-morphism $X_0 \to Y$ such that $[k[Y]] = [k[X_0]]^G$, and any fibre of $X_0 \to Y$ is a G-orbit. For more information on invariant theory the reader should consult [1, 2, 4, 7]. In this paper we are interested in pursuing the circle of ideas related to Rosenlicht's result by studying actions that are observable in codimension one. The references [1, 2] are especially suited to our discussion since they include many useful results about observable subgroups of algebraic groups.

If $\chi : G \to k^*$ is a character of G we let $k[X]_\chi = \{f \in k[G] \mid g \cdot f = \chi(g)f$ for all $g \in G\}$. Define

$$E_G[X] = \{\chi \in X(G) \mid k[X]_\chi \neq \{0\}\}.$$

$E_G[X]$ is a submonoid of $X(G)$. If $\chi \in E_G[X]$ and $f \in k[X]_\chi \setminus \{0\}$, then

$$X_f = \{x \in X \mid f(x) \neq 0\}$$

is a G-stable, affine, open subset of X. Furthermore,

$$E_G[X_f] = E_G[X][\chi^{-1}]$$

the smallest submonoid of $X(G)$ containing $\{\chi^{-1}\}$ and $E_G[X]$.

If $G \times X \to X$ is an action we let

$$X_{max} = \{x \in X \mid dim(Gx) \text{ assumes the maximal value}\}.$$

It is well-known that X_{max} is a nonempty, open, G-invariant subset of X.

We say that X is *locally factorial* if any Weil divisor of X is actually a Cartier divisor. In this situation any open subset of X, of the form $U = X \setminus \bigcup_i D_i$, where each $D_i \subseteq X$ is a divisor, is actually affine. Furthermore *any*, nonempty, open subset U of X is contained in a unique, affine, open subset U' obtained from U by adding a closed subset of codimension at least two.

Unless otherwise stated we assume that k is an algebraically closed field. An algebraic variety X is assumed to be affine and irreducible (unless otherwise stated).

2. Observability in codimension one

Let $G \times X \to X$ be an action of the connected affine algebraic group G on the irreducible, affine variety X.

Definition 2.1. We say that $G \times X \to X$ is *visible* if $k(X)^G = [k[X]^G]$. We say that $G \times X \to X$ is *observable in codimension one* if for any G-invariant, height-one prime ideal $\mathfrak{p}$ of $k[X]$, $\mathfrak{p}^G \neq (0)$.

Proposition 2.2. *Assume that $G \times X \to X$ is observable in codimension one. If X is normal, then $G \times X \to X$ is visible.*

Proof. If $r \in k(X)^G \setminus k$, then $I = \{f \in k[X] \mid fr \in k[X]\}$ is a G-invariant ideal of $k[X]$. Let $\{\mathfrak{p}_i \mid i = 1, ..., s\}$ be the set of height-one prime ideals of $k[X]$ such that

$$\nu_i(r) < 0, \quad i = 1, ..., s,$$

where ν_i is the discrete valuation associated with $\mathfrak{p}_i$. Let $\nu_i(r) = -m_i$, where $m_i > 0$. By Theorem 6 (page 25) of [2], $f \in I$ if and only if $\nu_i(f) \geq m_i$ for each i. Thus

$$I = \bigcap_i \mathfrak{p}_i^{(m_i)},$$

where $\mathfrak{p}_i^{(m_i)} = \{f \in \mathfrak{p}_i \mid \nu_i(f) \geq m_i\}$. One checks directly that $\mathfrak{p}_i = \sqrt{\mathfrak{p}_i^{(m_i)}}$, the radical of $\mathfrak{p}_i^{(m_i)}$. Thus the radical $J = \sqrt{I}$, of I, is the finite intersection, $J = \bigcap_i \mathfrak{p}_i$, of height-one prime ideals. Furthermore, J is also G-stable. Since G is connected it stabilizes each $\mathfrak{p}_i$. By assumption $\mathfrak{p}_i^G \neq (0)$. Hence if $f_i \in \mathfrak{p}_i^G \setminus \{0\}$ we let $f = f_1 \cdots f_m \in J$. Then, for some $n > 0$, $f^n \in I$ and therefore $r = \frac{f^n r}{f^n} \in [k[X]^G]$. $\square$

In Lemma 2.4 and Theorem 2.5 below we use the following well-known result of Chevalley.

Theorem 2.3 (Chevalley's Theorem)**.** *Let $\varphi : X \to Y$ be a dominant morphism of irreducible, algebraic varieties. Then there is a nonempty open subset U of Y contained in $\varphi(X)$ such that, for each $y \in U$, any irreducible component V of $\varphi^{-1}(y)$ has dimension $r = dim(X) - dim(Y)$.*

Lemma 2.4. *Let $G \times X \to X$ be an action and assume that*

(1) *$k[X]^G$ is finitely generated.*
(2) *$\forall\ x \in X/G = Spec(k[X]^G)$, $\exists\ y \in X_{max}$ s.t. $\pi^{-1}(x) = \overline{Gy}$ (here $\pi : X \to X/G$ is the canonical map induced by the inclusion $k[X]^G \subset k[X]$).*

Then for any divisor $D \subset X \setminus X_{max}$ the composite $\pi|_D : D \subset X \to X/G$ is dominant.

Proof. Let $D \subset X \setminus X_{max}$ be a divisor, and assume that $\pi|_D : D \to X/G$ is not dominant. Hence $dim(\overline{\pi(D)}) < dim(X/G)$. Thus, by Chevalley's theorem, the typical fibre of $\pi|_D$ has dimension $dim(D) - dim(\overline{\pi(D)})$. But

$$dim(D) - dim(\overline{\pi(D)}) = dim(X) - (1 + dim(\overline{\pi(D)})) \geq dim(X) - dim(X/G)$$

since $dim(\overline{\pi(D)}) < dim(X/G)$. But, again by Chevalley's theorem, $dim(Gy) = dim(X) - dim(X/G)$ if $y \in X_{max}$. This is impossible since, as noted, the typical fibre of $\pi|_D$ has dimension $dim(D) - dim(\overline{\pi(D)})$, while the typical fibre of π has dimension $dim(X) - dim(X/G)$. But by assumption (2), the typical fibre of $\pi|_D$ is in the closure of some Gy, $y \in X_{max}$, and contained in the boundary $\overline{Gy} \setminus Gy$, so has dimension less than $dim(X) - dim(X/G)$. The conclusion is that for any G-divisor $D \subseteq X \setminus X_{max}$, $\pi|_D : D \subset X \to X/G$ is dominant. □

Theorem 2.5. *Let $G \times X \to X$ be a visible action. Then the following are equivalent.*

(1) *$G \times X \to X$ is observable in codimension one.*
(2) *For any G-invariant, closed subset V of X, of pure codimension one, $I(V)^G \neq (0)$.*
(3) *There exists a nonzero invariant $f \in k[X]^G$ such that $codim_{X_f}(X_f \setminus (X_f)_{max}) \geq 2$.*
(4) *There is a dense open set of G-orbits $U \subseteq X$ such that for all $x \in U$ the closure $Z = \overline{G \cdot x}$ in X has the property $codim_Z(Z \setminus G \cdot x) \geq 2$.*

Proof. Before we prove the equivalence of these four conditions, we make the following observations.

Each condition has the property that "X satisfies the condition if and only if X_f satisfies the condition", where $f \in k[X]^G \setminus \{0\}$. For conditions (1) and (2) the proof is almost identical to the proof of Proposition 3.2 of [8]. For condition (3) the proof is obvious. If X satisfies condition (4), then so too does X_f using $U \cap X_f$, while if X_f satisfies condition (4), for some open subset $U \subseteq X_f$, then X also satisfies condition (4) with this same U, noting that if $x \in X_f$, then the closure $Z = \overline{G \cdot x}$, in X, lies entirely in X_f.

Recall now from Theorem 3.7 of [8] that there is a nonzero invariant $f \in k[X]^G$ such that $\pi : X_f \to (X/G)_f$ has the property that $k[X]^G$ is finitely generated, and

$$\forall\ x \in (X/G)_f,\ \exists\ y \in X_{max}\ s.t.\ \pi^{-1}(x) = \overline{Gy}.$$

If follows from our preliminary observations that X satisfies one of the four conditions if and only if X_f satisfies that condition. We shall therefore assume, without loss of generality, that $X = X_f$, so that X has this desirable (displayed) property.

Assume (1) and let $V \subseteq X$ be as in (2). Then $I(V) = \bigcap_i \mathfrak{p}_i$, a finite intersection of height-one prime ideals. Since G is connected each prime $\mathfrak{p}$ is G-stable. By

assumption, there is an invariant $f_i \in \mathfrak{p}_i^G \setminus \{0\}$ for each i. Then $0 \neq \prod_i f_i \in I(V)^G$. Thus (2) holds. The converse "(2) implies (1)" is obvious.

Assume (2), and assume there is a divisor $D \subset X \setminus X_{max}$. By our assumption, $\pi|_D : D \to X/G$ is not dominant, since $I(D)^G \neq \{0\}$. But this is impossible by Lemma 2.4. Hence there is no divisor in $X \setminus X_{max}$. Thus (3) holds.

Assume (3). Consider the composite

$$\varphi : X \setminus X_{max} \subset X \to X/G.$$

Case 1. φ is not dominant: I claim that $Gy = \overline{Gy}$ for all y in some nonempty open subset of X. Indeed, if $Gy \subsetneq \overline{Gy}$ for all $y \in U$, a dense subset of X, then $\varphi(X \setminus X_{max}) \subseteq X/G$ is dense. In this case the action is actually observable.

Case 2. φ is dominant: According to our preliminary observation we may assume that $codim_X(X \setminus X_{max}) \geq 2$. Let $V \subset X \setminus X_{max}$ be an irreducible component of largest dimension such that $\varphi|_V$ is dominant. Let m be the dimension of a typical fibre of $\varphi|_V$. Then by Chevalley's theorem, $dim(V) = m + dim(X/G)$, so that by assumption, $m + dim(X/G) \leq dim(X) - 2$. But by our choice of V, $m = dim(\overline{Gy} \setminus Gy)$ for most $y \in X_{max}$ (since $(\overline{Gy} \setminus Gy) \cap V = \varphi|_V^{-1}(\varphi(y))$ for most y). Thus, $dim(Gy) = dim(X) - dim(X/G) \geq m + 2$. We conclude that $dim(\overline{Gy} \setminus Gy) \leq dim(Gy) - 2$ for most $y \in X_{max}$. Thus (4) holds.

Assume (4). We show that (4) implies (3). Indeed the typical fibre of $\pi : X \to X/G$ has dimension $dim(Gy)$, where $y \in X_{max}$. But from Lemma 2.4 we know that if $D \subset X \setminus X_{max}$ is a divisor, then $\pi|_D : D \to X/G$ is dominant. Thus the typical fibre of $\pi|D$ has dimension $dim(\overline{Gy}) - 1 = dim(X) - dim(X/G) - 1$. But the typical fibre of $\pi|D$ is contained in $\overline{Gy} \setminus Gy$ and therefore (by assumption) has dimension less than or equal to $dim(\overline{Gy}) - 2$.

For the final step of the proof we show that (3) implies (1). Let $D \subset X$ be a G-divisor. By our assumption, $D \cap X_{max} \neq \emptyset$. Suppose that $D \subset X \to X/G$ is dominant. Thus $D \cap X_{max} \subset X \to X/G$ is dominant. Let $U \subset X/G$ be a nonempty open subset such that $U \subset \pi(D \cap X_{max})$. Then for all $x \in U$, $\pi^{-1}(x) \cap (D \cap X_{max}) \neq \emptyset$. But from our initial (displayed) assumption, $\pi^{-1}(x) \cap X_{max}$ is a G-orbit, while D is G-invariant. Thus $\pi^{-1}(x) \cap X_{max} \subset D$. Hence $\pi^{-1}(U) \cap X_{max} \subset D$. This is impossible since $\pi^{-1}(U) \cap X_{max}$ is a nonempty open subset of X. Thus $D \subset X \to X/G$ is not dominant or, what is the same, $I(D)^G \neq (0)$. □

We now recall that a closed subgroup H of G is called a *Grosshans subgroup* if G/H is quasiaffine and $k[G]^H$ is finitely generated as a k-algebra. Equivalently, $H < G$ is a Grosshans subgroup if there exists an action $G \times X \to X$ with a dense orbit $O \subseteq X$ such that $codim_X(X \setminus O) \geq 2$, and $G_x = H$ for some $x \in O$. For more details on this important class of subgroups, see §4 of [2].

Corollary 2.6. *Suppose that $G \times X \to X$ is observable in codimension one. Then the typical orbit satisfies $codim_Z(Z \setminus Gx) \leq 2$, where $Z = \overline{Gx}$. In particular, the typical isotropy subgroup is Grosshans.*

Proof. Let X' be the normalization of X, so that $k[X']$ is the integral closure of $k[X]$ in $k(X)$. The canonical map $\nu : X' \to X$ sends closed subsets of codimension one to closed subsets of codimension one. Using that $I(\nu(V)) = k[X] \cap I(V)$, we see that the induced action $G \times X' \to X'$ is also observable in codimension one. By Proposition 2.2, $G \times X' \to X'$ is also visible. But then by Theorem 2.5, the

typical orbit, for $x \in X'$, satisfies $codim_Z(Z \setminus Gx) \leq 2$, where $Z = \overline{Gx}$. Thus G_x is Grosshans. It follows from this that the same properties hold for the action of G on X. Indeed, ν is a finite birational morphism which takes closed sets to closed sets of the same dimension. □

Corollary 2.7. *Let $G \times X \to X$ be an action with a dense (open) orbit $O \subseteq X$. The following are equivalent.*

(1) *The action is observable in codimension one.*
(2) *$X \setminus O$ has codimension at least two in X.*
(3) *There is no closed, G-invariant subset $D \subset X$ of codimension one.*

Proof. If $G \times X \to X$ has a dense orbit, then $k(X)^G = k$. Indeed, if $f \in k(X) \setminus k$, then f is a nonconstant regular function on $U = \{x \in X \mid f \text{ is defined at } x\}$. But $O \subseteq U$ is dense, so there can be no nonconstant regular functions on U. Thus the action is visible.

Now (2) and (3) are equivalent since there is a dense G-orbit. If (1) holds, then, by condition (4) of Theorem 2.5, (2) holds. If (3) holds, then, vacuously, (1) holds. □

Corollary 2.8. *Let $H \subseteq G$ be a closed subgroup. Then the following are equivalent.*

(1) *There is an action $G \times X \to X$, observable in codimension one, such that G has a dense orbit on X and $G_x = H$ for some $x \in X$ in general position.*
(2) *The subgroup H is a Grosshans subgroup of G.*

Furthermore, the action $G \times X \to X$ is unique up to G-isomorphism if we assume that X is normal.

Proof. Let $G \times X \to X$ be as in (1). Then by Corollary 2.7 the typical isotropy subgroup is a Grosshans subgroup.

Conversely if $H \subset G$ is a Grosshans subgroup, then, by definition, there exists an action $G \times X \to X$ with a dense orbit $O \subseteq X$ such that $codim_X(X \setminus O) \geq 2$, and $G_x = H$ for some $x \in O$. But then by Corollary 2.7 this action is observable in codimension one.

If $H \subseteq G$ is a Grosshans subgroup, then $X = Spec(k[G]^H)$ is the desired normal variety. Any normal affine variety is determined by an open subset with complement of codimension at least two. In particular, X is unique. See §4 of [2] for the detailed discussion. □

Theorem 2.9. *The following are equivalent for $G \times X \to X$.*

(1) *$G \times X \to X$ is observable.*
(2) (a) *$G \times X \to X$ is visible,*
  (b) *$G \times X \to X$ is observable in codimension one, and*
  (c) *The typical G-orbit on X is affine.*

Proof. That (1) implies (2) is a direct consequence of Theorem 1.1. So assume (2). If Gx is affine, then either Gx is closed in X or else $dim(\overline{Gx} \setminus Gx) = dim(\overline{Gx}) - 1$. But from part (4) of Theorem 2.5 it must be the former for most G-orbits in X. But then $G \times X \to X$ is stable. By assumption this action is also visible. Thus, by Theorem 1.1, $G \times X \to X$ is observable. □

3. Special cases

In this section we discuss some of the consequences obtained by strengthening the assumptions on X, G or $G \times X \to X$. We discuss the following important cases: (1) The connectedness of G. (2) $G \times X \to X$ is χ-observable. (3) X is factorial. (4) G is reductive.

3.1. **Connectedness.** In this subsection we put to rest the issue of connectedness of G. If G is not necessarily connected and $G \times X \to X$ is an action on the normal irreducible variety we say the action is *observable in codimension one* if for any pure height-one G-ideal $I \subset k[X]$ we obtain $I^G \neq (0)$.

Lemma 3.1. *Let $G \times X \to X$ be an action, where X is affine and irreducible and G is not necessarily connected. The following are equivalent.*

(1) *$G \times X \to X$ is observable in codimension one.*
(2) *$G^0 \times X \to X$ is observable in codimension one.*

Proof. Suppose that $G \times X \to X$ is observable in codimension one and let $I \subseteq k[X]$ be a nonzero, height-one, G^0-stable ideal. Then

$$I_* = \bigcap_{g \in G} g(I) \subset I$$

is a nonzero ideal, since this is really a finite intersection. But I_* is also G-stable and pure height-one. Thus by assumption, $I_*^G \neq 0$. But $I_*^G \subseteq I^{G^0}$, so the latter is also nonzero. Hence $G^0 \times X \to X$ is observable.

Conversely assume that $G^0 \times X \to X$ is observable and let $I \subseteq k[X]$ be a nonzero, pure height-one, G-stable ideal. Then I is also G^0-stable. By assumption, $I^{G^0} \neq 0$. Let $f \in I^{G^0} \setminus \{0\}$. The G-orbit of f is finite, say $G \cdot f = \{f_1, ..., f_s\}$. But then $F = \prod_i f_i \in I^G \setminus \{0\}$. Thus $G \times X \to X$ is observable in codimension one. □

Remark 3.2. Similar results hold for $G \times X \to X$ observable or χ-observable (Definition 3.3 below).

3.2. **$G \times X \to X$ is χ-observable.**

Definition 3.3. We say that the action $G \times X \to X$ is *χ-observable* if, for any nonzero, G-stable ideal $I \subset k[X]$, there exists $\chi \in E_G[X]$ such that $I_\chi = I \cap k[X]_\chi \neq (0)$.

Notice that if there is a nonzero semi-invariant $f \in k[X]$ such that $G \times X_f \to X_f$ is χ-observable, then $G \times X \to X$ is χ-observable. The proof is similar to the proof of Proposition 3.2 of [8].

Any action $G \times X \to X$ of a solvable group G is χ-observable. This simple observation (essentially the Lie-Kolchin Theorem) was used by Magid in [3] as part of his introduction to the homological theory of $k[X]$-modules with compatible G-action. Presumably many of the results of [3] could be extended to any χ-observable action.

Another large and very interesting class of χ-observable actions is the following. Let G be a connected algebraic group and let $H < G$ be a closed subgroup.

Proposition 3.4. *Let G be a connected group and let $H \subseteq G$ be a closed subgroup. Then the action $H \times G \to G$, by left translations, is χ-observable.*

Proof. Assume first that H is connected. Let $H_0 = R_u(H)[H,H]$. Then H_0 is a closed, connected, normal subgroup of H such that $X(H_0) = \{1\}$ and H/H_0 is a torus. By a theorem of Chevalley, there exists a rational representation $\rho : G \to End(V)$ and a line $L \subseteq V$ such that $H_0 = \{g \in G \mid \rho(g)(L) = L\}$. Thus $H_0 = \{g \in G \mid \rho(g)(v) = v \text{ for all } v \in L\}$ so that G/H_0 is quasi-affine. Thus H_0 is an observable subgroup of G. Thus by Theorem 2.3 of [8] the action of H_0 on G, by left translation, is an observable action.

Now let $I \subseteq k[G]$ be an H-stable ideal of $k[G]$. Then I is H_0-stable. Thus $I^{H_0} \neq \{0\}$. But H/H_0, which is a torus, acts on I^{H_0} and thereby decomposes it into a direct sum of semi-invariant subspaces. Thus $H \times G \to G$ is χ-observable.

Now assume that H is not connected. By the above argument, $H^0 \times G \to G$ is χ-observable. Then, by a simple argument as in the proof of Lemma 3.1, $H \times G \to G$ is χ-observable. See also Remark 3.2. □

Theorem 3.5. *Let $G \times X \to X$ be χ-observable. Then the following are equivalent.*

(1) *$G \times X \to X$ is observable.*
(2) *$G \times X \to X$ is observable in codimension one.*
(3) *$E_G[X]$ is a group.*

Proof. The implication "(1) implies (2)" is obvious from the definitions. So assume (2) and let $\chi \in E_G[X]$. If $f \in k[X]_\chi \setminus \{0\}$, then $fk[X] \subset k[X]$ is a pure height-one, G-stable ideal of $k[X]$. Our assumption implies that $(fk[X])^G \neq \{0\}$. If $h \in (fk[X])^G \setminus \{0\}$, then $h = fg$ for some $g \in k[X]$. One checks that $g \in k[X]_{\chi^{-1}}$. So (2) implies (3). Finally assume (3). If $I \subset k[X]$ is a nonzero, G-stable ideal, then $I_\chi \neq \{0\}$ for some χ since, by assumption, $G \times X \to X$ is χ-observable. But $E_G[X]$ is a group so that $k[X]_{\chi^{-1}} \neq \{0\}$. Hence let $f \in I_\chi \setminus \{0\}$ and $g \in k[X]_{\chi^{-1}} \setminus \{0\}$. Then $fg \in I^G \setminus \{0\}$. So $G \times X \to X$ is observable. □

Corollary 3.6. *Let $G \times X \to X$ be χ-observable. Then there is a nonzero semi-invariant f such that $G \times X_f \to X_f$ is observable.*

Proof. Let $\chi \in E_G[X]$ be such that $E_G[X][\chi^{-1}]$ is a group. Then observe that $E_G[X_f] = E_G[X][\chi^{-1}]$ for any nonzero $f \in k[X]_\chi$. Now apply Theorem 3.5. □

Corollary 3.7. *Let $G \times X \to X$ be χ-observable. Then the orbit in general position is affine.*

Proof. Let $f \in k[X]$ be a nonzero semi-invariant such that $G \times X_f \to X_f$ is observable. By Theorem 1.1, $G \times X_f \to X_f$ is stable. Thus the typical G-orbit on X_f is closed in X_f, and therefore affine. □

3.3. X **is factorial.** In this section we assume that X is factorial. Theorem 3.10 is a generalization of the results of [6]. Corollary 3.9 was proved by Pommerening in [5] generalizing an earlier result of Grosshans.

Theorem 3.8. *$G \times X \to X$ is observable in codimension one if and only if $E_G[X]$ is a group.*

Proof. Assume that $E_G[X]$ is a group. Let $\mathfrak{p}$ be a G-stable height-one prime ideal of $k[X]$. Since X is factorial, $\mathfrak{p} = (f)$, where $f \in k[X]_\chi$ is a nonzero semi-invariant. So let $g \in k[X]_{\chi^{-1}} \setminus \{0\}$ and thus $fg \in \mathfrak{p}^G \setminus \{0\}$. Conversely, if $E_G[X]$ is not a group, let $\chi \in E_G[X]$ be a nonunit and let $f \in k[X]_\chi$ be nonzero. It follows that $(f)^G = (0)$, so that $G \times X \to X$ is not observable in codimension one. □

Corollary 3.9. *For any action $G \times X \to X$, with X affine and factorial, the typical isotropy subgroup is a Grosshans subgroup.*

Proof. By inverting the appropriate semi-invariant we may assume, using Theorem 3.8, that the action is observable in codimension one. Indeed, this does not change the generic isotropy subgroup. The generic isotropy subgroup is a Grosshans subgroup by Corollary 2.6. □

Theorem 3.10. *If the typical orbit is affine, then $G \times X \to X$ is χ-observable.*

Proof. Choose a nonzero semi-invariant $f \in k[X]$ such that $E_G[X_f]$ is a group. By Theorem 3.8, $G \times X_f \to X_f$ is observable in codimension one. But then from Theorem 2.9, $G \times X_f \to X_f$ is observable. Hence $G \times X \to X$ is χ-observable since f is a semi-invariant. □

Corollary 3.11. *Let $\rho : G \to End(V)$ be a rational representation, considered as an action $G \times V \to V$.*

(1) *$\rho : G \to End(V)$ is observable in codimension one if and only if the $E_G[V]$ is a group.*
(2) *$\rho : G \to End(V)$ is observable if and only if the orbit in general position is affine and $E_G[V]$ is a group.*

Proof. Part (1) follows from Theorem 3.8, and part (2) follows from Theorem 2.9. □

3.4. G **is reductive.** Let $G \times X \to X$ be an action, where G is reductive. We consider the *socle*

$$X_{soc} = \overline{\bigcup_{A \in C} A},$$

where C is the set of closed orbits of G on X. This notion was first discussed by Vinberg [11] in case $char(k) = 0$.

Proposition 3.12. *Let $G \times X \to X$ be an action, where G is reductive and let $Y \subseteq X$ be a G-invariant, closed subset of X. Then the following are equivalent.*

(1) *$Y \subset X \to X/G$ is dominant.*
(2) *$X_{soc} \subseteq Y$.*

Furthermore X_{soc} is irreducible and $G \times X_{soc} \to X_{soc}$ is stable. If we let $\mathfrak{p} = \{f \in k[X] \mid f|X_{soc} = 0\}$, then $\mathfrak{p}$ is the unique, maximal G-ideal of $k[X]$ such that $\mathfrak{p}^G = (0)$.

Proof. If $Y \subset X \to X/G$ is dominant, then it is surjective since closed invariant sets go to closed sets. Thus Y contains all the closed G-orbits, since every fibre of $X \to X/G$ contains a unique, closed G-orbit. The converse is obvious.

Let $Z \subseteq X_{soc}$ be an irreducible component such that $Z \subset X \to X/G$ is dominant. Then, by the above, $X_{soc} \subseteq Z$. Hence X_{soc} is irreducible. Let $J \subset k[X]$ be a G-ideal such that $J^G = (0)$. It follows that

$$V(J) \subset X \to X/G$$

is dominant, so that $X_{soc} \subseteq V(J)$. Hence $J \subseteq I(X_{soc}) = \mathfrak{p}$. □

Recall that if G is reductive, then any stable action $G \times X \to X$ is observable.

Theorem 3.13. *Suppose that G is reductive and $G \times X \to X$ is an action, where X is affine. Let U and V be affine, open, G-invariant subsets of X. If $A \subseteq X$, the notation $\overline{A}$ will always mean "Zariski closure of A in X".*

(1) *If $U \subseteq V$, then $\overline{V_{soc}} \subseteq \overline{U_{soc}}$.*
(2) *There is a unique maximal value M for $\overline{U_{soc}}$ as U ranges over affine, open G-invariant subsets of X.*
(3) *If X is locally factorial and $M = \overline{U_{soc}}$, where $U \subseteq X$ is affine, then for any irreducible G-divisor $D \subseteq U$, $M \cap (U \setminus D) \neq \emptyset$. What is the same, $\mathfrak{p}(D)^G \neq (0)$. Equivalently, $G \times U \to U$ is observable in codimension one.*
(4) *If X is locally factorial, then any affine, open, G-invariant subset U of X, with $M = \overline{U_{soc}}$, is observable in codimension one.*

Proof. The inclusion $j : U \subseteq V$ induces an inclusion $k[V] \subseteq k[U]$. Let $\mathfrak{p} \subset k[U]$ and $\mathfrak{q} \subset k[V]$ be the prime ideals corresponding to U_{soc} and V_{soc} respectively. Then $\mathfrak{p} \cap k[V] \subseteq \mathfrak{q}$ since $\mathfrak{p}$ is the maximal G-stable ideal of $k[U]$ with $\mathfrak{p}^G = 0$ (and similarly for $\mathfrak{q}$). But $V(\mathfrak{p} \cap k[V]) = \overline{j(U_{soc})}$ (by standard results relating subvarieties, ideals and morphisms), while $V(\mathfrak{q}) \subseteq V(\mathfrak{p} \cap k[V])$ since $\mathfrak{p} \cap k[V] \subseteq \mathfrak{q}$. Thus $V_{soc} \equiv V(\mathfrak{q}) \subseteq \overline{j(U_{soc})}$. This proves (1).

For (2) consider a G-invariant affine open subset U such that U_{soc} has maximal dimension. If V is any other G-invariant open affine subset, then, by (1), $\overline{V_{soc}} \subseteq \overline{(U \cap V)_{soc}}$. But also, $\overline{U_{soc}} \subseteq \overline{(U \cap V)_{soc}}$, and these are closed, irreducible subvarieties of X of the same dimension. Thus $\overline{U_{soc}} = \overline{(U \cap V)_{soc}}$. Hence $M = \overline{U_{soc}}$.

For (3) let $U \subseteq X$ be an affine, open, G-invariant subset such that $\overline{U_{soc}} = M$ takes its maximal value. If $D \subset U$ is a G-divisor, observe that $U \setminus D$ is affine. If further $M \cap (U \setminus D) = \emptyset$, then $M \subset \overline{(U \setminus D)_{soc}}$ is a proper inclusion, a contradiction. Thus $M \cap (U \setminus D) \neq \emptyset$. Therefore $U_{soc} = M \cap U$ is not contained in D. Now by Proposition 3.12, $D \subset U \to U/G$ is not dominant. Equivalently, the kernel of $k[U]^G \subset k[U] \to k[U]/\mathfrak{p}(D)$ is nonzero.

The fourth statement here follows directly from the third statement. □

Example 3.14. Let X be an affine torus embedding with torus action $T \times X \to X$. The T-invariant, open, affine subsets of X are the subsets of the form $U(Y) = \{x \in X \mid Y \subseteq \overline{Tx}\}$, where $Y \subseteq X$ is a T-orbit. It is clear that $U(Y)_{soc} = Y$.

We need the following lemma regarding semi-invariants.

Lemma 3.15. *Let $G \times X \to X$ be an action, where G is reductive. Let $0 \neq f \in k[X]_\chi$. Then χ is a unit of $E_G[X]$ if and only if $X_{soc} \cap X_f \neq \emptyset$.*

Proof. Assume that $\chi \in E_G[X]^*$. Then $fk[X]_{\chi^{-1}} = fk[X] \cap k[X]^G \neq \{0\}$. Then $V(f) \subset X \to X/G$ is not dominant. Thus $X_{soc} \not\subseteq V(f)$. Conversely, let $\chi \in E_G[X] \setminus E_G[X]^*$. Then $fk[X] \cap k[X]^G = \{0\}$. Hence, $V(f) \subset X \to X/G$ is dominant. Thus, by Proposition 3.12, $X_{soc} \subseteq V(f)$. □

Theorem 3.16. *As above, let $G \times X \to X$ be an action, where G is reductive. Assume that $\chi, \mu \in E_G[X]$ and $f \in k[X]_\chi \setminus \{0\}$ and $g \in k[X]_\mu \setminus \{0\}$.*

(1) *$\overline{(X_f)_{soc}} = \overline{(X_g)_{soc}}$ if and only if $E_G[X_f] = E_G[X_g]$.*
(2) *$E_G[X_f] \subseteq E_G[X_g]$ if and only if $\overline{(X_f)_{soc}} \subseteq \overline{(X_g)_{soc}}$.*

Proof. First notice that $\chi \in E_G[X_g]^*$ if and only if $E_G[X_f] \subseteq E_G[X_g]$. Now write $V_g(f) = V(f) \cap X_g$. Then, from Lemma 3.15, $\chi \in E_G[X_g]^*$ iff $(X_g)_{soc} \not\subseteq V_g(f)$ iff $(X_g)_{soc} \cap X_f \neq \emptyset$ iff $(X_g)_{soc} \cap X_{fg} \neq \emptyset$ iff $\overline{(X_g)_{soc}} = \overline{(X_{fg})_{soc}}$. Similarly, $\mu \in E_G[X_f]^*$ iff $\overline{(X_f)_{soc}} = \overline{(X_{fg})_{soc}}$. Finally recall from Theorem 3.13 that it is always the case that $\overline{(X_f)_{soc}} \subseteq \overline{(X_{fg})_{soc}}$. □

Finally we look at the case where X is factorial and G is reductive. We record the following useful corollary about how $E_G[X]$ determines the entire essence of all the socle-closures $\overline{(X_f)_{soc}}$ that arise from G-invariant, affine, open subsets of $U = X_f$ of X.

Corollary 3.17. *Let $G \times X \to X$ be an action, where G is reductive and X is factorial. Let $U, V \subseteq X$ be G-invariant, open, affine subsets of X. Then $\overline{(U)_{soc}} \subseteq \overline{(V)_{soc}}$ if and only if $E_G[U] \subseteq E_G[V]$.*

Proof. $U = X_f$ for some semi-invariant $f \in k[X]$, and similarly for V. So apply Theorem 3.16. □

Remark 3.18. In general the assignment

$$U \mapsto \overline{U_{soc}}$$

is better at differentiating affine open G-subsets than is the assignment

$$U \mapsto E_G[U].$$

E_G is enough to describe this "socle-identification" for the collection of open subsets of the form X_f, where f is a semi-invariant. But generally there are not enough semi-invariants to distinguish all subsets of the form $\overline{U_{soc}}$, with U open and affine, unless X is factorial or G is a torus.

4. One last question

Consider an action $H \times X \to X$, where $H \subseteq G$ is a closed subgroup and X is affine. We define the *induced space*

$$G *_H X = \{[g, x] \mid [g, x] = [gh^{-1}, hx]\},$$

as the geometric quotient of the action $H \times G \times X \to G \times X$, $(h, g, x) \mapsto (gh^{-1}, hx)$. In [8] we made the assumption that G/H is affine, and we then showed that

(1) $G *_H X$ is affine, and

(2) $H \times X \to X$ is observable if and only if $G \times G *_H X \to G *_H X$ is observable.

See Lemma 3.16 and Proposition 3.17 of [8].

Suppose now that $H \times X \to X$, and $H < G$ is such that G/H is quasi-affine and $k[G]^H$ is finitely generated over k (i.e. H is a Grosshans subgroup of G). We have three questions.

(1) Can we form $G \times_H X \subseteq Y$ so that Y is affine and $G \times_H X \subseteq Y$ is an open subset with complement in Y of codimension at least two? Is $Y = Spec(\mathcal{O}(G \times_H X))$?
(2) Can we then conclude that "If G is reductive, then $k[X]^H$ is finitely generated since $k[X]^H = \mathcal{O}(G \times_H X)^G = k[Y]^G$"?
(3) With Y as in (1) above, is it true that "$H \times X \to X$ is observable in codimension one if and only if $G \times Y \to Y$ is observable in codimension one"?

Acknowledgment

The author would like to thank the referee for his/her insight and many constructive suggestions for improvement.

References

[1] W. Ferrer-Santos, A. Rittatore, *"Actions and Invariants of Algebraic Groups"*, Series: Pure and Applied Math. 268, Dekker-CRC Press, Florida, 2005. MR2138858 (2006c:14067)

[2] F. Grosshans, *"Algebraic Homogeneous Spaces and Invariant Theory"*, Series: Lecture Notes in Mathematics v. 1673, Springer, 1997. MR1489234 (99b:13005)

[3] A. Magid, *Dimension in Rings with Solvable Algebraic Group Action*, Math. Scand. 47(1980), 21-28. MR600077 (82i:13015)

[4] D. Mumford, J. Fogarty, F. Kirwan, *"Geometric Invariant Theory"*, Third Ed., Ergebnisse Math., Springer, 1994. MR1304906 (95m:14012)

[5] K. Pommerening, *Invariants of Unipotent Groups*, in *"Invariant Theory"*, S. S. Koh (Editor), Lecture Notes in Math. v. 1278, Springer, 1987, 8-17. MR924162 (89a:14065)

[6] V. Popov, *On the stability of the action of an algebraic group on an algebraic variety*, Math. USSR Izv. 6(1973), 367-379. MR0301028 (46:188)

[7] V. Popov, E. Vinberg, *Invariant Theory, in: "Algebraic Geometry IV"*, Enc. Math. Sci., Vol. 55, Springer, Berlin, 1994, 123-284.

[8] L. E. Renner, A. Rittatore, *Observable actions of algebraic groups*, Transformation Groups, 14(2010), 985-999. MR2577204 (2011b:14102)

[9] M. Rosenlicht, *Some basic theorems on algebraic groups*, Amer. J. Math. 78(1956), 401-443. MR0082183 (18:514a)

[10] M. Rosenlicht, *A remark on quotient spaces*, An. Acad. Brasil. Ci. 35(1963), 487-489. MR0171782 (30:2009)

[11] E. B. Vinberg, *On stability of actions of reductive algebraic groups*, in: Fong, Yuen et al. (eds.), *"Lie Algebras, Rings and Related Topics"*, Papers of the 2nd Tainan-Moscow International Algebra Workshop 1997, Tainan, Taiwan, 1997, Springer-Verlag, Hong Kong, 2000, 188-202.

Department of Mathematics, University of Western Ontario, London, ON, N6A 5B7, Canada