CORRIGENDUM TO “MULTIPLIERS AND ESSENTIAL NORM ON THE DRURY-ARVESON SPACE”

QUANLEI FANG AND JINGBO XIA

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ABSTRACT. We give an elementary proof of the “one-function corona theorem” for the multipliers of the Drury-Arveson space.

Let $H^2_n$ be the Drury-Arveson space of analytic functions on the unit ball $B$ in $\mathbb{C}^n$, and let $\mathcal{M}$ denote the collection of multipliers of $H^2_n$. This corrigendum concerns the following lemma in our original article “Multipliers and essential norm on the Drury-Arveson space”, Proc. Amer. Math. Soc. 139 (2011), 2497-2504:

**Lemma 3.1.** Let $f \in \mathcal{M}$. If there is a $c > 0$ such that $|f(z)| \geq c$ for every $z \in B$, then $1/f$ is also a multiplier of $H^2_n$.

This lemma is, of course, an immediate consequence of the recently proved corona theorem for $\mathcal{M}$ [7]. But in our original article, we also asserted that Lemma 3.1 also follows from Theorem 2 in Chen’s paper [6], which claims that for $g \in H^2_n$, $\|g\|^2$ is equivalent to

$$
\|g\|^2_\# = |g(0)|^2 + \iint \frac{|g(z) - g(w)|^2}{|1 - \langle z, w \rangle|^{2n+1}} dv(z) dv(w).
$$

It has since been discovered that this theorem of Chen’s is false. We would like to thank Dechao Zheng for informing us of this fact, who in turn attributes his source to Carl Sundberg. Once one knows this, the fact that $\|g\|^2$ is not equivalent to $\|g\|^2_\#$ can be directly shown. For example, if one tries the special case where the complex dimension $n$ equals 2, one finds that

$$
\iint \frac{|z_1 - w_1|^2}{|1 - \langle z, w \rangle|^5} dv(z) dv(w) = \infty,
$$

where $z_1$ and $w_1$ denote the first components of $z$ and $w$, respectively.

Thus, the question arises as to whether there is a proof of Lemma 3.1 that does not invoke the corona theorem of Costea, Sawyer and Wick [7]. One might call Lemma 3.1 the “one-function corona theorem” for $\mathcal{M}$. We learned this term from Dechao Zheng, and we also learned that there is considerable interest in finding an elementary proof of the one-function corona theorem, a proof that does not involve hard analysis in the style of [7]. In this somewhat expanded version of a corrigendum, we will give an elementary proof of Lemma 3.1. The virtue of our proof is that it involves only soft analysis, and very little of it indeed. What
makes this proof particularly worth reporting is the fact that it is based on the very
essence of the theory of Drury-Arveson space, namely the von Neumann inequality
for commuting row contractions.

The proof begins with some elementary facts. For each \( f \in \mathcal{M} \), we write \( \| f \|_{\mathcal{M}} \)
for its multiplier norm. That is,
\[
\| f \|_{\mathcal{M}} = \sup \{ \| fg \| : g \in H^2_n, \| g \| \leq 1 \}.
\]

For each analytic function \( h \) on \( B \) and each \( 0 \leq r < 1 \), we define the function
\[
h_r(z) = h(rz), \quad z \in B.
\]

Lemma 1. Let \( f \in \mathcal{M} \). Then for each \( 0 \leq r < 1 \), we have \( f_r \in \mathcal{M} \) and
\[
\| f_r \|_{\mathcal{M}} \leq \| f \|_{\mathcal{M}}.
\]

Proof. Let \( T^n \) denote the \( n \)-dimensional torus \( \{ (\tau_1, \ldots, \tau_n) : |\tau_j| = 1, 1 \leq j \leq n \} \).
Let \( dm_n \) be the Lebesgue measure on \( T^n \) with the normalization \( m_n(T^n) = 1 \). For
each \( \tau = (\tau_1, \ldots, \tau_n) \in T^n \), define the unitary transformation \( U_\tau \) on \( C^n \) by the
formula
\[
U_\tau(z_1, \ldots, z_n) = (\tau_1 z_1, \ldots, \tau_n z_n).
\]

Let \( f \in \mathcal{M} \). Then we obviously have
\[
\| f \|_{\mathcal{M}} = \| f \circ U_\tau \|_{\mathcal{M}}, \quad \tau \in T^n.
\]
For each \( 0 \leq r < 1 \), define the function
\[
P_r(\tau_1, \ldots, \tau_n) = \prod_{j=1}^n \frac{1 - r^2}{|1 - r \bar{\tau_j}|^2}
\]
on \( T^n \). By the well-known properties of the Poisson kernel, we have
\[
M_{f_\tau} = \int M_{f \circ U_\tau} P_r(\tau) dm_n(\tau).
\]
Since the integral of \( P_r \) on \( T^n \) equals 1 and \( P_r \geq 0 \), the lemma follows. \( \square \)

For each real number \( -n \leq t < \infty \), let \( \mathcal{H}^{(t)} \) be the Hilbert space of analytic
functions on \( B \) with the reproducing kernel
\[
\frac{1}{(1 - \langle \zeta, z \rangle)^{n+1+t}}.
\]
Alternately, one can describe \( \mathcal{H}^{(t)} \) as the completion of \( C[z_1, \ldots, z_n] \) with respect
to the norm \( \| \cdot \|_t \) arising from the inner product \( \langle \cdot, \cdot \rangle_t \) defined according to the
following rules: \( \langle z^\alpha, z^\beta \rangle_t = 0 \) whenever \( \alpha \neq \beta \),
\[
\langle z^\alpha, z^\beta \rangle_t = \frac{\alpha!}{\prod_{j=1}^{|\alpha|} (n + t + j)}
\]
if \( \alpha \in \mathbb{Z}_1^n \setminus \{0\} \), and \( \langle 1, 1 \rangle_t = 1 \). Obviously, we have \( \mathcal{H}^{(-n)} = H^2_n \).
Moreover, \( \mathcal{H}^{(-1)} \) is the Hardy space \( H^2(S) \), and \( \mathcal{H}^{(0)} \) is the Bergman space on the unit ball.

Lemma 2. If \( f \in \mathcal{M} \), then \( f \) is also a multiplier for every \( \mathcal{H}^{(t)} \), \( -n < t < \infty \).
Moreover, for each \( -n < t < \infty \) we have
\[
\| fg \|_t \leq \| f \|_{\mathcal{M}} \| g \|_t \quad \text{whenever} \quad g \in \mathcal{H}^{(t)}.
\]
Proof. Let $N$ be the number operator introduced by Arveson [2]. That is, $Nz^\alpha = |\alpha|z^\alpha$, $\alpha \in \mathbb{Z}_2^n$. For each $-n < t < \infty$, let $M_{z_1}^{(t)}, \ldots, M_{z_n}^{(t)}$ denote the operators of multiplication by the coordinate functions on $H^{(t)}$. Using (1), it is straightforward to verify that

$$M_{z_1}^{(t)}M_{z_1}^{(t)*} + \cdots + M_{z_n}^{(t)}M_{z_n}^{(t)*} = N(n + t + N)^{-1}.$$ 

Thus for each $-n < t < \infty$, the commuting $n$-tuple $(M_{z_1}^{(t)}, \ldots, M_{z_n}^{(t)})$ is a row contraction. Consequently, the von Neumann inequality

$$\|p(M_{z_1}^{(t)}, \ldots, M_{z_n}^{(t)})\| \leq \|p\|_M$$

holds for every polynomial $p$. See [2], [8]. That is, for each polynomial $p$, we have

$$(2) \quad \|pq\|_t \leq \|p\|_M\|g\|_t \quad \text{whenever} \quad g \in H^{(t)}.$$ 

Thus our task is to show that (2) still holds if we replace $p$ by $f \in M$. For this we use the power series expansion

$$\frac{1}{(1-u)^n} = \sum_{j=0}^{\infty} c_j u^j, \quad \text{where} \quad c_j = \frac{(j + n - 1)!}{j!(n-1)!},$$

which holds when $|u| < 1$. Let any $f \in M$ be given. By the Cauchy integral formula

$$f(z) = \int \frac{f(\xi)}{(1 - (z, \xi))^n} d\sigma(\xi),$$

where $d\sigma$ is the spherical measure on $S = \{\xi \in \mathbb{C}^n : |\xi| = 1\}$, for each $0 \leq r < 1$ we have

$$f_r = \sum_{j=0}^{\infty} c_j r^j \psi_j, \quad \text{where} \quad \psi_j(z) = \int f(\xi) \langle z, \xi \rangle^j d\sigma(\xi).$$

For each $\xi \in S$, the operator of multiplication by the function $\langle z, \xi \rangle$ is obviously a contraction on $H^{(t)}$ as well as on $H^n_z$. Therefore $\|\psi_j\|_M \leq \int |f| d\sigma \leq \|f\|_\infty$ and, similarly, $\|M_{\psi_j}^{(t)}\| \leq \|f\|_\infty$ for each $j \geq 0$. Thus the operators

$$\sum_{j=0}^{k} c_j r^j M_{\psi_j}^{(t)}, \quad k = 0, 1, 2, \ldots,$$

form a Cauchy sequence with respect to the operator norm on $H^{(t)}$. Hence

$$\|M_{f_r}^{(t)}\| = \lim_{k \to \infty} \left\| \sum_{j=0}^{k} c_j r^j M_{\psi_j}^{(t)} \right\| \leq \liminf_{k \to \infty} \left\| \sum_{j=0}^{k} c_j r^j \psi_j \right\|_M,$$

where the $\leq$ follows from (2). Since the operators $\sum_{j=0}^{k} c_j r^j M_{\psi_j}$, $k = 0, 1, 2, \ldots$, converge to $M_{f_r}$ on $H^n_z$ with respect to the operator norm, we have

$$\|M_{f_r}^{(t)}\| \leq \lim_{k \to \infty} \left\| \sum_{j=0}^{k} c_j r^j \psi_j \right\|_M = \|f_r\|_M \leq \|f\|_M,$$

where the second $\leq$ follows from Lemma 1. Now, if $g \in H^{(t)}$, then $f_r g_r = (fg)_r$. Hence

$$\|(fg)_r\|_t = \|M_{f_r}^{(t)} g_r\|_t \leq \|f\|_M \|g_r\|_t \leq \|f\|_M \|g\|_t.$$ 

Since this holds for every $0 \leq r < 1$, we have $\|fg\|_t \leq \|f\|_M \|g\|_t$ as promised. □
Write \( R = z_1 \partial_1 + \cdots + z_n \partial_n \), the radial derivative in \( n \) variables. We now pick a fixed natural number \( m \) such that \( 2m - n \geq 0 \). For each integer \( 0 \leq k \leq m \), we define the norm \( \| \cdot \|_{*,k} \) by the formula

\[
\| g \|_{*,k}^2 = |g(0)|^2 + \int |(R^k g)(z)|^2 (1 - |z|^2)^{2m-n} dv(z).
\]

For each \( \alpha \in \mathbb{Z}_+^n \), we have \( Rz^\alpha = |\alpha| z^\alpha \) and

\[
\int |\alpha|^2 (1 - |z|^2)^{2m-n} dv(z) = \frac{n!(2m-n)!\alpha!}{(|\alpha| + 2m)!}.
\]

Comparing this with (1), for each \( 0 \leq k \leq m \) there are \( 0 < c_k \leq C_k < \infty \) such that

\[
(3) \quad c_k \| g \|_{*,k} \leq \| g \|_{2m-n-2k} \leq C_k \| g \|_{*,k} \quad \text{for every } g \in \mathcal{H}^{2m-n-2k}.
\]

In particular, \( \| \cdot \|_{*,m} \) is equivalent to the norm on the Drury-Arveson space \( H_n^2 \).

**Proposition 3.** Suppose that \( Y_1, \ldots, Y_K \) are operators satisfying the following conditions:

(a) For each \( 1 \leq j \leq K \), either \( Y_j = R \) or \( Y_j = M_{f_j} \) for some \( f_j \in \mathcal{M} \).

(b) \( Y_1 = R \).

(c) \( \text{card} \{ j : Y_j = R, 1 \leq j \leq K \} = m \).

Then there is a constant \( C = C(Y_1, \ldots, Y_K) \) such that

\[
(4) \quad \int |(Y_1 \cdots Y_K g)(z)|^2 (1 - |z|^2)^{2m-n} dv(z) \leq C \| g \|^2
\]

for every \( g \in H_n^2 \), where \( \| \cdot \| \) denotes the norm on \( H_n^2 \).

**Proof.** We use an induction on \( K \). If \( K = m \), then, of course, (4) holds by virtue of (3) in the case \( k = m \). Now suppose that \( \ell \geq m \) and that the proposition has been proved for all \( m \leq K \leq \ell \). Consider the case \( K = \ell + 1 \). Let \( B \) denote the collection of \( j \)'s in \( \{ 1, \ldots, K \} \) such that \( Y_j = M_{f_j} \) for some \( f_j \in \mathcal{M} \). Then \( B \neq \emptyset \).

Let \( j_0 \) be the smallest integer in \( B \). If \( B = \{ j : j_0 \leq j \leq K \} \), then

\[
Y_1 \cdots Y_K = R^m M_{f_{j_0} \cdots f_K},
\]

and (4) again follows from (3) and the assumption that \( f_{j_0}, \ldots, f_K \) are multipliers of \( H_n^2 \).

Let us suppose that \( B \neq \{ j : j_0 \leq j \leq K \} \). Then the induction hypothesis gives us a \( C_1 \) such that

\[
(5) \quad \int |(Y_1 \cdots Y_{j_0-1} Y_{j_0+1} \cdots Y_K h)(z)|^2 (1 - |z|^2)^{2m-n} dv(z) \leq C_1 \| h \|^2
\]

for every \( h \in H_n^2 \). Now, given a \( g \in H_n^2 \), write

\[
\tilde{g} = Y_{j_0+1} \cdots Y_K g.
\]

Then

\[
Y_1 \cdots Y_K g = R^{j_0-1}(f_{j_0} \tilde{g}).
\]
Since $B \neq \{ j : j_0 \leq j \leq K \}$, there is at least one $j \in \{ j_0 + 1, \ldots, K \}$ such that $Y_j = R$. Thus $\tilde{g}(0) = 0$. Applying (3) and Lemma 2, we have

$$
\begin{align*}
\int |(Y_1 \cdots Y_K g)(z)|^2 (1 - |z|^2)^{2m-n} dv(z) &= \int |(R^{j_0-1}(f_{j_0}\tilde{g}))(z)|^2 (1 - |z|^2)^{2m-n} dv(z) \\
&\leq c_{j_0-1}^{-2} ||f_{j_0}\tilde{g}||^2_{2m-n-2j_0+2} \\
&\leq (C_{j_0-1}/c_{j_0-1})^2 ||f_{j_0}||^2_M \int |(R^{j_0-1}\tilde{g})(z)|^2 (1 - |z|^2)^{2m-n} dv(z) \\
&= (C_{j_0-1}/c_{j_0-1})^2 ||f_{j_0}||^2_M \int |(Y_1 \cdots Y_{j_0-1}Y_{j_0+1} \cdots Y_K g)(z)|^2 (1 - |z|^2)^{2m-n} dv(z).
\end{align*}
$$

Combining this with (5), the induction on $K$ is complete. 

With the above preparation, we now have

An elementary proof of Lemma 3.1. Let $f$ be given as in the lemma. By (3), it suffices to show that there is a $0 < C < \infty$ such that

$$
\int |(R^m M_{f-1} g)(z)|^2 (1 - |z|^2)^{2m-n} dv(z) \leq C ||g||^2
$$

for every $g \in H^2_n$. By Proposition 3 and the assumption $|f| \geq c > 0$ on $B$, this inequality will follow if we can prove the following assertion: the operator $R^m M_{f-1}$ is the sum of a finite number of terms of the form

$$
a M_{f-1} Y_1 \cdots Y_K,
$$

where $a \in \mathbb{R}$, $\nu \in \mathbb{N}$, and the operators $Y_1, \ldots, Y_K$ satisfy the following conditions:

(a) for each $1 \leq j \leq K$, either $Y_j = R$ or $Y_j = M_f$;

(b) $Y_1 = R$;

(c) card$\{ j : Y_j = R, 1 \leq j \leq K \} = m$.

To prove this, note that for each natural number $k$, we have the commutation relation

$$
RM_{f-k} = (k+1)M_{f-k} R - kM_{f-k-1}RM_f.
$$

Obviously, the above assertion about $R^m M_{f-1}$ follows from this identity and an easy induction on $m$. 

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Department of Mathematics and Computer Science, Bronx Community College, CUNY, Bronx, New York 10453

E-mail address: fangquanlei@com

Department of Mathematics, State University of New York at Buffalo, Buffalo, New York 14260

E-mail address: jxia@acsu.buffalo.edu