ON RADIAL AND POLAR
BLASCHKE-MINKOWSKI HOMOMORPHISMS

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Abstract. Brunn-Minkowski type inequalities for radial Blaschke-Minkowski homomorphisms of star bodies and polar Blaschke-Minkowski homomorphisms of convex bodies are established.

The setting for this paper is \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) (\( n > 2 \)). Let \( \mathcal{C}^n \) denote the set of nonempty convex figures (compact, convex subsets) and let \( \mathcal{K}^n \) denote the subset of \( \mathcal{C}^n \) consisting of all convex bodies (compact, convex subsets with nonempty interiors) in \( \mathbb{R}^n \). We reserve the letter \( u \) for unit vectors, and the letter \( B \) is reserved for the unit ball centered at the origin. The surface of \( B \) is \( S^{n-1} \). The volume of the unit \( n \)-ball is denoted by \( \omega_n \).

For \( u \in S^{n-1} \), let \( E_u \) denote the hyperplane, through the origin, that is orthogonal to \( u \). We will use \( K_u \) to denote the image of \( K \) under an orthogonal projection onto the hyperplane \( E_u \).

We use \( V(K) \) for the \( n \)-dimensional volume of a convex body \( K \). Let \( h(K, \cdot) : S^{n-1} \to \mathbb{R} \) denote the support function of \( K \in \mathcal{K}^n \); i.e. for \( u \in S^{n-1} \), \( h(K, u) = \max\{u \cdot x : x \in K\} \), where \( u \cdot x \) denotes the usual inner product of \( u \) and \( x \) in \( \mathbb{R}^n \).

Let \( \delta \) denote the Hausdorff metric on \( \mathcal{K}^n \), i.e., for \( K, L \in \mathcal{K}^n \), \( \delta(K, L) = |h_K - h_L|_\infty \), where \( |\cdot|_\infty \) denotes the sup-norm on the space of continuous functions \( C(S^{n-1}) \).

Associated with a compact subset \( K \) of \( \mathbb{R}^n \), which is star-shaped with respect to the origin, is its radial function \( \rho(K, \cdot) : S^{n-1} \to \mathbb{R} \), defined for \( u \in S^{n-1} \), by \( \rho(K, u) = \max\{\lambda \geq 0 : \lambda u \in K\} \).

If \( \rho(K, \cdot) \) is positive and continuous, \( K \) will be called a star body. Let \( S^n \) denote the set of star bodies in \( \mathbb{R}^n \). Let \( \tilde{\delta} \) denote the radial Hausdorff metric, i.e., if \( K, L \in S^n \), then \( \tilde{\delta}(K, L) = |\rho_K - \rho_L|_\infty \).

If \( K \) and \( L \) are convex bodies in \( \mathbb{R}^n \), then there is a convex body \( K \hat{+} L \) such that \( S(K \hat{+} L, \cdot) = S(K, \cdot) + S(L, \cdot) \), where \( S(K, \cdot) \) denotes the surface area measure of \( K \). The operation \( \hat{+} \) is called the Blaschke sum (see e.g. [13]).

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If $K$ is a convex body that contains the origin in its interior, the polar body $K^*$ of $K$ is defined by

$$K^* := \{ x \in \mathbb{R}^n | x \cdot y \leq 1, y \in K \}.$$ 

1. **Dual mixed volumes**

The radial Minkowski linear combination, $\lambda_1 K_1 + \cdots + \lambda_r K_r$, is defined by

$$\lambda_1 K_1 + \cdots + \lambda_r K_r = \{ \lambda_1 x_1 + \cdots + \lambda_r x_r : x_i \in K_i, i = 1, \ldots, r \},$$

for $K_1, \ldots, K_r \in S^n$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$.

It has the following important property (see [13]):

$$\rho(\lambda K + \mu L, \cdot) = \lambda \rho(K, \cdot) + \mu \rho(L, \cdot),$$

for $K, L \in S^n$ and $\lambda, \mu \geq 0$.

For $K_1, \ldots, K_r \in S^n$ and $\lambda_1, \ldots, \lambda_r \geq 0$, the volume of the radial Minkowski linear combination $\lambda_1 K_1 + \cdots + \lambda_r K_r$ is a homogeneous polynomial of degree $n$ in the $\lambda_i$,

$$V(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum \hat{V}(K_{i_1}, \ldots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n},$$

where the sum is taken over all $n$-tuples $(i_1, \ldots, i_n)$ whose entries are positive integers not exceeding $r$. If we require the coefficients of the polynomial in (1.3) to be symmetric in their argument, then they are uniquely determined. The coefficient $\hat{V}(K_{i_1}, \ldots, K_{i_n})$ is nonnegative and depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$. It is called the dual mixed volume of $K_{i_1}, \ldots, K_{i_n}$.

If $K_1, \ldots, K_n \in S^n$, the dual mixed volume $\hat{V}(K_1, \ldots, K_n)$ can be represented in the form (see [14])

$$\hat{V}(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u) \cdots \rho(K_n, u) dS(u).$$

If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = L$, the dual mixed volume is written as $\hat{V}_i(K, L)$. If $L = B$, the dual mixed volume $\hat{V}_i(K, L) = \hat{V}(K, B)$ is written as $\hat{W}_i(K)$.

For $K, L \in S^n$, the $i$-th dual mixed volume of $K$ and $L$, $\hat{V}_i(K, L)$, can be extended to all $i \in \mathbb{R}$ by

$$\hat{V}_i(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} \rho(L, u)^i dS(u), \quad i \in \mathbb{R}.$$ 

Thus, if $K \in S^n$ and $i \in \mathbb{R}$, then (see [14])

$$\hat{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} dS(u), \quad i \in \mathbb{R}.$$ 

If $K$ and $L$ are star bodies in $\mathbb{R}^n$, $s \neq 0$ and $\lambda, \mu \geq 0$, then $\lambda \cdot K + s \mu \cdot L$ is the star body whose radial function is given by

$$\rho(\lambda \cdot K + s \mu \cdot L, \cdot)^s = \lambda \rho(K, \cdot)^s + \mu \rho(L, \cdot)^s.$$ 

The addition $\hat{+}_s$ is called the $L_s$ radial sum.

The $L_s$ dual Brunn-Minkowski inequality states: If $K, L \in S^n$, then

$$V(K \hat{+}_s L)^{-s/n} \geq V(K)^{-s/n} + V(L)^{-s/n},$$

with equality if and only if $K$ and $L$ are dilates.
2. Radial Blaschke-Minkowski homomorphisms

Definition 2.1 ([16]). A map $\Psi : S^n \to S^n$ is called a radial Blaschke-Minkowski homomorphism if it satisfies the following conditions:

(a) $\Psi$ is continuous.
(b) For all $K, L \in S^n$,
   \[ \Psi(K \hat{+} L) = \Psi(K) \hat{+} \Psi(L), \]
   where $\hat{+}$ denotes the $L_{n-1}$ radial sum of $K$ and $L$.
(c) For all $K, L \in S^n$ and every $\vartheta \in SO(n)$,
   \[ \Psi(\vartheta K) = \vartheta \Psi(K), \]
where $SO(n)$ is the group of rotations in $n$ dimensions.

In 2006, Schuster [16] established the following Brunn-Minkowski inequality for radial Blaschke-Minkowski homomorphisms of star bodies.

If $K$ and $L$ are star bodies in $\mathbb{R}^n$, then
\[(2.1) \quad V(\Psi(K \hat{+} L))^{1/n(n-1)} \leq V(\Psi K)^{1/n(n-1)} + V(\Psi L)^{1/n(n-1)},\]

with equality if and only if $K$ and $L$ are dilates.

In fact a more general version of the Brunn-Minkowski inequality for radial Blaschke-Minkowski homomorphisms holds (see [16]): If $K$ and $L$ are star bodies in $\mathbb{R}^n$ and $0 \leq i \leq n - 1$, $0 \leq j < n - 2$, then
\[(2.2) \quad \bar{W}_i(\Psi_j(K \hat{+} L))^{1/(n-i)(n-j-1)} \leq \bar{W}_i(\Psi_j K)^{1/(n-i)(n-j-1)} + \bar{W}_i(\Psi_j L)^{1/(n-i)(n-j-1)},\]

with equality if and only if $K$ and $L$ are dilates. Here $\Psi_j$ denotes the mixed radial Blaschke-Minkowski homomorphism defined by:

Theorem 2.2 (see [16]). Let $\Psi : S^n \to S^n$ be a radial Blaschke-Minkowski homomorphism. There is a continuous operator $\Psi : S^n \times \cdots \times S^n \to S^n$, symmetric in its arguments such that, for $K_1, \ldots, K_m \in S^n$ and $\lambda_1, \ldots, \lambda_m \leq 0$,
\[ \Psi(\lambda_1 K_1 \hat{+} \cdots \hat{+} \lambda_m K_m) = \sum_{i_1, \ldots, i_{n-1}} \lambda_{i_1} \cdots \lambda_{i_{n-1}} \Psi(K_{i_1}, \ldots, K_{i_{n-1}}). \]

Clearly, Theorem 2.2 generalizes the notion of radial Blaschke-Minkowski homomorphisms. We call $\Psi : S^n \times \cdots \times S^n \to S^n$ the mixed radial Blaschke-Minkowski homomorphism induced by $\Psi$. The mixed radial Blaschke-Minkowski homomorphisms were first studied in more detail in [17]–[18]. If $K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = B$, we write $\Psi_i K$ for $\Psi(K, \ldots, K, B, \ldots, B)$ and call $\Psi_i K$ the mixed Blaschke-Minkowski homomorphism of order $i$ of $K$. We write $\Psi_i(K, L)$ for $\Psi(K, \ldots, K, L, \ldots, L)$.

The aim of this paper is to establish the following new Brunn-Minkowski inequality for mixed radial Blaschke-Minkowski homomorphisms.
Theorem 2.3. If $K, L \in S^n$ and $i, j \in \mathbb{R}, s \in \mathbb{N}$ satisfy $i \leq n - 1 \leq j \leq n, 0 \leq s \leq n - 1$, then
\begin{equation}
\left( \frac{\tilde{W}_i(\Psi_s(K+L))}{\tilde{W}_j(\Psi_s(K+L))} \right)^{1/(j-i)} \leq \left( \frac{\tilde{W}_i(\Psi_sK)}{\tilde{W}_j(\Psi_sK)} \right)^{1/(j-i)} + \left( \frac{\tilde{W}_i(\Psi_sL)}{\tilde{W}_j(\Psi_sL)} \right)^{1/(j-i)},
\end{equation}
with equality if and only if $K$ and $L$ are dilates.

3. Polar Blaschke-Minkowski homomorphisms

Definition 3.1 (see [16]). A map $\Phi : \mathcal{K}^n \rightarrow \mathcal{K}^n$ is called a Blaschke-Minkowski homomorphism if it satisfies the following conditions:

(a) $\Phi$ is continuous.

(b) For all $K, L \in \mathcal{K}^n$,
\[ \Phi(K+L) = \Phi(K) + \Phi(L). \]

(c) For all $K, L \in \mathcal{K}^n$ and every $\vartheta \in SO(n)$,
\[ \Phi(\vartheta K) = \vartheta \Phi(K). \]

In [16] it was shown that the polar body $(\Phi K)^*$ is well defined for every Blaschke-Minkowski homomorphism $\Phi$ and $K \in \mathcal{K}^n$. In the following we simply write $\Phi^*K$ rather than $(\Phi K)^*$.

In 2006, Schuster [16] also established the following Brunn-Minkowski inequality for polars of even Blaschke-Minkowski homomorphisms $\Phi$ of convex bodies.

If $K$ and $L$ are convex bodies in $\mathbb{R}^n$, then
\begin{equation}
V(\Phi^*(K+L))^{-1/n(n-1)} \geq V(\Phi^*K)^{1/(n(n-1))} + V(\Phi^*L)^{1/n(n-1)},
\end{equation}
with equality if and only if $K$ and $L$ are homothetic.

In fact a more general version of the Brunn-Minkowski inequality for polars of even Blaschke-Minkowski homomorphisms holds (see [16]): If $K$ and $L$ are convex bodies in $\mathbb{R}^n$ and $0 \leq j \leq n - 3$, then
\begin{equation}
V(\Phi^j(K+L))^{-1/(n-j-1)} \geq V(\Phi^jK)^{1/(n-j-1)} + V(\Phi^jL)^{1/(n-j-1)},
\end{equation}
with equality if and only if $K$ and $L$ are homothetic. Here, $\Phi^jK$ denotes the mixed Blaschke-Minkowski homomorphism induced by $\Phi$ defined by:

Theorem 3.2 (see [16]). There is a continuous operator $\Phi : \mathcal{K}^n \times \cdots \times \mathcal{K}^n$ symmetric in its arguments such that, for $K_1, \ldots, K_r$ and $\lambda_1, \ldots, \lambda_r \geq 0$,
\[ \Phi(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum_{i_1, \ldots, i_{n-1}} \lambda_{i_1} \cdots \lambda_{i_{n-1}} \Phi(K_{i_1}, \ldots, K_{i_{n-1}}). \]

Clearly, Theorem 3.2 generalizes the notion of Blaschke-Minkowski homomorphisms. We call $\Phi : \mathcal{K}^n \times \cdots \times \mathcal{K}^n \rightarrow \mathcal{K}^n$ the mixed Blaschke-Minkowski homomorphism induced by $\Phi$. Mixed Blaschke-Minkowski homomorphisms were first studied in more detail in [18]. If $K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = B$, we write $\Phi_i K$ for $\Phi(K, \ldots, K, B, \ldots, B)$ and call $\Phi_i K$ the mixed Blaschke-Minkowski homomorphism of order $i$. We write $\Phi_i(K, L)$ for $\Phi(K, \ldots, K, L, \ldots, L)$ and write $\Phi_0 K$ as $\Phi K$. 
Blaschke-Minkowski homomorphisms are an important notion in the theory of convex body-valued valuations (see, e.g., [5–6], [9–10], [15], [19], [22] and [1–2], [7–8], [11–12], [20]). They are natural duals to radial Blaschke-Minkowski homomorphisms which are important examples of star body-valued valuations.

Another aim of this paper is to establish the following Brunn-Minkowski inequality for polars of even Blaschke-Minkowski homomorphisms.

**Theorem 3.3.** Let $K, L$ be convex bodies in $\mathbb{R}^n$ and $i, j \in \mathbb{R}$ satisfy $i \geq n + 1 \geq j \geq n$. Then

\[
\left( \frac{W_i(\Phi^*(K + L))}{W_j(\Phi^*(K + L))} \right)^{1/(i-j)} \leq \left( \frac{W_i(\Phi^*K)}{W_j(\Phi^*K)} \right)^{1/(i-j)} + \left( \frac{W_i(\Phi^*L)}{W_j(\Phi^*L)} \right)^{1/(i-j)},
\]

with equality if and only if $K$ and $L$ are homothetic.

### 4. BRUNN–MINKOWSKI TYPE INEQUALITIES FOR RADIAL AND POLAR BLASCHKE–MINKOWSKI HOMOMORPHISMS

An extension of Beckenbach’s inequality (see [3], p. 27) was obtained by Dresher [4] by means of moment-space techniques:

**Lemma 4.1** (The Beckenbach-Dresher inequality). If $p \geq 1 \geq r \geq 0$, $f, g \geq 0$, and $\phi$ is a distribution function, then

\[
\left( \frac{\int (f + g)^p \, d\phi}{\int (f + g)^r \, d\phi} \right)^{1/(p-r)} \leq \left( \frac{\int f^p \, d\phi}{\int f^r \, d\phi} \right)^{1/(p-r)} + \left( \frac{\int g^p \, d\phi}{\int g^r \, d\phi} \right)^{1/(p-r)},
\]

with equality if and only if the functions $f$ and $g$ are proportional.

**Lemma 4.2** (see [16]). If $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a Blaschke-Minkowski homomorphism, then there is a function $g \in \mathcal{C}(S^{n-1}, \hat{e})$ such that

\[
h(\Phi K, \cdot) = S_{n-1}(K, \cdot)^{n-1} * g,
\]

where $\mathcal{C}(S^{n-1}, \hat{e})$ denotes the set of continuous zonal functions on $S^{n-1}$.

As a consequence of Lemma 4.2, we have for the mixed Blaschke-Minkowski homomorphism induced by $\Phi$,

\[
h(\Phi(K_1, \ldots, K_{n-1}), \cdot) = S(K_1, \ldots, K_{n-1}; \cdot) * g,
\]

where $S(K_1, \ldots, K_{n-1}; \cdot)$ is the mixed surface area measure of $K_1, \ldots, K_{n-1}$.

Let $\Phi : \mathcal{K}^n \times \cdots \times \mathcal{K}^n \to \mathcal{K}^n$ be a mixed Blaschke-Minkowski homomorphism.

If $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$, then (see [16])

\[
\rho(\Phi^*(K_1, \ldots, K_{n-1}), \cdot)^{-1} = h(\Phi^*(K_1, \ldots, K_{n-1}), \cdot).
\]

**Lemma 4.3** (see [16]). A map $\Psi : S^n \to S^n$ is a radial Blaschke-Minkowski homomorphism if and only if there is a measure $\mu \in \mathcal{M}_+(S^{n-1}, \hat{e})$ such that

\[
\rho(\Psi K, \cdot) = \rho(K, \cdot)^{n-1} * \mu,
\]

where $\mathcal{M}_+(S^{n-1}, \hat{e})$ denotes the set of nonnegative zonal measures on $S^{n-1}$. 

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For the mixed radial Blaschke-Minkowski homomorphism induced by $\Psi$, we have
\begin{equation}
\rho(\Psi(K_1, \ldots, K_{n-1}), \cdot) = \rho(K_1, \cdot) \cdots \rho(K_{n-1}, \cdot) * \mu.
\end{equation}

We are now in a position to prove Theorem 2.3. The following statement is just a slight reformulation of Theorem 2.3.

**Theorem 4.4.** If $K, L \in S^n$ and $p, r \in \mathbb{R}, j \in \mathbb{N}$ satisfy $0 \leq r \leq 1 \leq p$, $0 \leq j \leq n - 1$, then
\begin{equation}
\left( \frac{\tilde{W}_{n-p}(\Psi_j(K + jL))}{\tilde{W}_{n-r}(\Psi_j(K + jL))} \right)^{1/(p-r)} \leq \left( \frac{\tilde{W}_{n-p}(\Psi_jK)}{\tilde{W}_{n-r}(\Psi_jK)} \right)^{1/(p-r)} + \left( \frac{\tilde{W}_{n-p}(\Psi_jL)}{\tilde{W}_{n-r}(\Psi_jL)} \right)^{1/(p-r)},
\end{equation}
with equality if and only if $K$ and $L$ are dilates.

**Proof.** From (1.7), we have
\[ \rho(K + sL, \cdot)^s * \mu = \rho(K, \cdot)^s * \mu + \rho(L, \cdot)^s * \mu, \quad s \neq 0, \]
where $\mu$ is defined in Lemma 4.3. Hence, from (1.2) and (4.6), we obtain
\[ \rho(\Psi_s(K + sL), \cdot) = \rho(\Psi_sK, \cdot) + \rho(\Psi_sL, \cdot) = \rho(\Psi_sK + \Psi_sL, \cdot). \]
Namely,
\[ \Psi_s(K + sL) = \Psi_sK + \Psi_sL. \]

Therefore, from (1.2) and (1.6), we have
\begin{equation}
\begin{aligned}
\tilde{W}_{n-p}(\Psi_j(K + jL)) &= \frac{1}{n} \int_{S^{n-1}} \rho(\Psi_j(K + jL), u)^p dS(u) \\
&= \frac{1}{n} \int_{S^{n-1}} \rho(\Psi_jK + \Psi_jL, u)^p dS(u) = \frac{1}{n} \int_{S^{n-1}} (\rho(\Psi_jK, u) + \rho(\Psi_jL, u))^p dS(u)
\end{aligned}
\end{equation}

and
\begin{equation}
\tilde{W}_{n-r}(\Psi_j(K + jL)) = \frac{1}{n} \int_{S^{n-1}} (\rho(\Psi_jK, u) + \rho(\Psi_jL, u))^r dS(u).
\end{equation}

From (4.8) and (4.9) and in view of Lemma 4.1, we obtain
\begin{equation}
\left( \frac{\tilde{W}_{n-p}(\Psi_j(K + jL))}{\tilde{W}_{n-r}(\Psi_j(K + jL))} \right)^{\frac{1}{p-r}} = \left( \frac{\int_{S^{n-1}} (\rho(\Psi_jK, u) + \rho(\Psi_jL, u))^p dS(u)}{\int_{S^{n-1}} (\rho(\Psi_jK, u) + \rho(\Psi_jL, u))^r dS(u)} \right)^{\frac{1}{p-r}}
\end{equation}

\begin{equation}
\leq \left( \frac{\int_{S^{n-1}} \rho(\Psi_jK, u)^p dS(u)}{\int_{S^{n-1}} \rho(\Psi_jK, u)^r dS(u)} \right)^{\frac{1}{p-r}} + \left( \frac{\int_{S^{n-1}} \rho(\Psi_jL, u)^p dS(u)}{\int_{S^{n-1}} \rho(\Psi_jL, u)^r dS(u)} \right)^{\frac{1}{p-r}}
\end{equation}

\begin{equation}
= \left( \frac{\tilde{W}_{n-p}(\Psi_jK)}{\tilde{W}_{n-r}(\Psi_jK)} \right)^{\frac{1}{p-r}} + \left( \frac{\tilde{W}_{n-p}(\Psi_jL)}{\tilde{W}_{n-r}(\Psi_jL)} \right)^{\frac{1}{p-r}}.
\end{equation}

Equality holds if and only if the functions $\rho(\Psi_jK, u)$ and $\rho(\Psi_jL, u)$ are proportional. Namely, $\rho(\Psi_jK, u) = \lambda \rho(\Psi_jL, u)$. From (4.5), we obtain $\rho(\Psi_jK, u) = \rho(\Psi_j(\lambda^{1/(n-j-1)}L), u)$. Hence, equality holds if and only if $K$ and $L$ are dilates.

Let $p = n - i$ and $r = n - j$. Since $0 \leq r \leq 1 \leq p$, we have
\[ r \leq 1 \leq p \Rightarrow i \leq n - 1 \leq j, \quad 0 \leq r \Rightarrow j \leq n. \]
Therefore,
\begin{equation}
(4.10) \quad i \leq n - 1 \leq j \leq n.
\end{equation}

Taking for \( p = n - i \) and \( r = n - j \) in (4.7) and using (4.10), we see that (4.7) changes to the inequality in Theorem 2.3.

Taking for \( p = n - i, j = 0 \) and \( r = 1 \) in (4.7), (4.7) changes to the following inequality:
\begin{equation}
(4.11) \quad \left( \frac{\tilde{W}_i(\Psi(K + L))}{\tilde{W}_{n-1}(\Psi(K + L))} \right)^{1/(n-i-1)} \leq \left( \frac{\tilde{W}_i(\Psi K)}{\tilde{W}_{n-1}(\Psi K)} \right)^{1/(n-i-1)} + \left( \frac{\tilde{W}_i(\Psi L)}{\tilde{W}_{n-1}(\Psi L)} \right)^{1/(n-i-1)},
\end{equation}
with equality if and only if \( K \) and \( L \) are dilates.

For \( K \in S^n \), there is a unique star body \( IK \) whose radial function satisfies for \( u \in S^{n-1} \),
\[ \rho(IK, u) = v(K \cap E_u). \]

It is called the \textit{intersection bodies} of \( K \). The volume of intersection bodies is given by
\[ V(IK) = \frac{1}{n} \int_{S^{n-1}} v(K \cap E_u)^n dS(u). \]

The mixed intersection body of \( K_1, \ldots, K_{n-1} \in S^n \), \( I(K_1, \ldots, K_{n-1}) \), is defined by
\[ \rho(I(K_1, \ldots, K_{n-1}), u) = \tilde{v}(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u), \]
where \( \tilde{v} \) is \((n - 1)\)-dimensional dual mixed volume.

If \( K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = L \), then \( I(K_1, \ldots, K_{n-1}) \) is written as \( I_i(K, L) \). If \( L = B \), then \( I_i(K, L) \) is written as \( I_iK \) and called the \( i \)th intersection body of \( K \). For \( I_0K \) we simply write \( IK \).

If \( \Psi : S^n \times \cdots \times S^n \to S^n \) is the mixed intersection operator \( I : S^n \times \cdots \times S^n \to S^n \) in (4.7), we obtain

**Corollary 4.5.** If \( K, L \in S^n \) and \( p, r \in \mathbb{R}, j \in \mathbb{N} \) satisfy \( 0 \leq r \leq 1 \leq p, 0 \leq j \leq n - 1 \), then
\begin{equation}
(4.12) \quad \left( \frac{\tilde{W}_{n-p}(I_j(K +_j L))}{\tilde{W}_{n-r}(I_j(K +_j L))} \right)^{1/(p-r)} \leq \left( \frac{\tilde{W}_{n-p}(I_j K)}{\tilde{W}_{n-r}(I_j K)} \right)^{1/(p-r)} + \left( \frac{\tilde{W}_{n-p}(I_j L)}{\tilde{W}_{n-r}(I_j L)} \right)^{1/(p-r)},
\end{equation}
with equality if and only if \( K \) and \( L \) are dilates.

The following statement is just a slight reformulation of Theorem 3.3.

**Theorem 4.6.** If \( K, L \in K^n \) and \( p, r \in \mathbb{R} \) satisfy \( p \leq -1 \leq r \leq 0 \), then
\begin{equation}
(4.13) \quad \left( \frac{\tilde{W}_{n-p}(\Phi^*(K + L))}{\tilde{W}_{n-r}(\Phi^*(K + L))} \right)^{1/(r-p)} \leq \left( \frac{\tilde{W}_{n-p}(\Phi^* K)}{\tilde{W}_{n-r}(\Phi^* K)} \right)^{1/(r-p)} + \left( \frac{\tilde{W}_{n-p}(\Phi^* L)}{\tilde{W}_{n-r}(\Phi^* L)} \right)^{1/(r-p)},
\end{equation}
with equality if and only if \( K \) and \( L \) are homothetic.
Proof. From (1.6), (4.4) and in view of Definition 3.1, we have
\[
\tilde{W}_{n-p}(\Phi^*(K+L)) = \frac{1}{n} \int_{S^{n-1}} \rho(\Phi^*(K+L), u)^p dS(u)
\]
\[
= \frac{1}{n} \int_{S^{n-1}} h(\Phi(K+L), u)^{-p} dS(u) = \frac{1}{n} \int_{S^{n-1}} h(\Phi K + \Phi L, u)^{-p} dS(u)
\]
(4.14) \[
= \frac{1}{n} \int_{S^{n-1}} (h(\Phi K, u) + h(\Phi L, u))^{-p} dS(u)
\]
and
\[
\tilde{W}_{n-r}(\Phi^*(K+L)) = \frac{1}{n} \int_{S^{n-1}} (h(\Phi K, u) + h(\Phi L, u))^{-r} dS(u).
\]
From (4.14), (4.15), and Lemma 4.1, we obtain
\[
\left( \frac{\tilde{W}_{n-p}(\Phi^*(K+L))}{\tilde{W}_{n-r}(\Phi^*(K+L))} \right)^{1/(r-p)} = \left( \frac{\int_{S^{n-1}} (h(\Phi K, u) + h(\Phi L, u))^{-p} dS(u)}{\int_{S^{n-1}} (h(\Phi K, u) + h(\Phi L, u))^{-r} dS(u)} \right)^{1/(r-p)}
\]
\[
\leq \left( \frac{\int_{S^{n-1}} h(\Phi K, u)^{-p} dS(u)}{\int_{S^{n-1}} h(\Phi K, u)^{-r} dS(u)} \right)^{1/(r-p)} + \left( \frac{\int_{S^{n-1}} h(\Phi L, u)^{-p} dS(u)}{\int_{S^{n-1}} h(\Phi L, u)^{-r} dS(u)} \right)^{1/(r-p)}
\]
\[
= \left( \frac{\tilde{W}_{n-p}(\Phi^*K)}{\tilde{W}_{n-r}(\Phi^*K)} \right)^{1/(r-p)} + \left( \frac{\tilde{W}_{n-p}(\Phi^*L)}{\tilde{W}_{n-r}(\Phi^*L)} \right)^{1/(r-p)}.
\]
Equality holds if and only if the functions $h(\Phi^*K, u)$ and $h(\Phi^*L, u)$ are proportional, namely, $h(\Phi^*K, u) = \lambda h(\Phi^*L, u)$, and from (4.3), we obtain $h(\Phi^*K, u) = \lambda h(\Phi^*(\lambda^{-1/(n-1)}L), u)$. Hence, equality holds if and only if $K$ and $L$ are homothetic.

Let $p = n - i$ and $r = n - j$. Since $p \leq -1 \leq r \leq 0$, we have
\[
p \leq -1 \leq r \leq 0 \Rightarrow i \geq n + 1 \geq j \geq n.
\]
Taking for $p = n - i$ and $r = n - j$ in (4.13) and using (4.16), we see that (4.13) changes to the inequality in Theorem 3.3.

Taking $p = -n, r = -1$ and $s = 0$ in (4.13), we have
\[
\left( \frac{\tilde{W}_{2n}(\Phi^*(K+L))}{\tilde{W}_{n+1}(\Phi^*(K+L))} \right)^{1/(n-1)} \leq \left( \frac{\tilde{W}_{2n}(\Phi^*K)}{\tilde{W}_{n+1}(\Phi^*K)} \right)^{1/(n-1)} + \left( \frac{\tilde{W}_{2n}(\Phi^*L)}{\tilde{W}_{n+1}(\Phi^*L)} \right)^{1/(n-1)},
\]
with equality if and only if $K$ and $L$ are homothetic.

If $K_1, \ldots, K_r \in K^n$ and $\lambda_1, \ldots, \lambda_r \geq 0$, then the projection body of the Minkowski linear combination $\lambda_1 K_1 + \cdots + \lambda_r K_r \in K^n$ can be written as a symmetric homogeneous polynomial of degree $(n - 1)$ in the $\lambda_i$ (see [15]):
\[
\Pi(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum \lambda_i \cdots \lambda_{i-1} \Pi_{i_1 \cdots i_{n-1}},
\]
where the sum is a Minkowski sum taken over all $(n - 1)$-tuples $(i_1, \ldots, i_{n-1})$ of positive integers not exceeding $r$. The body $\Pi_{i_1 \cdots i_{n-1}}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_{n-1}}$, and is uniquely determined by (4.18). It is called the mixed projection bodies of $K_{i_1}, \ldots, K_{i_{n-1}}$, and is written as $\Pi(K_{i_1}, \ldots, K_{i_{n-1}}).$
If $K_1 = \cdots = K_{n-1-i} = K$ and $K_{n-i} = \cdots = K_{n-1} = L$, then $\Pi(K_1, \ldots, K_{n-1})$ will be written as $\Pi_i(K, L)$. If $L = B$, then $\Pi_i(K, L)$ is denoted by $\Pi_i K$ and when $i = 0$, $\Pi_i K$ is denoted by $\Pi K$, where $\Pi K$ is the projection body of $K$. □

If $\Phi : \mathcal{K}^n \times \cdots \times \mathcal{K}^n \to \mathcal{K}^n$ is the mixed projection operator $\Pi : \mathcal{K}^n \times \cdots \times \mathcal{K}^n \to \mathcal{K}^n$ in (4.13), we obtain

**Corollary 4.7.** If $K, L \in \mathcal{K}^n$ and $p, r \in \mathbb{R}$ satisfy $p \leq -1 \leq r \leq 0$, then

\[
\left( \frac{W_{n-p}(\Pi^*(K + L))}{W_{n-r}(\Pi^*(K + L))} \right)^{1/(r-p)} \leq \left( \frac{W_{n-p}(\Pi^* K)}{W_{n-r}(\Pi^* K)} \right)^{1/(r-p)} + \left( \frac{W_{n-p}(\Pi^* L)}{W_{n-r}(\Pi^* L)} \right)^{1/(r-p)},
\]

with equality if and only if $K$ and $L$ are homothetic.

We finally remark that inequalities for the intersection operator $I$ were also established in [24], [27], for the $L_p$-intersection operator $I_p$ in [21] and for the polar projection body operator $\Pi^*$ in [25], [26].

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