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ON RADIAL AND POLAR BLASCHKE-MINKOWSKI HOMOMORPHISMS

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Abstract. Brunn-Minkowski type inequalities for radial Blaschke-Minkowski homomorphisms of star bodies and polar Blaschke-Minkowski homomorphisms of convex bodies are established.

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$ $(n > 2)$. Let $\mathcal{C}^n$ denote the set of nonempty convex figures (compact, convex subsets) and let $\mathcal{K}^n$ denote the subset of $\mathcal{C}^n$ consisting of all convex bodies (compact, convex subsets with nonempty interiors) in $\mathbb{R}^n$. We reserve the letter u for unit vectors, and the letter B is reserved for the unit ball centered at the origin. The surface of B is S^{n-1}. The volume of the unit n-ball is denoted by ω_n. For $u \in S^{n-1}$, let E_u denote the hyperplane, through the origin, that is orthogonal to u. We will use K^u to denote the image of K under an orthogonal projection onto the hyperplane E_u.

We use $V(K)$ for the n-dimensional volume of a convex body K. Let $h(K,\cdot) : S^{n-1} \to \mathbb{R}$ denote the support function of $K \in \mathcal{K}^n$; i.e. for $u \in S^{n-1}$,

$$h(K,u) = \mathrm{Max}\{u \cdot x : x \in K\},$$

where $u \cdot x$ denotes the usual inner product of u and x in $\mathbb{R}^n$.

Let δ denote the Hausdorff metric on $\mathcal{K}^n$, i.e., for $K, L \in \mathcal{K}^n$, $\delta(K,L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$.

Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin, is its radial function $\rho(K,\cdot) : S^{n-1} \to \mathbb{R}$, defined for $u \in S^{n-1}$, by

$$\rho(K,u) = \mathrm{Max}\{\lambda \geq 0 : \lambda u \in K\}.$$

If $\rho(K,\cdot)$ is positive and continuous, K will be called a star body. Let $\mathcal{S}^n$ denote the set of star bodies in $\mathbb{R}^n$. Let $\tilde{\delta}$ denote the radial Hausdorff metric, i.e., if $K, L \in \mathcal{S}^n$, then $\tilde{\delta}(K,L) = |\rho_K - \rho_L|_\infty$.

If K and L are convex bodies in $\mathbb{R}^n$, then there is a convex body $K \ddot{+} L$ such that

$$S(K \ddot{+} L, \cdot) = S(K,\cdot) + S(L,\cdot),$$

where $S(K,\cdot)$ denotes the surface area measure of K. The operation $\ddot{+}$ is called the Blaschke sum (see e.g. [13]).

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If K is a convex body that contains the origin in its interior, the polar body K^* of K is defined by

$$K^* := \{x \in \mathbb{R}^n | x \cdot y \leq 1, y \in K\}.$$

1. Dual mixed volumes

The radial Minkowski linear combination, $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$, is defined by

$$\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r = \{\lambda_1 x_1 \tilde{+} \cdots \tilde{+} \lambda_r x_r : x_i \in K_i, i = 1, \ldots, r\}, \tag{1.1}$$

for $K_1, \ldots, K_r \in \mathcal{S}^n$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$.

It has the following important property (see [13]):

$$\rho(\lambda K \tilde{+} \mu L, \cdot) = \lambda \rho(K, \cdot) + \mu \rho(L, \cdot), \tag{1.2}$$

for $K, L \in \mathcal{S}^n$ and $\lambda, \mu \geq 0$.

For $K_1, \ldots, K_r \in \mathcal{S}^n$ and $\lambda_1, \ldots, \lambda_r \geq 0$, the volume of the radial Minkowski linear combination $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$ is a homogeneous polynomial of degree n in the λ_i,

$$V(\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r) = \sum \tilde{V}(K_{i_1}, \ldots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}, \tag{1.3}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ whose entries are positive integers not exceeding r. If we require the coefficients of the polynomial in (1.3) to be symmetric in their argument, then they are uniquely determined. The coefficient $\tilde{V}(K_{i_1}, \ldots, K_{i_n})$ is nonnegative and depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$. It is called the dual mixed volume of $K_{i_1}, \ldots, K_{i_n}$.

If $K_1, \ldots, K_n \in \mathcal{S}^n$, the dual mixed volume $\tilde{V}(K_1, \ldots, K_n)$ can be represented in the form (see [14])

$$\tilde{V}(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u) \cdots \rho(K_n, u) dS(u). \tag{1.4}$$

If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = L$, the dual mixed volume is written as $\tilde{V}_i(K, L)$. If $L = B$, the dual mixed volume $\tilde{V}_i(K, L) = \tilde{V}_i(K, B)$ is written as $\tilde{W}_i(K)$.

For $K, L \in \mathcal{S}^n$, the i-th dual mixed volume of K and L, $\tilde{V}_i(K, L)$, can be extended to all $i \in \mathbb{R}$ by

$$\tilde{V}_i(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} \rho(L, u)^i dS(u), \ i \in \mathbb{R}. \tag{1.5}$$

Thus, if $K \in \mathcal{S}^n$ and $i \in \mathbb{R}$, then (see [14])

$$\tilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} dS(u), \ i \in \mathbb{R}. \tag{1.6}$$

If K and L are star bodies in $\mathbb{R}^n$, $s \neq 0$ and $\lambda, \mu \geq 0$, then $\lambda \cdot K \tilde{+}_s \mu \cdot L$ is the star body whose radial function is given by

$$\rho(\lambda \cdot K \tilde{+}_s \mu \cdot L, \cdot)^s = \lambda \rho(K, \cdot)^s + \mu \rho(L, \cdot)^s. \tag{1.7}$$

The addition $\tilde{+}_s$ is called the L_s radial sum.

The L_s dual Brunn-Minkowski inequality states: If $K, L \in \mathcal{S}^n$, then

$$V(K \tilde{+}_s L)^{-s/n} \geq V(K)^{-s/n} + V(L)^{-s/n},$$

with equality if and only if K and L are dilates.

2. Radial Blaschke-Minkowski homomorphisms

Definition 2.1 ([16]). A map $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ is called a radial Blaschke-Minkowski homomorphism if it satisfies the following conditions:

(a) Ψ is continuous.

(b) For all $K, L \in \mathcal{S}^n$,

$$\Psi(K \hat{+} L) = \Psi(K) \tilde{+} \Psi(L),$$

where $\hat{+}$ denotes the L_{n-1} radial sum of K and L.

(c) For all $K, L \in \mathcal{S}^n$ and every $\vartheta \in SO(n)$,

$$\Psi(\vartheta K) = \vartheta \Psi(K),$$

where $SO(n)$ is the group of rotations in n dimensions.

In 2006, Schuster [16] established the following Brunn-Minkowski inequality for radial Blaschke-Minkowski homomorphisms of star bodies.

If K and L are star bodies in $\mathbb{R}^n$, then

$$V(\Psi(K \tilde{+} L))^{1/n(n-1)} \leq V(\Psi K)^{1/n(n-1)} + V(\Psi L)^{1/n(n-1)}, \tag{2.1}$$

with equality if and only if K and L are dilates.

In fact a more general version of the Brunn-Minkowski inequality for radial Blaschke-Minkowski homomorphisms holds (see [16]): If K and L are star bodies in $\mathbb{R}^n$ and $0 \leq i \leq n-1$, $0 \leq j < n-2$, then

$$\tilde{W}_i(\Psi_j(K \tilde{+} L))^{1/(n-i)(n-j-1)} \leq \tilde{W}_i(\Psi_j K)^{1/(n-i)(n-j-1)} + \tilde{W}_i(\Psi_j L)^{1/(n-i)(n-j-1)}, \tag{2.2}$$

with equality if and only if K and L are dilates. Here Ψ_j denotes the mixed radial Blaschke-Minkowski homomorphism defined by:

Theorem 2.2 (see [16]). *Let $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ be a radial Blaschke-Minkowski homomorphism. There is a continuous operator $\Psi : \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$, symmetric in its arguments such that, for $K_1, \ldots, K_m \in \mathcal{S}^n$ and $\lambda_1, \ldots, \lambda_m \leq 0$,*

$$\Psi(\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_m K_m) = \sum_{i_1, \ldots, i_{n-1}} \lambda_{i_1} \cdots \lambda_{n-1} \Psi(K_{i_1}, \ldots, K_{i_{n-1}}).$$

Clearly, Theorem 2.2 generalizes the notion of radial Blaschke-Minkowski homomorphisms. We call $\Psi : \mathcal{S}^n \times \cdots \times \mathcal{S}^n \to \mathcal{S}^n$ the mixed radial Blaschke-Minkowski homomorphism induced by Ψ. The mixed radial Blaschke-Minkowski homomorphisms were first studied in more detail in [17]–[18]. If $K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = B$, we write $\Psi_i K$ for $\Psi(\underbrace{K, \ldots, K}_{n-i-1}, \underbrace{B, \ldots, B}_{i})$ and call $\Psi_i K$ the mixed Blaschke-Minkowski homomorphism of order i of K. We write $\Psi_i(K, L)$ for $\Psi(\underbrace{K, \ldots, K}_{n-i-1}, \underbrace{L, \ldots, L}_{i})$.

The aim of this paper is to establish the following new Brunn-Minkowski inequality for mixed radial Blaschke-Minkowski homomorphisms.

Theorem 2.3. *If $K, L \in \mathcal{S}^n$ and $i, j \in \mathbb{R}, s \in \mathbb{N}$ satisfy $i \le n-1 \le j \le n, 0 \le s \le n-1$, then*

$$\left(\frac{\tilde{W}_i(\Psi_s(K\tilde{+}_sL))}{\tilde{W}_j(\Psi_s(K\tilde{+}_sL))}\right)^{1/(j-i)} \le \left(\frac{\tilde{W}_i(\Psi_sK)}{\tilde{W}_j(\Psi_sK)}\right)^{1/(j-i)} + \left(\frac{\tilde{W}_i(\Psi_sL)}{\tilde{W}_j(\Psi_sL)}\right)^{1/(j-i)}, \tag{2.3}$$

with equality if and only if K and L are dilates.

3. Polar Blaschke-Minkowski homomorphisms

Definition 3.1 (see [16])**.** A map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is called a Blaschke-Minkowski homomorphism if it satisfies the following conditions:

(a) Φ is continuous.

(b) For all $K, L \in \mathcal{K}^n$,

$$\Phi(K\ddot{+}L) = \Phi(K) + \Phi(L).$$

(c) For all $K, L \in \mathcal{K}^n$ and every $\vartheta \in SO(n)$,

$$\Phi(\vartheta K) = \vartheta\Phi(K).$$

In [16] it was shown that the polar body $(\Phi K)^*$ is well defined for every Blaschke-Minkowski homomorphism Φ and $K \in \mathcal{K}^n$. In the following we simply write Φ^*K rather than $(\Phi K)^*$.

In 2006, Schuster [16] also established the following Brunn-Minkowski inequality for polars of *even* Blaschke-Minkowski homomorphisms Φ of convex bodies.

If K and L are convex bodies in $\mathbb{R}^n$, then

$$V(\Phi^*(K+L))^{-1/n(n-1)} \ge V(\Phi^*K)^{1/(n(n-1)} + V(\Phi^*L)^{1/n(n-1)}, \tag{3.1}$$

with equality if and only if K and L are homothetic.

In fact a more general version of the Brunn-Minkowski inequality for polars of even Blaschke-Minkowski homomorphisms holds (see [16]): If K and L are convex bodies in $\mathbb{R}^n$ and $0 \le j \le n-3$, then

$$V(\Phi_j^*(K+L))^{-1/n(n-j-1)} \ge V(\Phi_j^*K)^{1/(n(n-j-1)} + V(\Phi_j^*L)^{1/n(n-j-1)}, \tag{3.2}$$

with equality if and only if K and L are homothetic. Here, Φ_jK denotes the mixed Blaschke-Minkowski homomorphism induced by Φ defined by:

Theorem 3.2 (see [16])**.** *There is a continuous operator $\Phi : \underbrace{\mathcal{K}^n \times \cdots \times \mathcal{K}^n}_{n-1}$, symmetric in its arguments such that, for $K_1, \ldots, K_r$ and $\lambda_1, \ldots, \lambda_r \ge 0$,*

$$\Phi(\lambda_1K_1 + \cdots + \lambda_rK_r) = \sum_{i_1,\ldots,i_{n-1}} \lambda_{i_1}\cdots\lambda_{n-1}\Phi(K_{i_1}, \ldots, K_{i_{n-1}}).$$

Clearly, Theorem 3.2 generalizes the notion of Blaschke-Minkowski homomorphisms. We call $\Phi : \mathcal{K}^n \times \cdots \times \mathcal{K}^n \to \mathcal{K}^n$ the mixed Blaschke-Minkowski homomorphism induced by Φ. Mixed Blaschke-Minkowski homomorphisms were first studied in more detail in [18]. If $K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = B$, we write Φ_iK for $\Phi(\underbrace{K, \ldots, K}_{n-i-1}, \underbrace{B, \ldots, B}_{i})$ and call Φ_iK the mixed Blaschke-Minkowski homomorphism of order i. We write $\Phi_i(K, L)$ for $\Phi(\underbrace{K, \ldots, K}_{n-i-1}, \underbrace{L, \ldots, L}_{i})$ and write Φ_0K as ΦK.

Blaschke-Minkowski homomorphisms are an important notion in the theory of convex body-valued valuations (see, e.g., [5]–[6], [9]–[10], [15], [19], [22] and [1]–[2], [7]–[8], [11]–[12], [20]–[21]). They are natural duals to radial Blaschke-Minkowski homomorphisms which are important examples of star body-valued valuations.

Another aim of this paper is to establish the following Brunn-Minkowski inequality for polars of even Blaschke-Minkowski homomorphisms.

Theorem 3.3. *Let K, L be convex bodies in $\mathbb{R}^n$ and $i, j \in \mathbb{R}$ satisfy $i \geq n+1 \geq j \geq n$. Then*

$$\left(\frac{\tilde{W}_i(\Phi^*(K \ddot{+} L))}{\tilde{W}_j(\Phi^*(K \ddot{+} L))}\right)^{1/(i-j)} \leq \left(\frac{\tilde{W}_i(\Phi^* K)}{\tilde{W}_j(\Phi^* K)}\right)^{1/(i-j)} + \left(\frac{\tilde{W}_i(\Phi^* L)}{\tilde{W}_j(\Phi^* L)}\right)^{1/(i-j)}, \tag{3.3}$$

with equality if and only if K and L are homothetic.

4. Brunn-Minkowski type inequalities for radial and polar Blaschke-Minkowski homomorphisms

An extension of Beckenbach's inequality (see [3], p. 27) was obtained by Dresher [4] by means of moment-space techniques:

Lemma 4.1 (The Beckenbach-Dresher inequality)**.** *If $p \geq 1 \geq r \geq 0$, $f, g \geq 0$, and ϕ is a distribution function, then*

$$\left(\frac{\int (f+g)^p d\phi}{\int (f+g)^r d\phi}\right)^{1/(p-r)} \leq \left(\frac{\int f^p d\phi}{\int f^r d\phi}\right)^{1/(p-r)} + \left(\frac{\int g^p d\phi}{\int g^r d\phi}\right)^{1/(p-r)}, \tag{4.1}$$

with equality if and only if the functions f and g are proportional.

Lemma 4.2 (see [16])**.** *If $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a Blaschke-Minkowski homomorphism, then there is a function $g \in \mathcal{C}(S^{n-1}, \hat{e})$ such that*

$$h(\Phi K, \cdot) = S_{n-1}(K, \cdot)^{n-1} * g, \tag{4.2}$$

where $\mathcal{C}(S^{n-1}, \hat{e})$ denotes the set of continuous zonal functions on S^{n-1}.

As a consequence of Lemma 4.2, we have for the mixed Blaschke-Minkowski homomorphism induced by Φ,

$$h(\Phi(K_1, \ldots, K_{n-1}), \cdot) = S(K_1, \cdots, K_{n-1}; \cdot) * g, \tag{4.3}$$

where $S(K_1, \cdots, K_{n-1}; \cdot)$ is the mixed surface area measure of $K_1, \cdots, K_{n-1}$.

Let $\Phi : \underbrace{\mathcal{K}^n \times \cdots \times \mathcal{K}^n}_{n-1} \to \mathcal{K}^n$ be a mixed Blaschke-Minkowski homomorphism. If $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$, then (see [16])

$$\rho(\Phi^*(K_1, \ldots, K_{n-1}), \cdot)^{-1} = h(\Phi^*(K_1, \ldots, K_{n-1}), \cdot). \tag{4.4}$$

Lemma 4.3 (see [16])**.** *A map $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ is a radial Blaschke-Minkowski homomorphism if and only if there is a measure $\mu \in \mathcal{M}_+(S^{n-1}, \hat{e})$ such that*

$$\rho(\Psi K, \cdot) = \rho(K, \cdot)^{n-1} * \mu, \tag{4.5}$$

where $\mathcal{M}_+(S^{n-1}, \hat{e})$ denotes the set of nonnegative zonal measures on S^{n-1}.

For the mixed radial Blaschke-Minkowski homomorphism induced by Ψ, we have

$$\rho(\Psi(K_1,\dots,K_{n-1}),\cdot)=\rho(K_1,\cdot)\cdots\rho(K_{n-1},\cdot)*\mu. \tag{4.6}$$

We are now in a position to prove Theorem 2.3. The following statement is just a slight reformulation of Theorem 2.3.

Theorem 4.4. *If $K,L\in\mathcal{S}^n$ and $p,r\in\mathbb{R}, j\in\mathbb{N}$ satisfy $0\le r\le 1\le p$, $0\le j\le n-1$, then*

$$\left(\frac{\tilde{W}_{n-p}(\Psi_j(K\tilde{+}_jL))}{\tilde{W}_{n-r}(\Psi_j(K\tilde{+}_jL))}\right)^{1/(p-r)}\le\left(\frac{\tilde{W}_{n-p}(\Psi_jK)}{\tilde{W}_{n-r}(\Psi_jK)}\right)^{1/(p-r)}+\left(\frac{\tilde{W}_{n-p}(\Psi_jL)}{\tilde{W}_{n-r}(\Psi_jL)}\right)^{1/(p-r)}, \tag{4.7}$$

with equality if and only if K and L are dilates.

Proof. From (1.7), we have

$$\rho(K\tilde{+}_sL,\cdot)^s*\mu=\rho(K,\cdot)^s*\mu+\rho(L,\cdot)^s*\mu,\ \ s\neq 0,$$

where μ is defined in Lemma 4.3. Hence, from (1.2) and (4.6), we obtain

$$\rho(\Psi_s(K\tilde{+}_sL),\cdot)=\rho(\Psi_sK,\cdot)+\rho(\Psi_sL,\cdot)=\rho(\Psi_sK\tilde{+}\Psi_sL,\cdot).$$

Namely,

$$\Psi_s(K\tilde{+}_sL)=\Psi_s(K)\tilde{+}\Psi_s(L).$$

Therefore, from (1.2) and (1.6), we have

$$\begin{aligned}\tilde{W}_{n-p}(\Psi_j(K\tilde{+}_jL))&=\frac{1}{n}\int_{S^{n-1}}\rho(\Psi_j(K\tilde{+}_jL),u)^pdS(u)\\&=\frac{1}{n}\int_{S^{n-1}}\rho(\Psi_jK\tilde{+}\Psi_jL,u)^pdS(u)=\frac{1}{n}\int_{S^{n-1}}(\rho(\Psi_jK,u)+\rho(\Psi_jL,u))^pdS(u)\end{aligned} \tag{4.8}$$

and

$$\tilde{W}_{n-r}(\Psi_j(K\tilde{+}_jL))=\frac{1}{n}\int_{S^{n-1}}(\rho(\Psi_jK,u)+\rho(\Psi_jL,u))^rdS(u). \tag{4.9}$$

From (4.8) and (4.9) and in view of Lemma 4.1, we obtain

$$\begin{aligned}\left(\frac{\tilde{W}_{n-p}(\Psi_j(K\tilde{+}_jL))}{\tilde{W}_{n-r}(\Psi_j(K\tilde{+}_jL))}\right)^{\frac{1}{p-r}}&=\left(\frac{\int_{S^{n-1}}(\rho(\Psi_jK,u)+\rho(\Psi_jL,u))^pdS(u)}{\int_{S^{n-1}}(\rho(\Psi_jK,u)+\rho(\Psi_jL,u))^rdS(u)}\right)^{\frac{1}{p-r}}\\&\le\left(\frac{\int_{S^{n-1}}\rho(\Psi_jK,u)^pdS(u)}{\int_{S^{n-1}}\rho(\Psi_jK,u)^rdS(u)}\right)^{\frac{1}{p-r}}+\left(\frac{\int_{S^{n-1}}\rho(\Psi_jL,u)^pdS(u)}{\int_{S^{n-1}}\rho(\Psi_jL,u)^rdS(u)}\right)^{\frac{1}{p-r}}\\&=\left(\frac{\tilde{W}_{n-p}(\Psi_jK)}{\tilde{W}_{n-r}(\Psi_jK)}\right)^{\frac{1}{p-r}}+\left(\frac{\tilde{W}_{n-p}(\Psi_jL)}{\tilde{W}_{n-r}(\Psi_jL)}\right)^{\frac{1}{p-r}}.\end{aligned}$$

Equality holds if and only if the functions $\rho(\Psi_jK,u)$ and $\rho(\Psi_jL,u)$ are proportional. Namely, $\rho(\Psi_jK,u)=\lambda\rho(\Psi_jL,u)$. From (4.5), we obtain $\rho(\Psi_jK,u)=\rho(\Psi_j(\lambda^{1/(n-j-1)}L),u)$. Hence, equality holds if and only if K and L are dilates.

Let $p=n-i$ and $r=n-j$. Since $0\le r\le 1\le p$, we have

$$r\le 1\le p\Rightarrow i\le n-1\le j,\ 0\le r\Rightarrow j\le n.$$

Therefore,

$$i \leq n-1 \leq j \leq n. \tag{4.10}$$

Taking for $p = n-i$ and $r = n-j$ in (4.7) and using (4.10), we see that (4.7) changes to the inequality in Theorem 2.3.

Taking for $p = n-i, j = 0$ and $r = 1$ in (4.7), (4.7) changes to the following inequality:

$$\left(\frac{\tilde{W}_i(\Psi(K\tilde{+}L))}{\tilde{W}_{n-1}(\Psi(K\tilde{+}L))}\right)^{1/(n-i-1)} \leq \left(\frac{\tilde{W}_i(\Psi K)}{\tilde{W}_{n-1}(\Psi K)}\right)^{1/(n-i-1)} + \left(\frac{\tilde{W}_i(\Psi L)}{\tilde{W}_{n-1}(\Psi L)}\right)^{1/(n-i-1)}, \tag{4.11}$$

with equality if and only if K and L are dilates.

For $K \in \mathcal{S}^n$, there is a unique star body $\mathbf{I}K$ whose radial function satisfies for $u \in S^{n-1}$,

$$\rho(\mathbf{I}K, u) = v(K \cap E_u).$$

It is called the *intersection bodies* of K. The volume of intersection bodies is given by

$$V(\mathbf{I}K) = \frac{1}{n}\int_{S^{n-1}} v(K \cap E_u)^n dS(u).$$

The mixed intersection body of $K_1, \ldots, K_{n-1} \in \mathcal{S}^n$, $\mathbf{I}(K_1, \ldots, K_{n-1})$, is defined by

$$\rho(\mathbf{I}(K_1, \ldots, K_{n-1}), u) = \tilde{v}(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u),$$

where $\tilde{v}$ is $(n-1)$-dimensional dual mixed volume.

If $K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = L$, then $\mathbf{I}(K_1, \ldots, K_{n-1})$ is written as $\mathbf{I}_i(K, L)$. If $L = B$, then $\mathbf{I}_i(K, L)$ is written as $\mathbf{I}_i K$ and called the ith intersection body of K. For $\mathbf{I}_0 K$ we simply write $\mathbf{I}K$. □

If $\Psi : \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$ is the mixed intersection operator $\mathbf{I} : \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$ in (4.7), we obtain

Corollary 4.5. *If $K, L \in \mathcal{S}^n$ and $p, r \in \mathbb{R}, j \in \mathbb{N}$ satisfy $0 \leq r \leq 1 \leq p$, $0 \leq j \leq n-1$, then*

$$\left(\frac{\tilde{W}_{n-p}(\mathbf{I}_j(K\tilde{+}_j L))}{\tilde{W}_{n-r}(\mathbf{I}_j(K\tilde{+}_j L))}\right)^{1/(p-r)} \leq \left(\frac{\tilde{W}_{n-p}(\mathbf{I}_j K)}{\tilde{W}_{n-r}(\mathbf{I}_j K)}\right)^{1/(p-r)} + \left(\frac{\tilde{W}_{n-p}(\mathbf{I}_j L)}{\tilde{W}_{n-r}(\mathbf{I}_j L)}\right)^{1/(p-r)}, \tag{4.12}$$

with equality if and only if K and L are dilates.

The following statement is just a slight reformulation of Theorem 3.3.

Theorem 4.6. *If $K, L \in \mathcal{K}^n$ and $p, r \in \mathbb{R}$ satisfy $p \leq -1 \leq r \leq 0$, then*

$$\left(\frac{\tilde{W}_{n-p}(\Phi^*(K\ddot{+}L))}{\tilde{W}_{n-r}(\Phi^*(K\ddot{+}L))}\right)^{1/(r-p)} \leq \left(\frac{\tilde{W}_{n-p}(\Phi^* K)}{\tilde{W}_{n-r}(\Phi^* K)}\right)^{1/(r-p)} + \left(\frac{\tilde{W}_{n-p}(\Phi^* L)}{\tilde{W}_{n-r}(\Phi^* L)}\right)^{1/(r-p)}, \tag{4.13}$$

with equality if and only if K and L are homothetic.

Proof. From (1.6), (4.4) and in view of Definition 3.1, we have

$$\begin{aligned}\tilde{W}_{n-p}(\Phi^*(K\ddot{+}L)) &= \frac{1}{n}\int_{S^{n-1}} \rho(\Phi^*(K\ddot{+}L),u)^p dS(u)\\ &= \frac{1}{n}\int_{S^{n-1}} h(\Phi(K\ddot{+}L),u)^{-p} dS(u) = \frac{1}{n}\int_{S^{n-1}} h(\Phi K+\Phi L,u)^{-p} dS(u)\\ &= \frac{1}{n}\int_{S^{n-1}} (h(\Phi K,u)+h(\Phi L,u))^{-p} dS(u) \qquad (4.14)\end{aligned}$$

and

$$\tilde{W}_{n-r}(\Phi^*(K\ddot{+}L)) = \frac{1}{n}\int_{S^{n-1}} (h(\Phi K,u)+h(\Phi L,u))^{-r} dS(u). \qquad (4.15)$$

From (4.14), (4.15), and Lemma 4.1, we obtain

$$\begin{aligned}\left(\frac{\tilde{W}_{n-p}(\Phi^*(K\ddot{+}L))}{\tilde{W}_{n-r}(\Phi^*(K\ddot{+}L))}\right)^{1/(r-p)} &= \left(\frac{\int_{S^{n-1}}(h(\Phi K,u)+h(\Phi L,u))^{-p}dS(u)}{\int_{S^{n-1}}(h(\Phi K,u)+h(\Phi L,u))^{-r}dS(u)}\right)^{1/(r-p)}\\ &\le \left(\frac{\int_{S^{n-1}} h(\Phi K,u)^{-p}dS(u)}{\int_{S^{n-1}} h(\Phi K,u)^{-r}dS(u)}\right)^{1/(r-p)} + \left(\frac{\int_{S^{n-1}} h(\Phi L,u)^{-p}dS(u)}{\int_{S^{n-1}} h(\Phi L,u)^{-r}dS(u)}\right)^{1/(r-p)}\\ &= \left(\frac{\int_{S^{n-1}} \rho(\Phi^* K,u)^{p}dS(u)}{\int_{S^{n-1}} \rho(\Phi^* K,u)^{r}dS(u)}\right)^{1/(r-p)} + \left(\frac{\int_{S^{n-1}} \rho(\Phi^* L,u)^{p}dS(u)}{\int_{S^{n-1}} \rho(\Phi^* L,u)^{r}dS(u)}\right)^{1/(r-p)}\\ &= \left(\frac{\tilde{W}_{n-p}(\Phi^* K)}{\tilde{W}_{n-r}(\Phi^* K)}\right)^{1/(r-p)} + \left(\frac{\tilde{W}_{n-p}(\Phi^* L)}{\tilde{W}_{n-r}(\Phi^* L)}\right)^{1/(r-p)}.\end{aligned}$$

Equality holds if and only if the functions $h(\Phi^*K,u)$ and $h(\Phi^*L,u)$ are proportional, namely, $h(\Phi^*K,u) = \lambda h(\Phi^*L,u)$, and from (4.3), we obtain $h(\Phi^*K,u) = \lambda h(\Phi^*(\lambda^{-1/(n-1)}L),u)$. Hence, equality holds if and only if K and L are homothetic.

Let $p = n-i$ and $r = n-j$. Since $p \le -1 \le r \le 0$, we have

$$p \le -1 \le r \le 0 \Rightarrow i \ge n+1 \ge j \ge n. \qquad (4.16)$$

Taking for $p = n-i$ and $r = n-j$ in (4.13) and using (4.16), we see that (4.13) changes to the inequality in Theorem 3.3.

Taking $p = -n, r = -1$ and $s = 0$ in (4.13), we have

$$\left(\frac{\tilde{W}_{2n}(\Phi^*(K\ddot{+}L))}{\tilde{W}_{n+1}(\Phi^*(K\ddot{+}L))}\right)^{1/(n-1)} \le \left(\frac{\tilde{W}_{2n}(\Phi^*K)}{\tilde{W}_{n+1}(\Phi^*K)}\right)^{1/(n-1)} + \left(\frac{\tilde{W}_{2n}(\Phi^*L)}{\tilde{W}_{n+1}(\Phi^*L)}\right)^{1/(n-1)}, \qquad (4.17)$$

with equality if and only if K and L are homothetic.

If $K_1,\dots,K_r \in \mathcal{K}^n$ and $\lambda_1,\dots,\lambda_r \ge 0$, then the projection body of the Minkowski linear combination $\lambda_1K_1+\cdots+\lambda_rK_r \in \mathcal{K}^n$ can be written as a symmetric homogeneous polynomial of degree $(n-1)$ in the λ_i (see [15]):

$$\mathbf{\Pi}(\lambda_1K_1+\cdots+\lambda_rK_r) = \sum \lambda_{i_1}\dots\lambda_{i_{n-1}}\mathbf{\Pi}_{i_1\cdots i_{n-1}}, \qquad (4.18)$$

where the sum is a Minkowski sum taken over all $(n-1)$-tuples $(i_1,\dots,i_{n-1})$ of positive integers not exceeding r. The body $\mathbf{\Pi}_{i_1\dots i_{n-1}}$ depends only on the bodies $K_{i_1},\dots,K_{i_{n-1}}$ and is uniquely determined by (4.18). It is called *the mixed projection bodies* of $K_{i_1},\dots,K_{i_{n-1}}$, and is written as $\mathbf{\Pi}(K_{i_1},\dots,K_{i_{n-1}})$.

If $K_1 = \cdots = K_{n-1-i} = K$ and $K_{n-i} = \cdots = K_{n-1} = L$, then $\mathbf{\Pi}(K_{i_1}, \ldots, K_{i_{n-1}})$ will be written as $\mathbf{\Pi}_i(K, L)$. If $L = B$, then $\mathbf{\Pi}_i(K, L)$ is denoted by $\mathbf{\Pi}_i K$ and when $i = 0$, $\mathbf{\Pi}_i K$ is denoted by $\mathbf{\Pi} K$, where $\mathbf{\Pi} K$ is the projection body of K. □

If $\Phi : \underbrace{\mathcal{K}^n \times \cdots \times \mathcal{K}^n}_{n-1} \to \mathcal{K}^n$ is the mixed projection operator $\mathbf{\Pi} : \underbrace{\mathcal{K}^n \times \cdots \times \mathcal{K}^n}_{n-1} \to \mathcal{K}^n$ in (4.13), we obtain

Corollary 4.7. *If $K, L \in \mathcal{K}^n$ and $p, r \in \mathbb{R}$ satisfy $p \leq -1 \leq r \leq 0$, then*

(4.19)
$$\left(\frac{\tilde{W}_{n-p}(\mathbf{\Pi}^*(K \ddot{+} L))}{\tilde{W}_{n-r}(\mathbf{\Pi}^*(K \ddot{+} L))}\right)^{1/(r-p)} \leq \left(\frac{\tilde{W}_{n-p}(\mathbf{\Pi}^* K)}{\tilde{W}_{n-r}(\mathbf{\Pi}^* K)}\right)^{1/(r-p)} + \left(\frac{\tilde{W}_{n-p}(\mathbf{\Pi}^* L)}{\tilde{W}_{n-r}(\mathbf{\Pi}^* L)}\right)^{1/(r-p)},$$

with equality if and only if K and L are homothetic.

We finally remark that inequalities for the intersection operator $\mathbf{I}$ were also established in [23], [27], for the L_p-intersection operator $\mathbf{I_p}$ in [24] and for the polar projection body operator $\mathbf{\Pi}^*$ in [25]–[26].

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