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SYMMETRY PROBLEM

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ABSTRACT. A novel approach to an old symmetry problem is developed. A new proof is given for the following symmetry problem, studied earlier: if $\Delta u = 1$ in $D \subset \mathbb{R}^3$, $u = 0$ on S, the boundary of D, and $u_N = const$ on S, then S is a sphere. It is assumed that S is a Lipschitz surface homeomorphic to a sphere. This result has been proved in different ways by various authors. Our proof is based on a simple new idea.

1. INTRODUCTION

Symmetry problems are of interest both theoretically and in applications. A well-known, and still unsolved, symmetry problem is the Pompeiu problem (see [9], [10], and the references therein). In modern formulation this problem consists of proving the following conjecture:

If $D \subset \mathbb{R}^n$, $n \geq 2$, is a domain homeomorphic to a ball, and the boundary S of D is smooth ($S \in C^{1,\lambda}$, $\lambda > 0$, is sufficient), and if the problem

$$(1) \qquad (\nabla^2 + k^2)u = 0 \quad in \quad D, \qquad u\big|_S = c \neq 0, \quad u_N\big|_S = 0, \quad k^2 = const > 0,$$

where c is a constant, has a solution, then S is a sphere.

A similar problem (*Schiffer's conjecture*) is also unsolved (see also [4]):

If the problem

$$(2) \qquad (\nabla^2 + k^2)u = 0 \quad in \quad D, \qquad u\big|_S = 0, \quad u_N\big|_S = c \neq 0, \quad k^2 = const > 0$$

has a solution, then S is a sphere.

Standing assumptions. *In this paper we assume that $D \subset \mathbb{R}^3$ is a bounded domain homeomorphic to a ball, with a sufficiently smooth boundary S (S is Lipschitz suffices).*

We use the following notation: $D' = \mathbb{R}^3 \setminus D$, $B_R = \{x : |x| \leq R\}$, $B_R \supset D$, $\mathcal{H}$ is the set of all harmonic functions in B_R, $R > 0$ is an arbitrary large number, such that the ball B_R contains D, $|D|$ and $|S|$ are the volume of D and the surface area of S, respectively.

In [12] it is proved that if

$$(3) \qquad \int_D \frac{dy}{4\pi|x-y|} = \frac{c}{|x|}, \qquad \forall x \in B_R', \quad c = const > 0,$$

then D is a ball. The proof in [12] is based on an idea similar to the one we are using in this paper.

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In [13] a symmetry problem of interest in elasticity theory is studied by A.D. Alexandrov's method of a moving plane ([1]), used also in [14]. The result in [14], which is formulated below in Theorem 1, was proved in [15] by a method, different from the one given in [14], and discussed also in [2]. The argument in [2] remained unclear to the author.

In [5] another symmetry problem of potential theory was studied.

Our goal is to give a new proof of Theorem 1. The result of Theorem 1 was obtained in [14] for $\mathbb{R}^n$, $n \geq 2$.

Theorem 1. *If $D \supset \mathbb{R}^3$ is a bounded domain, homeomorphic to a ball, S is its Lipschitz boundary, and the problem*

$$\Delta u = 1 \quad in \quad D, \qquad u\big|_S = 0, \quad u_N\big| = c := \frac{|D|}{|S|} > 0 \tag{4}$$

has a solution, then S is a sphere.

This result is equivalent to the following result:

If

$$\int_D h(x)dx = c \int_S h(s)ds, \qquad \forall h \in \mathcal{H}, \qquad c := \frac{|D|}{|S|}, \tag{5}$$

then S is a sphere.

The equivalence of (4) and (5) can be proved as follows.

Suppose (4) holds. Multiply (4) by an arbitrary $h \in \mathcal{H}$, integrate by parts and get

$$\int_D h(x)dx = c \int_S h(s)ds. \tag{6}$$

If $h = 1$ in (6), then one gets $c = \frac{|D|}{|S|}$, so (6) is identical to (5).

Suppose (5) holds. Then (6) holds. Let v solve the problem $\Delta v = 1$ in D, $v\big|_S = 0$. This v exists and is unique. Using (6), the equation $\Delta h = 0$ in D, and the Green's formula, one gets

$$c \int_S h(s)ds = \int_D h(x)dx = \int_D h(x)\Delta v dx = \int_S h(s) v_N ds. \tag{7}$$

Thus,

$$\int_S h(s)[v_N - c]ds = 0, \qquad \forall h \in \mathcal{H}. \tag{8}$$

We will need the following lemma:

Lemma A. *The set of restrictions on S of all harmonic functions in D is dense in $L^2(S)$.*

Proof of Lemma A. We give a proof for the convenience of the reader. The proof is borrowed from [12]. Suppose that $g \in L^2(S)$, and $\int_S g(s)h(s)ds = 0 \quad \forall h \in \mathcal{H}$. Since $(4\pi|x-y|)^{-1}$ is in $\mathcal{H}$ if $y \in D'$, one gets

$$w(y) := \int_S g(s)(4\pi|s-y|)^{-1}ds = 0 \quad \forall y \in D'.$$

Thus, a single layer potential w, with L^2 density g, vanishes in D', and, by continuity, on S. Since w is a harmonic function in D vanishing on S, it follows that $w = 0$ in D. By the jump formula for the normal derivative of the single-layer potential across a Lipschitz boundary, one gets $g = 0$. □

Thus, (8) implies $v_N\big|_S = c$. Therefore, (4) holds.

A result, related to equation (5), was studied in [7] for a two-dimensional problem. The arguments in [7] were not quite clear to the author.

Our main result is a new proof of Theorem 1. The proof is simple, and the method of the proof is new. This method can be used in other problems (see [5], [10], [12], [11]).

2. Proofs

Proof of Theorem 1. We denote by D' the complement of D in $\mathbb{R}^3$, by S^2 the unit sphere, by $[a, b]$ the cross product of two vectors, by $g = g(\phi)$ the rotation about an axis, directed along a vector $\alpha \in S^2$, by the angle ϕ, and note that if $h(x)$ is a harmonic function in any ball B_R, containing D, then $h(gx)$ is also a harmonic function in B_R.

Take $h = h(g(\phi)x)$ in (5), differentiate with respect to ϕ and then set $\phi = 0$. This yields:

$$\int_D \nabla h(x) \cdot [\alpha, x] dx = c \int_S \nabla h(s) \cdot [\alpha, s] ds.$$

Using the divergence theorem, one rewrites this as

$$\alpha \cdot \int_S [s, N] h(s) ds = \alpha \cdot \int_S [s, c\nabla h(s)] ds.$$

Since $\alpha \in S^2$ is arbitrary, one gets

$$\int_S [s, N] h(s) ds = \int_S [s, c\nabla h(s)] ds, \qquad \forall h \in \mathcal{H}, \tag{9}$$

where $N = N_s$ is a unit normal to S at the point $s \in S$, pointing into D'.

Let $y \in B'_R$ be an arbitrary point, and $h(x) = \frac{1}{|x-y|} \in \mathcal{H}$, where $x \in B_R$. Then equation (9) implies that

$$v(y) := \int_S \frac{[s, N] ds}{|s - y|} = c[\nabla \int_S \frac{ds}{|s - y|}, y], \qquad \forall y \in B'_R, \tag{10}$$

because

$$c \int_S [s, \nabla_s \frac{1}{|s - y|}] ds = c \int_S [\frac{s}{|s - y|^3}, y] ds = c[\nabla_y \int_S \frac{ds}{|s - y|}, y]. \tag{11}$$

Relation (11) actually holds for all $y \in D'$, because of the analyticity of its left and right sides in D'. Let

$$w(y) := \int_S |s - y|^{-1} ds.$$

Denote $y^0 := y/|y|$. It is known (see, e.g., [3]) that

$$|y - s|^{-1} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{4\pi}{2n+1} Y_{nm}(y^0) \overline{Y_{nm}(s^0)} |s|^n |y|^{-(n+1)}, \qquad |y| > |s|, \tag{12}$$

where the overline stands for the complex conjugate, y^0 is the unit vector characterized by the angles θ, ϕ in spherical coordinates, Y_{nm} are normalized spherical harmonics:

$$Y_{nm}(y^0) = Y_{nm}(\theta, \phi) = \gamma_{nm} P_{n,|m|}(\cos\theta) e^{im\phi}, \qquad -n \le m \le n,$$

$\gamma_{nm} = [\frac{(2n+1)(n-m)!}{4\pi(n+m)!}]^{1/2}$ are normalizing constants:

$$(Y_{nm}(y^0), Y_{pq}(y^0))_{L^2(S^2)} = \delta_{np}\delta_{mq},$$

and

$$P_{n,|m|}(\cos\theta) = (\sin\theta)^{|m|}(\frac{d}{d\cos\theta})^{|m|} P_n(\cos\theta)$$

are the associated Legendre functions, where $P_n(\cos\theta)$ are the Legendre polynomials.

If $z = \cos\theta$, then

$$P_{n,m}(z) = (z^2-1)^{m/2}(\frac{d}{dz})^m P_n(z), \quad m = 1, 2, ...,$$

$$P_n(z) = (2^n n!)^{-1}(\frac{d}{dz})^n (z^2-1)^n, \quad P_0(z) = 1$$

(see [3]). The definitions of $P_{n,m}(z)$ in various books can differ by a factor $(-1)^m$.

Using formula (12), one obtains

(13)

$$w(y) = \sum_{n=0}^{\infty} \frac{4\pi}{2n+1} \sum_{m=-n}^{n} Y_{nm}(y^0)|y|^{-(n+1)} c_{nm}, \qquad c_{nm} := \int_S |s|^n \overline{Y_{nm}(s^0)} ds.$$

Substitute this in (10), equate the terms in front of $|y|^{-(n+1)}$, and define vectors

$$a_{nm} := \int_S [s, N]|s|^n \overline{Y_{nm}(s^0)} ds \tag{14}$$

to obtain

$$\sum_{m=-n}^{n} Y_{nm}(y^0) a_{nm} = \sum_{m=-n}^{n} cc_{nm}[e_\theta \partial_\theta Y_{nm}(y^0) + e_\phi (\sin\theta)^{-1}\partial_\phi Y_{nm}(y^0), e_r], \tag{15}$$

where e_θ,e_ϕ, and e_r are orthogonal unit vectors of the spherical coordinate system, $[e_\phi, e_r]$ is the cross product, $[e_\phi, e_r] = e_\theta$, $[e_\theta, e_r] = -e_\phi$, $y = ry^0$, $r = |y|$, $y^0 = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, $\partial_\theta = \frac{\partial}{\partial\theta}$.

Therefore, formula (15) can be rewritten as

$$\sum_{m=-n}^{n} Y_{nm}(y^0) a_{nm} = \sum_{m=-n}^{n} cc_{nm}\Big(- e_\phi \partial_\theta Y_{nm}(y^0) + e_\theta (\sin\theta)^{-1}\partial_\phi Y_{nm}(y^0)\Big). \tag{16}$$

From (16) we want to derive that

$$a_{nm} = 0, \qquad n \ge 0, \ -n \le m \le n. \tag{17}$$

If (17) is established, then it follows from (14) and from the completeness in $L^2(S)$ of the system $\{|s|^n Y_{nm}(s^0)\}_{n\ge 0, -n\le m\le n}$ that $[s, N] = 0$ on S, and this implies that S is a sphere, as follows from Lemma 1 formulated and proved below. Consequently, Theorem 1 is proved as soon as relations (17) are established. The completeness of the system $\{|s|^n Y_{nm}(s^0)\}_{n\ge 0, -n\le m\le n}$ in $L^2(S)$ follows from Lemma B:

The functions $|x|^n Y_{nm}(x^0)$, $n \ge 0$, $-n \le m \le n$, are harmonic in any ball, centered at the origin.

Lemma B. *The set of restrictions of the above functions to any Lipschitz surface homeomorphic to a sphere is complete in $L^2(S)$.*

Proof of Lemma B. The proof is given for completeness. It is similar to the proof of Lemma A. Suppose that $g \in L^2(S)$ and

$$\int_S g(s)|s|^n Y_{nm}(s^0)ds = 0, \quad \forall n \geq 0, |m| \leq n.$$

This and (12) imply that

$$\int_S g(s)(4\pi|s-y|)^{-1}ds = 0 \qquad \forall y \in D',$$

and the argument, given in the proof of Lemma A, yields the desired conclusion $g = 0$. □

Vector a_{nm} is written in the Cartesian basis $\{e_j\}_{1\leq j\leq 3}$ as

$$a_{nm} = \sum_{j=1}^{3} a_{nm,j} e_j.$$

The relation between the components F_1, F_2, F_3 of a vector F in Cartesian coordinates and its components F_r, F_θ, F_ϕ in spherical coordinates can be found, e.g., in [6], Section 6.5:

$$\begin{aligned} F_1 &= F_r \sin\theta\cos\phi + F_\theta \cos\theta\cos\phi - F_\phi \sin\phi, \\ F_2 &= F_r \sin\theta\sin\phi + F_\theta \cos\theta\sin\phi + F_\phi \cos\phi, \\ F_3 &= F_r \cos\theta - F_\theta \sin\theta. \end{aligned}$$

Using these relations one derives from (16) the following formulas:

$$\sum_{m=-n}^{n} a_{nm,1} Y_{nm}(y^0) = \sum_{m=-n}^{n} cc_{nm} \Big(\partial_\theta Y_{nm}(y^0)\sin\phi + \partial_\phi Y_{nm}(y^0)\cot\theta\cos\phi\Big), \tag{18}$$

$$\sum_{m=-n}^{n} a_{nm,2} Y_{nm}(y^0) = \sum_{m=-n}^{n} cc_{nm} \left(-\partial_\theta Y_{nm}(y^0)\cos\phi + \partial_\phi Y_{nm}(y^0)\cot\theta\sin\phi\right), \tag{19}$$

$$\sum_{m=-n}^{n} a_{nm,3} Y_{nm}(y^0) = -\sum_{m=-n}^{n} cc_{nm}\partial_\phi Y_{nm}(y^0). \tag{20}$$

From formulas (18)-(20) one derives (17).

If $n = 0$, then $a_{00} = 0$, as the following calculation shows:

$$a_{00} = \frac{1}{(4\pi)^{1/2}} \int_S [s, N]ds = -\frac{1}{(4\pi)^{1/2}} \int_D [\nabla, x]dx = 0.$$

If $n > 0$, then multiply equation (20) by $e^{-im\phi}$, integrate with respect to ϕ over $[0, 2\pi]$, write $P_{n,m}$ for $P_{n,m}(\cos\theta)$, and obtain

$$a_{nm,3}P_{n,m} = -cc_{n,m}imP_{n,m}, \qquad c_{n,m} := c_{nm}. \tag{21}$$

One concludes that $a_{n0,3} = 0$ and $a_{nm,3} = -imcc_{n,m}$.

If one derives from (18)-(19) that $c_{n,m} = 0$, then equation (17) follows, and Theorem 1 is proved.

From (18) and (19) one derives analogs of (21):

$$\begin{aligned} 2ia_{nm,1}\gamma_{nm}P_{n,m} &= cc_{n,m-1}\gamma_{n,m-1}\left(\partial_\theta P_{n,m-1} - (m-1)\cot\theta P_{n,m-1}\right) \\ &\quad - cc_{n,m+1}\gamma_{n,m+1}\left(\partial_\theta P_{n,m+1} + (m+1)\cot\theta P_{n,m+1}\right), \end{aligned} \tag{22}$$

$$\begin{aligned} 2a_{nm,2}\gamma_{nm}P_{n,m} &= cc_{n,m-1}\gamma_{n,m-1}\left(-\partial_\theta P_{n,m-1} + (m-1)\cot\theta P_{n,m-1}\right) \\ &\quad - cc_{n,m+1}\gamma_{n,m+1}\left(\partial_\theta P_{n,m+1} + (m+1)\cot\theta P_{n,m+1}\right). \end{aligned} \tag{23}$$

Let us take $\theta \to 0$ in the above equations. It is known (see [3], Section 3.9.2, formula (4)) that

$$P_{n,m}(z) \sim b(n,m)(z-1)^{m/2}, \quad z \to 1, \qquad b(n,m) := \frac{(n+m)!}{2^{m/2}m!(n-m)!}. \tag{24}$$

Equation (22) can be considered as a linear combination

$$\sum_{j=1}^{3} A_j f_j(z) = 0, \tag{25}$$

where the A_j are constants:

$$A_1 = 2ia_{nm,1}\gamma_{nm}, \quad A_2 = -cc_{n,m-1}\gamma_{n,m-1}, \quad A_3 = cc_{n,m+1}\gamma_{n,m+1},$$

and

$$\begin{aligned} f_1(z) &= P_{n,m}(z), \\ f_2(z) &= -(1-z^2)^{1/2}P'_{n,m-1}(z) - (m-1)\frac{z}{(1-z^2)^{1/2}}P_{n,m-1}(z), \\ f_3(z) &= -(1-z^2)^{1/2}P'_{n,m+1}(z) - (m+1)\frac{z}{(1-z^2)^{1/2}}P_{n,m+1}(z), \quad z = \cos\theta. \end{aligned}$$

If the system of functions $\{f_j(z)\}_{j=1}^3$ is linearly independent on the interval $[-1,1]$, then all $A_j = 0$ in (25), that is, $A_1 = 0$, $A_2 = 0$, and $A_3 = 0$. This implies that

$$a_{nm,1} = c_{n,m} = 0, \quad -n \le m \le n.$$

The quantities $a_{nm,2}$ and $a_{nm,3}$ are proportional to $c_{n,m}$. Since $c_{n,m} = 0$, it follows that

$$a_{nm,2} = a_{nm,3} = 0, \quad -n \le m \le n,$$

and Theorem 1 is proved.

Thus, to complete the proof of Theorem 1 it is sufficient to verify the linear independence of the system of functions $\{f_j(z)\}_{j=1}^3$ on the interval $z \in [-1,1]$.

From formula (24) it follows that these functions have the following main terms of their asymptotics as $z \to 1$:

$$f_1(z) \sim B_1(z-1)^{m/2}, \quad f_2(z) \sim B_2\frac{(z-1)^{(m+1)/2}}{(1-z^2)^{1/2}}, \quad f_3(z) \sim B_3\frac{(z-1)^{(m+3)/2}}{(1-z^2)^{1/2}},$$

where the constants $B_j \ne 0$, $1 \le j \le 3$, depend on n, m. The linear independence of the system $\{f_j(z)\}_{j=1}^3$ holds because the system

$$\{(z-1)^{m/2}, \quad \frac{(z-1)^{(m+1)/2}}{(1-z^2)^{1/2}}, \quad \frac{(z-1)^{(m+3)/2}}{(1-z^2)^{1/2}}\}$$

is linearly independent. The linear independence of this system holds if the system

$$\{1, \quad (1+z)^{-0.5}, \quad (z-1)(1+z)^{-0.5}\}$$

is linearly independent on the interval $[-1,1]$. The linear independence of this system on the interval $[-1,1]$ is obvious.

Theorem 1 is proved. $\square$

Lemma 1. *If S is a C^2-smooth closed surface and $[s,N_s]=0$ on S, then S is a sphere.*

Proof of Lemma 1. Let $s=r(u,v)$ be a parametric equation of S. Then the vectors r_u and r_v are linearly independent and N_s is directed along the vector $[r_u,r_v]$. Thus, the assumption $[s,N_s]=0$ on S implies that

$$[r,[r_u,r_v]]=r_u(r,r_v)-r_v(r,r_u)=0.$$

Since the vectors r_u and r_v are linearly independent, it follows that $(r,r_v)=(r,r_u)=0$. Thus, $(r,r)=R^2$, where R^2 is a constant. This means that S is a sphere. Lemma 1 is proved. $\square$

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