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SIMULTANEOUS EMBEDDINGS OF FINITE DIMENSIONAL DIVISION ALGEBRAS

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ABSTRACT. L. Small asked whether two finite dimensional division algebras containing a common central subfield F are embeddable in a common division algebra. Although we have a counterexample, the question is answered affirmatively for division algebras whose centers are finitely generated over a common perfect subfield.

A celebrated theorem of P.M. Cohn [C] says that for any two division rings (not necessarily finite dimensional) over a field F, their amalgamated product over F is a domain which can be embedded in a division ring. Note that even with the two initial division rings being finite dimensional over their centers, the resulting division ring is **never** finite dimensional over its center. Perhaps this led Lance Small to ask the following question:

We say D/K is a **division algebra** when D is a division ring finite dimensional over its center K. Assume that K_1 and K_2 are fields with a given common subfield F_0. Small asked whether any two division algebras D_1/K_1 and D_2/K_2 can be embedded in some third division algebra E.

Let us frame this in several versions, in order of increasing strength:

(1) Is there some E containing both D_1 and D_2?
(2) Is there some K-division algebra E containing both D_1 and D_2 in such a way that $K \subset K_i$ for $i = 1, 2$?
(3) If D_1/K_1 and D_2/K_2 are division algebras of respective degrees p^{t_1} and p^{t_2}, then can D_1/K_1 and D_2/K_2 be embedded into a single division algebra E/K of degree p^t, and is there a bound for t in terms of t_1 and t_2? What would be the best bound?

Question (3) is natural to ask, since every division algebra is a tensor product of division algebras of prime power degree. In this paper, we focus on Question (1). We start with a surprisingly straightforward counterexample in the next section, but then show that a positive solution exists for division algebras finitely generated over a common subfield which is either algebraically closed or the prime subfield (Theorem 2.6).

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When L/F is a cyclic Galois extension of dimension n and $a \in F$, $\Delta(L/F, a)/F$ denotes the F-central cyclic algebra having maximal subfield L, together with some element z inducing the automorphism generating $\mathrm{Gal}(L/F)$, satisfying $z^n = a$. If F_1/F and F_2/F are field extensions, then an **amalgamation** of the F_i over F is the field of fractions of a quotient of $F_1 \otimes_F F_2$. If $F_1 \otimes_F F_2$ is itself a domain, then its field of fractions is called the **free amalgamation** of the F_i over F. Recall from [J] or [La1, Proposition 4 of §X.6] that any finitely generated, separably generated field extension is a finite separable extension of a purely transcendental field extension. In particular, this is the case for any finitely generated field extension in characteristic 0.

1. A counterexample

Suppose, first of all, that D_1/K_1 is an f.d. division algebra embedded in the division algebra E/K, so K_1 and K share the same prime subfield P. There is a tower of subalgebras $K \subseteq K_1K \subset D_1K \subset E$, where K_1K must be an amalgamation of K_1 and K over P.

More specifically, suppose $p_1 \neq p_2$ are primes. Let G be the infinite cyclic profinite p_2-group, the inverse limit of all $\mathbb{Z}/{p_2}^n\mathbb{Z}$. Take $K_1/\mathbb{Q}$ Galois with group G. (For example, K_1 could be contained in the infinite extension of $\mathbb{Q}$ obtained by adjoining all p_2^n roots of 1.) Note that G has no finite subgroups. The field extension K_1K/K has Galois group a subgroup of G and must be finite dimensional (being inside E/K), and so must be trivial. That is, $K_1K = K$, implying $K_1 \subseteq K$.

Next, take $D_1'/\mathbb{Q}$ of degree p_1 and let $D_1 = D_1' \otimes_{\mathbb{Q}} K_1$, a division algebra since $K_1/\mathbb{Q}$ is a pro-p_2 extension, and let $D_2/\mathbb{Q}$ be any division algebra split by K_1. For example, there is a cyclic degree p_2 extension $L/\mathbb{Q}$ such that $L \subset K_1$. By class field theory (cf. [La2, Theorem 4 of §X.3]), there is some element of $a \in \mathbb{Q}$ which is not a norm from L, so by Wedderburn's criterion, the cyclic algebra $\Delta(L/\mathbb{Q}, a)$ is a degree p_2 division algebra with maximal subfield L.

Proposition 1.1. *There is no division algebra E/K containing both D_1 and D_2.*

Proof. If E/K contains both D_1 and D_2, then $K_1 \subseteq K$ is central, by the above paragraph, so E contains D_2K_1, which is a division algebra but also a homomorphic image of the split algebra $D_2 \otimes_{\mathbb{Q}} K_1$ and thus is commutative, a contradiction. $\square$

The rationale for this example is that the centers are incompatible in some sense.

2. Positive results

Now we turn to the positive result, where we observe that the above incompatibility is impossible when the K_i are both finitely generated over an algebraically closed or prime field. In fact, we observe that it suffices for the K_i both to be finitely generated over a common perfect field (Theorem 2.6).

We approach the problem via [Sa]. First let us fix some notation. All algebras are over a given field F_0. We write $\mathrm{index}(D)$ for the (Schur) index of a division algebra D. Fixing $r > 1$, let $F \supset F_0$ be a field, and let $UD(F, n)/Z(F, n)$ denote the generic division algebra of degree n over F in r indeterminates. We write Z for $Z(F, n)$. The letter t always denotes an indeterminate. We begin with some lemmas.

Lemma 2.1. *There is a field $K(t) \supset Z$ and a degree n cyclic extension $L/K(t)$ such that L/F is rational and $UD(F,n) \otimes_Z K(t) = \Delta(L/K(t), t)$.*

Proof. We may assume that $r = 2$. Write $Z = F(X \oplus Y)^{S_n}$ as usual (see [Sa], p. 322). Let $C_n \subset S_n$ be generated by the n cycle $(1, 2, \ldots, n)$. Over C_n, $Y \cong M \oplus \mathbb{Z}$ where M is a free C_n lattice. We can set $L = F(X \oplus M)$, $K = F(X)^{C_n}$ so that $K(t) = L^{C_n}$, and take t to be the generator of $\mathbb{Z}$. □

Lemma 2.2. *Suppose F is a field, D/F is a division algebra, and $b \in \mathbb{N}$. Set*

$$A = D \otimes_F UD(F,n)^b = (D \otimes_F Z) \otimes_Z UD(F,n)^b.$$

Then $\mathrm{index}(A)$ *is the degree of D times $n/(n,b) =$* $\mathrm{index}(UD(F,n)^b)$.

Proof. Since $UD(F,n)$ has index equal exponent, the index of any power is equal to the exponent. In fact, if $b = b'(n,b)$ and $n = n'(n,b)$, then $(b', n') = 1$. Thus $UD(F,n)^b = (UD(F,n)^{(n,b)})^{b'}$ and the index and exponent of $UD(F,n)^b$ is the same as that of $(UD(F,n)^{(n,b)})$. That is, we may assume $b|n$.

By Lemma 2.1, there is a field $K(t) \supset Z$ and a degree n cyclic extension $L/K(t)$ such that $D \otimes_F L$ is a division algebra and $UD(F,n) \otimes_Z K(t) = \Delta(L/K(t), t)$. Of course, by Galois theory, $\Delta(L/K(t), t)^b$ is equal in the Brauer group to $\Delta(L'/K(t), t)$ where L/L' has dimension b. We conclude by showing that $(D \otimes_F K(t)) \otimes_{K(t)} \Delta(L'/K(t), t)$ is a division algebra. Indeed, taking $y \in \Delta(L'/K(t), t)$ such that $y^{n/b} = t$ and $yxy^{-1} = \sigma(x)$ for all $x \in L'$, where σ generates the Galois group of L' over $K(t)$, we note that the skew polynomial algebra $D[y; \sigma]$ satisfies a polynomial identity, so it has a division algebra of fractions obtained by inverting the central elements, and this is precisely $(D \otimes_F K(t)) \otimes_{K(t)} \Delta(L'/K(t), t)$. □

We are in the game of embedding division algebras into bigger division algebras. The key method is the following.

Theorem 2.3. *Suppose D/K is a division algebra of degree a and K/F is a field extension of dimension b. Assume E/F is a division algebra of degree $N = nab$. Then D is isomorphic to a subalgebra of E over F if and only if $(E \otimes_F K) \otimes_K D^{op}$ has (Schur) index dividing n. Furthermore, if this index divides n, then it is equal to n.*

Proof. Suppose $D \subset E$. In particular, $K \subset E$ and so $E \otimes_F K$ has index N/b, and we set E'/K to be the associated division algebra which is the centralizer of K in E. Then $D \subset E'$ and we take D' to be its centralizer, implying $E' = D \otimes_K D'$. Since D' has degree n, we have proven one direction.

Conversely, suppose $\mathrm{index}((E \otimes_F K) \otimes_K D^{op})$ divides n. Then $\mathrm{index}(E \otimes_F K)$ divides na, implying $\mathrm{index}(E \otimes_F K) = na$ since $\mathrm{index}(E \otimes_F K) \geq \frac{N}{b} = na$. Thus, $(E \otimes_F K) \otimes_K D^{op} \sim A$, where A/K has degree n, implying $[E \otimes_F K]$ is equal in the Brauer group to $[D][A]$.

Let E' be the centralizer of K in E. Since the degrees agree, $E' \cong D \otimes_K A$. □

We are going to force one algebra inside another by using partial splitting fields and Weil transfers. More specifically, let A/K be a central simple algebra and n be an integer dividing the degree of A/K. Let $V_n(A)$ be the variety of rank n left ideals of A and let $K_n(A)$ be its field of fractions. Then for any field $K' \supset K$, $V_n(A)$ has a K' point if and only if $A \otimes_K K'$ has index dividing n.

Next we set $W_n(A)$ to be the Weil transfer to F of $V_n(A)$, so for $F' \supset F$, $W_n(A)$ has an F' point if and only if $V_n(A)$ has a $K \otimes_F F'$ point (and in fact there is a natural correspondence). Let $F_n(A)$ denote the field of rational functions of $W_n(A)$. Then $\mathrm{index}(A \otimes_F KF_n(A))$ divides n.

The important tool for using this construction is the following result ([Sa]) about index reduction, for which we need to introduce more notation. Let K/F be finite separable with Galois closure $\bar{K}/F$. Let G be the Galois group of $\bar{K}/F$ and $H \subset G$ be the subgroup corresponding to $\bar{K}/K$. If r is the degree of A/K, then we can define an "action" of the G module $R = (\mathbb{Z}/r\mathbb{Z})[G/H]$ as follows. Let $\bar{A} = A \otimes_K \bar{K}$, so H has a natural semilinear action on $\bar{A}$ and for any $g \in G$ we can define the g twist $g(\bar{A})$. Of course, for $g \in G$, gHg^{-1} has a natural semilinear action on $g(\bar{A})$.

For $\alpha \in R$, define $H_\alpha = \{g \in G | g\alpha = \alpha\}$. Define $K(\alpha) = \bar{K}^{H_\alpha}$. Write

$$\alpha = \sum n_{gH}\, gH;$$

then the n_{gH} are constant on H_α-orbits. Fix a coset gH and set $e = n_{gH}$. Let $L \subset H_\alpha$ be the stabilizer of gH. Let $\mathcal{O} = \{g_i H\}$ be the orbit of H_α containing gH so $e = n_{g_i H}$ for all i. Then L acts naturally on $g(\bar{A})$ and H_α acts on B_{gH}, which is the tensor product over $\bar{K}$ of $g_i(\bar{A})^e$, one for each $g_i H$ in $\mathcal{O}$.

Now we let gH vary, one for each H_α orbit. Tensor over $\bar{K}$ all the B_{gH} defined above and call the resulting $\bar{K}$ algebra B. Note that $\bar{K}$ is the center of B. Define A^α to be the H_α invariant subring of B. Then A^α has center $K(\alpha)$.

Finally, for $\alpha = \sum n_{gH} gH$ as above, define

$$|\alpha| = \prod_{gH} \frac{n}{(n, n_{gH})}. \tag{1}$$

Theorem 2.4 ([Sa], p. 332)**.** *With notation as above, suppose B/F is any central simple algebra (over F). Then the index of $B \otimes_F F_n(A)$ is the gcd of all the integers*

$$\mathrm{index}(B \otimes_F A^\alpha)[K(\alpha) : F]|\alpha|,$$

taken over all $\alpha \in R$.

We actually need a double version of the above result. Let us assume that K/F and K'/F are finite separable with Galois closures $\bar{K}/F$ and $\bar{K}'/F$ and corresponding groups $G \supset H$ and $G' \supset H'$. For convenience we may assume that $\bar{K}/F$ and $\bar{K}'/F$ are linearly disjoint. Let A/K and A'/K' be central simple algebras and let $F_{n,n'}(A, A')$ denote the free amalgamation of the fields $F_n(A)$ and $F_{n'}(A')$ over F. If A'/K' has degree r', set $R' = (\mathbb{Z}/r'\mathbb{Z})[G'/H']$ as above. If $\alpha \in R$ and $\beta \in R'$, set $K(\alpha, \beta) = K(\alpha) \otimes_F K'(\beta)$. Finally, write $\beta = \sum_{gH'} m_{gH'} gH' \in R'$ and set

$$|\beta| = \prod_{gH'} \frac{n'}{(n', m_{gH'})}.$$

Theorem 2.5. *Suppose B/F is a central simple algebra and set*

$$B(\alpha, \beta) = B \otimes_F K(\alpha, \beta).$$

Then the (Schur) index $\mathbf{i} := \mathrm{index}(B \otimes_F F_{n,n'}(A, A'))$ is the gcd of the integers

$$\mathrm{index}\left((B(\alpha,\beta) \otimes_{K(\alpha,\beta)} (A^\alpha \otimes_{K(\alpha)} K(\alpha,\beta)) \otimes_{K(\alpha,\beta)} (A'^\beta \otimes_{K'(\beta)} K(\alpha,\beta))\right) \times [K(\alpha,\beta) : F]|\alpha||\beta|,$$

ranging over all $\alpha \in R$ and $\beta \in R'$.

Proof. The basic idea here is to apply Theorem 2.4 twice, noting that $[K(\alpha,\beta):F]=[K(\alpha):F][K'(\beta):F]$. Put $B'=B\otimes_F F_n(A)$. Then, by Theorem 2.4, using the fact that $[F_nK'(\beta):F_n(A)]=[K'(\beta):F]$, $\mathbf{i}$ is the gcd of all integers

$$\operatorname{index}\Big((B'\otimes_{F_n(A)}F_nK'(\beta))\otimes_{F_nK'(\beta)}({A'}^{\beta}\otimes_{K'(\beta)}F_nK'(\beta))\Big)\,[K'(\beta):F]|\beta|,$$

where $F_nK'(\beta)$ is the join of $F_n(A)$ and $K'(\beta)$ over F. Note that $F_nK'(\beta)$ is the function field of $W_n(A\otimes_F K(\beta))$, which is the $K(\alpha,\beta)/K'(\beta)$ transfer of $V_n(A\otimes_F K'(\beta))$. Now by Theorem 2.4 again, using B' instead of B, each

$$\operatorname{index}\big((B'\otimes_{F_n(A)}F_nK'(\beta))\otimes_{F_nK'(\beta)}({A'}^{\beta}\otimes_{K'(\beta)}F_nK'(\beta))\big)$$

is the gcd of

$$\operatorname{index}\big((B\otimes_F A^{\alpha})\otimes_{K(\alpha)}({A'}^{\beta}\otimes_{F_nK'(\beta)}K(\alpha,\beta))\big)\,[K(\alpha,\beta):K'(\beta)]|\alpha|,$$

and the result follows. $\square$

We are ready for a result generalizing [FSS, Theorem 2.8], which considers the problem of embedding a K_1-division algebra into a K-division algebra, where K_1 is a finite dimensional field extension of K. Now suppose we already know that D_1/K_1 and D_2/K_2 are division algebras of degrees d_i and K_i/F is separable of degree e_i. Set $m_i=d_ie_i$ and let N be any multiple of the lcm of m_1^2 and m_2^2.

Theorem 2.6. *There is a finitely generated field extension $F'\supset F$ such that F'/F is separably generated and F is algebraically closed in F', together with a division algebra E'/F' of degree N such that $D_i\subset E'$ is compatible with $F\subset F'$ for $i=1,2$.*

Proof. Set $n_i=N/m_i$, noting that n_i is a multiple of m_i. Set

$$E=UD(F,N)$$

with center Z. We will extend Z so that the D_i embed in the base extension of E. To achieve this we set $F'=F_{n_1,n_2}(A_1,A_2)$ where

$$A_i=(D_i^{\circ}\otimes_{K_i}K_iZ)\otimes_{K_iZ}(E\otimes_Z K_iZ).$$

Note that $(E\otimes_Z K_iZ)$ is just the generic division algebra over K_i. Also note that $E'=E\otimes_Z F'$ has both D_i embedded, by Theorem 2.3, because, in view of Lemma 2.2, we have suitably reduced the index of both A_i. The problem is to show that E' is a division algebra, i.e., that $\operatorname{index}(E')=N$, and for this we apply Theorem 2.5. In applying this theorem, note that the degree of A_1 is Nd_1, and so is a multiple of n_1. We make a similar comment about the degree of A_2. $\square$

To apply Theorem 2.5, we need to get a handle on

$$\operatorname{index}((E\otimes_Z ZK(\alpha,\beta))^{1+\alpha+\beta}\otimes(D_1^{-\alpha}\otimes_{K(\alpha)}ZK(\alpha,\beta))\otimes(D_2^{-\beta}\otimes_{K'(\beta)}ZK(\alpha,\beta))),$$

where the unsubscripted tensors are over $ZK(\alpha,\beta)$. Write $\alpha=\sum n_{gH}gH$ and set $a=\sum n_{gH}$ and similarly for β and b. Note that E is not moved by either Galois group, so $(E\otimes_Z K(\alpha,\beta)Z)^{1+\alpha+\beta}$ is $E^{1+a+b}\otimes_Z K(\alpha,\beta)Z$, which has index $N/(N,1+a+b)$, which we define to be $N_{a,b}$. By Lemma 2.2 the above index is

$$\operatorname{index}\big((D_1^{-\alpha}\otimes_{K(\alpha)}K(\alpha,\beta))\big)\otimes_{K(\alpha,\beta)}(D_2^{-\beta}\otimes_{K'(\beta)}K(\alpha,\beta))N_{a,b},$$

and so we want to show that N divides the expression
(2)
$$\operatorname{index}\big((D_1^\alpha \otimes_{K(\alpha)} K(\alpha,\beta)\big) \otimes_{K(\alpha,\beta)} \big(D_2^\beta \otimes_{K'(\beta)} K(\alpha,\beta)\big)[K(\alpha,\beta):F]|\alpha||\beta|N_{a,b}.$$

We show the needed divisibility prime by prime. So assume for p prime that p^s divides N exactly, in the sense that $\frac{N}{p^s}$ is prime to p. Likewise, assume that p^{t_i} divides n_i exactly. Since $N = n_i d_i e_i$ and n_i is a multiple of $d_i e_i$, we have $2t_i \geq s$ and also $t_1 + t_2 \geq s$. If $1 + a + b$ is prime to p we are done. Thus we assume p divides

$$1 + \sum_{gH} n_{gH} + \sum_{gH'} m_{gH'}, \tag{3}$$

and this implies that at least one summand in

$$\sum_{gH} n_{gh} + \sum_{gH'} m_{gH'} \tag{4}$$

is prime to p. If any term in (4), say n_{gH}, is prime to p, then $\frac{n}{(n,n_{gH})}$ is divisible by p^{t_1}. Thus, if two terms in (4) are prime to p, then p^{2t_1} or $p^{t_1+t_2}$ or p^{2t_2} divides $|\alpha||\beta|$, and again we are done.

Thus we assume that p is prime to exactly one summand in (4). Replacing α by $g^{-1}\alpha$, we assume that only n_H is prime to p. It follows that H_α fixes the trivial coset H and so $H_\alpha \subseteq H$, implying $K(\alpha) \supseteq K$. Set $\alpha' = \alpha - n_H H$; thus,

$$|\alpha'| = \prod_{gH \neq H} \frac{n}{(n, n_{gH})}.$$

We know that $K(\alpha) = [K(\alpha):K][K:F]$. Write $s = s_1 + s_2 + s_3$ where p^{s_1} is the exact power of p dividing n, p^{s_2} is the exact power dividing d_1, and p^{s_3} is the exact power dividing e_1. Note that p^{s_1} divides $\frac{n}{(n,n_H)}$, and of course p^{s_3} divides $[K:F]$. Thus it suffices to show that p^{s_2} divides

$$\operatorname{index}(D'')[K(\alpha,\beta):K]|\alpha'||\beta|, \tag{5}$$

where

$$D'' = (D_1^{\alpha} \otimes_{K(\alpha)} K(\alpha,\beta)) \otimes_{K(\alpha,\beta)} (D_2^{\beta} \otimes_{K'(\beta)} K(\alpha,\beta)).$$

We will prove in fact that (5) is divisible by d_1. Note that

$$(D_1^{\alpha} \otimes_{K(\alpha)} K(\alpha,\beta)) = (D_1 \otimes_K K(\alpha,\beta)) \otimes_{K(\alpha,\beta)} (D_1^{\alpha'} \otimes_{K(\alpha)} K(\alpha,\beta)).$$

We need to estimate some indices. Of course D_1 has index d_1, and so over $\bar{K}\bar{K}'$, $g(D_1)^{n_{gH}}$ has index dividing $\frac{d_1}{(d_1,n_{gH})}$. Then $D_1^{\alpha'} \otimes_{K(\alpha)} K(\alpha,\beta)$ has index dividing $\prod_{gH\neq H} \frac{d_1}{(d_1,n_{gH})}$. Similarly $D_2^{\beta} \otimes_{K'(\beta)} K(\alpha,\beta)$ has index dividing $\prod_{gH'} \frac{d_2}{(d_2,m_{gH'})}$.

We need a trivial lemma.

Lemma 2.7. *Suppose a divides b. Then $a/(a,d)$ divides $b/(b,d)$.*

Proof. For any prime p, the power of p dividing $\frac{a}{(a,d)}$ is less than or equal to the power of p dividing $\frac{b}{(b,d)}$.

Let p^{u_1} be the exact power of p dividing index$(D_1 \otimes_K K(\alpha,\beta))$ and let

$$D^{\#} = (D_1^{\alpha'} \otimes_{K(\alpha)} K(\alpha,\beta)) \otimes_{K(\alpha,\beta)} (D_2^{\beta} \otimes_{K'(\beta)} K(\alpha,\beta)).$$

Also let p^{u_2} be the exact power of p dividing index$(D^{\#})$. It follows from Lemma 2.7 that p^{u_2} divides $|\alpha'||\beta|$. Also, $D'' = (D_1^{n_H} \otimes_K K(\alpha,\beta)) \otimes_{K(\alpha,\beta)} D^{\#}$. Now let p^{u_3} be the exact power of p dividing

$$\mathrm{index}(D_1^{n_H} \otimes_K K(\alpha,\beta)).$$

This is the same as the exact power of p dividing

$$\mathrm{index}(D_1 \otimes_K K(\alpha,\beta)).$$

Set p^{u_4} to be the exact power of p dividing $[K(\alpha,\beta) : K]$. Then index(D'') is a multiple of $p^{u_1-u_2}$. Thus index$(D'')|\alpha'||\beta|$ is a multiple of p^{u_1} and

$$\mathrm{index}(D'')|\alpha'||\beta|[K(\alpha,\beta) : K]$$

is a multiple of $p^{u_1+u_4}$ which is the exact power of p dividing

$$\mathrm{index}\,(D_1 \otimes_K K(\alpha,\beta))\,[K(\alpha,\beta) : K],$$

a multiple of d_1. This proves Theorem 2.6. □

Theorem 2.8. *Suppose D_i/K_i are division algebras with the K_i finite separable extensions of a purely transcendental field extension of a field F_0. Then the D_i can be embedded into a common division algebra E finite over its center.*

Proof. Tensoring the base fields by purely transcendental extensions still will leave the D_i division algebras, so we may assume that the K_i are finite separable extensions of a common subfield purely transcendental over F_0. Thus, we may apply Theorem 2.6, where here D_i takes the role of D_i' and E takes the role of E'. □

The referee has pointed out that the same techniques can be applied to Question (2):

Theorem 2.9. *Suppose D_i/K_i are division algebras with the K_i stably isomorphic or there is a field unirational over both K_1 and K_2. Then the D_i can be embedded into a common division algebra E finite over its center K, such that $K_i \subseteq K$ for $i = 1, 2$.*

Proof. Tensoring the base fields by purely transcendental extensions provides them with the same center, to which we can apply Theorem 2.6. □

The question remains: What is the lowest possible bound for $\deg E$?

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