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COMMENT ON A RESULT BY ALPIN, CHIEN, AND YEH

A. MELMAN

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Abstract. Theorem 3 in Alpin, Yu. A.; Chien, Mao-Ting; and Yeh, Lina, *The numerical radius and bounds for zeros of a polynomial*, Proc. Amer. Math. Soc., 131 (2003), no. 3, 725–730, is incorrect. In this note, we identify the error and present a correctly stated theorem.

1. Introduction

This note concerns the paper [1] by Alpin, Chien, and Yeh. One of their results, namely, Theorem 3, is erroneous. We provide a counterexample and then identify the flaw in the argument of the proof before stating an amended theorem. Most of our notation will be the same as in [1].

Let

$$\begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{pmatrix} \tag{1.1}$$

be the companion matrix of the monic polynomial

$$p(t) = t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \cdots + a_1 t + a_0 \ , \tag{1.2}$$

and define for a complex $n \times n$ matrix $A = (a_{ij})$:

$$R_i'(A) = \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \ \text{ and } \ g_i(A) = \frac{1}{2}\left(R_i'(A) + R_i'(A^*)\right) \ , \ 1 \le i \le n \ , \tag{1.3}$$

where A^* is the Hermitian conjugate of A. We denote the spectrum of A by $\sigma(A)$ and its numerical range by $W(A)$, i.e.,

$$W(A) = \{z^* A z \,:\, z \in \mathbb{C}^n, |z| = 1\} \ .$$

The set of points $z \in \mathbb{C}$ satisfying $|z-a||z-b| = \rho$ for $a, b \in \mathbb{C}$ and $\rho \ge 0$ is known as an oval of Cassini.

We now state Theorem 3 in [1].

Theorem 1.1 ([1, Theorem 3])**.** *Let p be the polynomial* (1.2) *and $p(z) = 0$, and let A be the matrix defined by* (1.1)*. Let $\alpha = \max_{1\le i<n} g_i(A)$. Then*

$$|z| \le |a_{n-1}|/2 + \left(|a_{n-1}|^2/4 + \left(\alpha^2 g_n^2(A) + |a_{n-1}|^2\right)^{1/2}\right)^{1/2} \ .$$

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The following claim presents a counterexample to this theorem.

Claim 1.2. The upper bound from Theorem 1 is not an upper bound for the modulus of any zero of $q(t) = t^9 + 0.01t^8 + 0.2$.

Proof. For the companion matrix A of $q(t)$, we have

$$\begin{aligned}
&R_1'(A) = 0.2\ ,\ R_2'(A) = R_3'(A) = \cdots = R_8'(A) = 1\ ,\ R_9'(A) = 1\ ,\\
&R_1'(A^*) = 1\ ,\ R_2'(A^*) = R_3'(A^*) = \cdots = R_8'(A^*) = 1\ ,\ R_9'(A^*) = 0.2\ ,\\
&g_1(A) = 0.6\ ,\ g_2(A) = g_3(A) = \cdots = g_8(A) = 1\ ,\ g_9(A) = 0.6\ .
\end{aligned}$$

Therefore, $\alpha = \max_{1\le i<9} g_i(A) = 1$, and the upper bound in Theorem 1 is given by

$$0.01/2 + \left(0.01^2/4 + \left(1^2 \cdot 0.6^2 + 0.01^2\right)^{1/2}\right)^{1/2} = 0.7797.$$

However, any zero z of $q(t)$ satisfies $0.8352 \le |z| \le 0.8374$, with the entire range being larger than 0.7797. □

2. Corrected theorem

The error in the proof of Theorem 3 in [1] was caused by the following incorrect inequality:

$$\alpha \le |a_{n-1}|/2 + \left(|a_{n-1}|^2/4 + \left(\alpha^2 g_n^2(A) + |a_{n-1}|^2\right)^{1/2}\right)^{1/2}\ . \tag{2.1}$$

To see why this inequality cannot be correct in general, assume, e.g., that for the coefficients of p in (1.2): $a_1 = a_2 = \cdots = a_{n-2} = 0$, $a_{n-1} = \beta$, and $a_0 = \epsilon$ with $\beta, \epsilon > 0$ and $\beta, \epsilon \ll 1$. Then $\alpha = 1$ and $g_n = (1+\epsilon)/2$. For β sufficiently small, inequality (2.1) becomes

$$\begin{aligned}
1 &\le \frac{\beta}{2} + \left(\frac{\beta^2}{4} + \left(\frac{(1+\epsilon)^2}{4} + \beta^2\right)^{1/2}\right)^{1/2}\\
&= \frac{\beta}{2} + \left(\frac{1+\epsilon}{2} + \mathcal{O}\left(\beta^2\right)\right)^{1/2}\\
&= \left(\frac{1+\epsilon}{2}\right)^{1/2} + \mathcal{O}(\beta)\ .
\end{aligned}$$

Clearly, as $\beta, \epsilon \longrightarrow 0$, this inequality cannot hold. The following theorem corrects Theorem 3 in [1].

Theorem 2.1. *Let p be the polynomial* (1.2) *and $p(z) = 0$, and let A be the matrix defined by* (1.1). *Let α_1 and α_2 be the largest and (possibly equal) second largest of the numbers $\{g_i(A)\}_{i=1}^{n-1}$. Then*

$$|z| \le \max\left\{\sqrt{\alpha_1\alpha_2}\ ,\ \frac{|a_{n-1}|}{2} + \left(\frac{|a_{n-1}|^2}{4} + \left(\alpha_1^2 g_n^2(A) + |a_{n-1}|^2\right)^{1/2}\right)^{1/2}\right\}\ .$$

Proof. The proof follows, just as in Theorem 3 in [1], by applying Theorem 2 in [1] to the companion matrix A of p. The closed interiors of the ovals of Cassini defined in that theorem are then the collection of points $z \in \mathbb{C}$ for which

$$|z| \leq (g_i(A)g_j(A))^{1/2} \ , \ 1 \leq i \neq j < n \ , \tag{2.2}$$

and those for which

$$|z||z + a_{n-1}| \leq \left(g_i^2(A)g_n^2(A) + |a_{n-1}|^2\right)^{1/2} \ , \ 1 \leq i < n \ . \tag{2.3}$$

The points defined by (2.2) are all contained in the disk defined by $|z| \leq (\alpha_1\alpha_2)^{1/2}$, and the points defined by (2.3) are all contained in the disk defined by

$$|z| \leq \frac{|a_{n-1}|}{2} + \left(\frac{|a_{n-1}|^2}{4} + \left(\alpha_1^2 g_n^2(A) + |a_{n-1}|^2\right)^{1/2}\right)^{1/2} \ .$$

This, together with the fact that $\sigma(A) \subset W(A)$, completes the proof. □

For our counterexample with $q(t) = t^9 + 0.01t^8 + 0.2$, we obtain $\alpha_1 = \alpha_2 = 1$, so that Theorem 2 states, correctly, that for any zero z of q: $|z| \leq 1$.

References

1. Alpin, Yu. A., Chien, Mao-Ting, and Yeh, Lina. *The numerical radius and bounds for zeros of a polynomial.* Proc. Amer. Math. Soc., 131 (2003), no. 3, 725–730. MR1937409 (2003h:26021)

Department of Applied Mathematics, School of Engineering, Santa Clara University, Santa Clara, California 95053

E-mail address: amelman@scu.edu