ON BOREL SETS BELONGING TO EVERY INVARIANT CCC $\sigma$-IDEAL ON $2^\mathbb{N}$

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(Communicated by Julia Knight)

Abstract. Let $I_{ccc}$ be the $\sigma$-ideal of subsets of the Cantor group $2^\mathbb{N}$ generated by Borel sets which belong to every translation-invariant $\sigma$-ideal on $2^\mathbb{N}$ satisfying the countable chain condition (ccc). We prove that $I_{ccc}$ strongly violates ccc. This generalizes a theorem of Balcerzak-Roslanowski-Shelah stating the same for the $\sigma$-ideal on $2^\mathbb{N}$ generated by Borel sets $B \subseteq 2^\mathbb{N}$ which have perfectly many pairwise disjoint translates. We show that the last condition does not follow from $B \in I_{ccc}$ even if $B$ is assumed to be compact. Various other conditions which for a Borel set $B$ imply that $B \in I_{ccc}$ are also studied.

As a consequence we prove in particular that:

• If $A_n$ are Borel sets, $n \in \mathbb{N}$, and $2^\mathbb{N} = \bigcup_n A_n$, then there is $n_0$ such that every perfect set $P \subseteq 2^\mathbb{N}$ has a perfect subset $Q$, a translate of which is contained in $A_{n_0}$.
• CH is equivalent to the statement that $2^\mathbb{N}$ can be partitioned into $\aleph_1$ many disjoint translates of a closed set.

1. Introduction

A $\sigma$-ideal on an uncountable Polish space $X$ is a family $I \subseteq \mathcal{P}(X)$ which is closed under taking subsets and countable unions. Throughout the paper we assume that $I$ is proper, i.e., $X \not\in I$, contains all singletons and every set from $I$ is covered by Borel sets from $I$. We say that a $\sigma$-ideal $I$ on $X$ is ccc if there is no uncountable family of disjoint Borel subsets of $X$ outside $I$.

Our starting point is a question stated in [2] by Balcerzak, Roslanowski and Shelah. Looking at $\sigma$-ideals that are not ccc they asked what the reasons for the failure of ccc there could be. They considered properties (M) and (D), introduced and investigated earlier by Balcerzak in [1].

We say that a $\sigma$-ideal $I$ in a Polish space $X$ has property (M) if there is a Borel surjective function $f : X \to 2^\mathbb{N}$ such that all fibers of $f$ are not in $I$.

If $(G, +)$ is a Polish abelian group, following [22] we denote by $\mathcal{F}_0(G)$ the family of all Borel sets $B \subseteq G$ such that there exists a perfect set $P \subseteq G$ with $(B + x) \cap (B + y) = \emptyset$ for $x, y \in P$ and $x \neq y$. We say that an invariant $\sigma$-ideal $I$ on $G$ has property (D) if there is a Borel set $B \in \mathcal{F}_0(G) \setminus I$.

Balcerzak (see [1]) observed that (D) implies (M) for every invariant $\sigma$-ideal $I$ on $G$ and posed the problem whether the converse holds true. Bukovsky reformulated Balcerzak’s question about $(M) \Rightarrow (D)$ by considering the $\sigma$-ideal $I_0$ on $2^\mathbb{N}$ (where $2^\mathbb{N}$ is considered with the coordinatewise addition modulo 2 and referred to as the...
Cantor group) generated by \( \mathcal{F}_0(2^\mathbb{N}) \) and asking if \( I_0 \) has property (M). The positive answer to the latter is the content of [2] Theorem 2.1.

A \( \sigma \)-ideal \( I \) on a Polish abelian group \( G \) is translation-invariant (shortly: invariant) if

\[
\forall x \in G \forall A \subseteq G \ (A \in I \Rightarrow x + A \in I).
\]

The idea behind (D) is to single out a particular property (namely, \( \mathcal{F}_0 \)) of a Borel subset of, say, \( 2^\mathbb{N} \) that prevents it from being a member of any invariant ccc \( \sigma \)-ideal and then to ask whether a failure of ccc of an invariant \( \sigma \)-ideal \( I \) (even in the strong form of (M)) is always witnessed by an \( I \)-positive Borel set with the property under consideration. Balcerzak, Roslanowski and Shelah gave the negative answer in the case of property \( \mathcal{F}_0 \). Generalizing this result, we prove that the idea indicated above cannot be accomplished by any property. Namely, in Section 2 we show (see Theorem 4.3) that \( C \) is equivalent to the statement that \( 2^\mathbb{N} \) has property (M). This is done by adapting the main idea of a simplified proof of the Balcerzak-Roslanowski-Shelah theorem, due to Solecki [22]. The point is that it is actually easier to prove that \( I_{ccc} \) has (M) than it is to prove that so has \( I_0 \). In effect, our argument provides a still simpler proof of the latter as well.

In Section 4 we consider the question which Borel sets belong to \( I_{ccc} \). In particular, we give an example (see Theorem 3.8) of a compact set \( C \in I_{ccc} \setminus \mathcal{F}_0(2^\mathbb{N}) \) (in fact no two different translates of \( C \) are disjoint). An example of a compact subset of \( \mathbb{R} \) which is not in \( \mathcal{F}_0(\mathbb{R}) \) but belongs to every invariant ccc \( \sigma \)-ideal on \( \mathbb{R} \) is the standard middle 1/3 Cantor set (see Remark 3.10).

Finally, in Section 4 we indicate some links between the subject of this paper and that of covering \( 2^\mathbb{N} \) (or, more generally, an uncountable Polish abelian group) by less than \( \kappa \) many translates of its closed nowhere dense subset. The latter has been recently dealt with by many authors (see [6, 7, 8, 9, 19, 15]). We prove (see Theorem 4.3) that CH is equivalent to the statement that \( 2^\mathbb{N} \) can be partitioned into \( \aleph_1 \) many disjoint translates of a closed set.

2. Property (M)

Let \( I_{ccc} \) be the \( \sigma \)-ideal on \( 2^\mathbb{N} \) generated by Borel sets which belong to every invariant ccc \( \sigma \)-ideal on \( 2^\mathbb{N} \).

**Theorem 2.1.** \( I_{ccc} \) has property (M).

**Proof.** Following Solecki [22], for each \( n \in \mathbb{N} \) we fix a partition of \( 2^\mathbb{N} \) into an \( F_\sigma \)-set \( A_n^0 \) of measure 1 and a dense \( G_\delta \)-set \( A_n^1 \). Next we define \( f : (2^\mathbb{N})^N \to 2^N \) by \( f((x_n))(i) = 0 \) if \( x_i \in A_n^0 \) and \( f((x_n))(i) = 1 \) if \( x_i \in A_n^1 \). Using the canonical topological group isomorphism we view \( f \) as a function with domain \( 2^\mathbb{N} \). Our goal is to show that the fibers of \( f \) are not only outside \( I_0 \), as shown in [22], but actually even outside \( I_{ccc} \).

To that end, for each \( y \in 2^N \) we shall find an invariant ccc \( \sigma \)-ideal \( I_y \) on \( 2^\mathbb{N} \) such that \( f^{-1}(y) \not\subseteq I_y \).

So fix \( y \in 2^N \). Let \( Y_0 = \{ n \in \mathbb{N} : y(n) = 0 \} \) and \( Y_1 = \{ n \in \mathbb{N} : y(n) = 1 \} \).

First assume that \( Y_0 \neq \emptyset \) and \( Y_1 \neq \emptyset \). Using the canonical topological group isomorphism, identify \( 2^\mathbb{N} \) with the product group \( G = G_0 \times G_1 \), where \( G_0 = (2^\mathbb{N})^{Y_0} \) and \( G_1 = (2^\mathbb{N})^{Y_1} \). Then \( f^{-1}(y) = \prod_{n \in Y_0} A_n^0 \times \prod_{n \in Y_1} A_n^1 \), and it suffices to find a
G-invariant ccc σ-ideal \( I_y \) on \( G \) such that \( f^{-1}(y) \notin I_y \). Let \( \mathcal{N}(G_0) \) be the σ-ideal of null sets in \( G_0 \) (with respect to the product of ordinary measures on \( 2^{\aleph_0} \)) and let \( \mathcal{M}(G_1) \) be the σ-ideal of meager subsets of \( G_1 \). Note that \( \prod_{n \in \mathbb{N}} A_n^0 \notin \mathcal{N}(G_0) \) (in fact it is of measure 1) and \( \prod_{n \in \mathbb{N}} A_n^1 \notin \mathcal{M}(G_1) \) (in fact it is comeager). It follows that \( f^{-1}(y) \notin \mathcal{N}(G_0) \otimes \mathcal{M}(G_1) \), where \( \mathcal{N}(G_0) \otimes \mathcal{M}(G_1) \), the Fubini product of \( \mathcal{N}(G_0) \) and \( \mathcal{M}(G_1) \), is the σ-ideal generated by the Borel sets \( B \subseteq G_0 \times G_1 \) with \( \{ x \in G_1 : B_x \notin \mathcal{M}(G_1) \} \in \mathcal{N}(G_0) \). By a theorem of Gavalec [12] (see also [11] for a more general result), the σ-ideal \( \mathcal{N}(G_0) \otimes \mathcal{M}(G_1) \) is ccc. As it is also easy to see that it is \( G \)-invariant, it suffices to let \( I_y = \mathcal{N}(G_0) \otimes \mathcal{M}(G_1) \).

If \( Y_0 = \emptyset \) or \( Y_1 = \emptyset \), then it suffices to let \( I_y = \mathcal{M}(G_1) \) in the first case and \( I_y = \mathcal{N}(G_0) \) in the second case. \( \square \)

The following corollary was originally proved by Balcerzak, Rosłanowski and Shelah [2, Theorem 2.1] and a simplified proof was found by Solecki [22]. Our approach, though closely related to [22], provides a still simpler proof of it.

**Corollary 2.2.** \( I_0 \) has property \((M)\).

**Proof.** As, clearly, \( I_0 \subseteq I_{ccc} \), the corollary is an immediate consequence of Theorem 2.1. \( \square \)

**Remark 2.3.** Exactly as in [22], Theorem 2.1 can be generalized to the case of a countable product of compact, metric, uncountable abelian groups.

### 3. Which Borel sets are in \( I_{ccc} \)?

For a cardinal \( 0 < \kappa \leq \aleph_1 \), we say that a subset \( A \) of an abelian, uncountable group \( G \) is \( \kappa \)-small if there is an uncountable set \( P \subseteq G \), called a witness (of \( \kappa \)-smallness of \( A \)), such that \( |(g + A) \cap P| < \kappa \) for every \( g \in G \) (cf. [6] and [8]). If, moreover, \( G \) is Polish and there exists a perfect witness, then we say that \( A \) is perfectly \( \kappa \)-small. Clearly, \( \emptyset \) is the only 1-small subset of \( G \) and if \( 0 < \lambda < \kappa \) \( \leq \aleph_1 \) and \( A \) is \( \lambda \)-small, then it is also \( \kappa \)-small with the same witness. We say that \( A \) is small (perfectly small, respectively) if it is \( \aleph_1 \)-small (perfectly \( \aleph_1 \)-small, respectively).

Perfectly small subsets of \( \mathbb{R} \) have been studied by Darji and Keleti [6], who proved (see [6] Theorem 2.5) that every compact subset of \( \mathbb{R} \) with packing dimension less than 1 is perfectly small. An example of such a set is the standard middle 1/3 Cantor set.

Following an idea from [6] let us consider the action of \( G \) on the product group \( G^{\kappa} \) by coordinate-wise translations, i.e., the function \( F_\kappa : G \times G^{\kappa} \to G^{\kappa} \) defined by

\[
F_\kappa(g, \langle g_\alpha : \alpha < \kappa \rangle) = \langle g + g_\alpha : \alpha < \kappa \rangle.
\]

For a set \( P \subseteq G \) let

\[
IS_\kappa(P) = \{ \langle g_\alpha : \alpha < \kappa \rangle \in P^{\kappa} : \forall \alpha, \beta (\alpha \neq \beta \Rightarrow g_\alpha \neq g_\beta) \}
\]

be the set of all injective sequences of length \( \kappa \) of elements of \( P \).

The following simple lemma gives useful characterizations of smallness (cf. [6] Lemma 2.3).

**Lemma 3.1.** For a non-zero cardinal \( \kappa \leq \aleph_1 \) and subsets \( A \) and \( P \), \(|P| \geq \aleph_1 \), of an uncountable abelian group \( G \) the following are equivalent:

1. \( P \) is a witness that \( A \) is \( \kappa \)-small,
(2) $\bigcap_{\alpha<\kappa} (-p_\alpha + A) = \emptyset$ for every $\kappa$ many different elements $p_\alpha \in P$, 
(3) $F_\kappa[G \times A^\kappa] \cap IS_\kappa(P) = \emptyset$.

Proof. Note that $P$ is not a witness that $A$ is $\kappa$-small iff for a certain $g \in G$ there are $p_\alpha \in P$, $\alpha < \kappa$, with $\alpha \neq \beta \Rightarrow p_\alpha \neq p_\beta$ such that

(A) $\{p_\alpha : \alpha < \kappa\} \subseteq g + A$.

But (A) is equivalent to

(B) $-g \in \bigcap_{\alpha<\kappa} (-p_\alpha + A)$

and also to

(C) $\{p_\alpha : \alpha < \kappa\} \in F_\kappa[\{g\} \times A^\kappa]$. \hfill $\Box$

In particular, by (2), $A$ is 2-small with a witness of cardinality $\lambda$ iff there are $\lambda$ many pairwise disjoint translations of $A$. Likewise, a Borel subset $A$ of a Polish abelian group $G$ is perfectly 2-small iff $A \in \mathcal{F}_0(G)$.

Also, $A$ is $\aleph_0$-small ($\aleph_1$-small, respectively) iff there is an uncountable set $P \subseteq G$ such that the family $\{g + A : g \in P\}$ is point-countable (point-finite, respectively).

In the terminology of Kubiš [14] and Mátrai [10], condition (3) says that $P$ is $C$-homogeneous, where $C$ is the complement of $F_\kappa[G \times A^\kappa]$ in $G^\kappa$.

Given a group $G$ and a non-empty set $A \subseteq G$, the supremum of cardinalities $\lambda$ such that there are $\lambda$ many pairwise disjoint translations of $A$ is sometimes called (see [3]) the packing index of $A$ and denoted by $\text{ind}_P(A)$. Therefore, $A$ is 2-small iff $\text{ind}_P(A)$ is uncountable. The packing indices of analytic subsets of the Polish group have been studied by Banakh, Lyaskovska and Repovs [3].

The following result sheds some light on interesting (and to some extent still unclear) connections between the various notions of smallness introduced above. The particular case of $n = 2$ is implicit in [3, Theorem 1] (cf. Remark 3.3).

**Proposition 3.2.** Let $A$ be a subset of a Polish, abelian group $G$.

1. Assume that $G$ is $\sigma$-compact. If $A$ is $F_\sigma$ and $n$-small for a certain $n$, $0 < n < \aleph_0$, then $A$ is perfectly $n$-small.
2. If $A$ is analytic and $\aleph_0$-small with a non-meager witness, then $A$ is perfectly $\aleph_0$-small.
3. Let $\aleph_1 < \lambda \leq \mathfrak{c}$. It is consistent with ZFC that if $A$ is analytic and $\aleph_0$-small with a witness of cardinality $\lambda$, then $A$ is perfectly $\aleph_0$-small.

Proof. In order to deal with $\kappa$-small sets, where $0 < \kappa \leq \aleph_0$, let $C = G^\kappa \setminus F_\kappa[G \times A^\kappa]$. By Lemma 3.1 (3), in each case assuming the existence of a $C$-homogeneous set with certain properties we want to prove the existence of a perfect one.

In case (1) ($\kappa = n$), $G$ and $A$ being $\sigma$-compact and $F_n$ being continuous, $C$ is a $G_\delta$ subset of $G^n$ and the existence of an uncountable $C$-homogeneous set implies the existence of a perfect one, by a theorem of Kubiš [14, Corollary 2.3].

In cases (2) and (3) ($\kappa = \aleph_0$), $A$ being analytic, $C$ is a co-analytic subset of $G^{\aleph_0}$, and the existence of a perfect $C$-homogeneous set is guaranteed by Theorems 1.2 and 4.10 of Mátrai [16], respectively. \hfill $\Box$

As mentioned above, the particular case of 2-small and perfectly 2-small subsets of Polish groups has been studied by Banakh, Lyaskovska and Repovs [3] in the context of packing indices and earlier by Balcerzak [1] in the context of property (D). The following characterization of compact perfectly 2-small sets (equivalently,
compact members of $\mathcal{F}_0(G)$ is implicit in [3] (see [3, Lemma 1] and the proof of [3, Theorem 1]; see also [1, Proposition 3.1]).

**Remark 3.3.** For a compact subset $C$ of a Polish abelian group $G$ the following are equivalent:

1. $C$ is perfectly 2-small,
2. $C - C$ does not contain a neighbourhood of the neutral element of $G$,
3. $C - C$ has empty interior.

Leaving aside interrelations between $\kappa$-small and perfectly $\kappa$-small sets (see, however, Proposition 3.13), in the next two results we show that all Borel subsets of $2^\omega$ with any of the properties under consideration, possibly apart from ($\mathfrak{m}$)smallness, are in the $\sigma$-ideal $I_{ccc}$. The statement that every small Borel subset of $2^\omega$ is in $I_{ccc}$ is independent from ZFC.

Recall that $S \subseteq 2^\omega$ is a Sierpiński set if $S$ is uncountable but $|S \cap B| \leq \mathfrak{m}$ for every null-set $B \subseteq 2^\omega$.

**Proposition 3.4.** Let $B$ be a Borel subset of $2^\omega$.

1. If $B$ is $\aleph_0$-small, then $B \in I_{ccc}$.
2. Under $\text{MA} + \neg\text{CH}$, if $B$ is small, then $B \in I_{ccc}$.
3. If there exists a Sierpiński set $S \subseteq 2^\omega$, then every Borel null-set $B \subseteq 2^\omega$ is small with a witness $S$. In particular, there is one not in $I_{ccc}$.

**Proof.** To prove (1) and (2), assume that $B$ is small and let $I$ be an invariant ccc $\sigma$-ideal on $2^\omega$. Suppose that $B \notin I$.

If $B$ is, moreover, $\aleph_0$-small with a witness $P$, $|P| \geq \mathfrak{m}$, then, by Proposition 3.13, $\{-g + B : g \in P\}$ is a point-finite family of Borel subsets of $2^\omega$ not in $I$. But, $I$ being ccc and $P$ being uncountable, this is impossible by a result of Fremlin (see [10, Lemma 1.E(b)]), and this contradiction ends the proof of (1).

Analogically, if $B$ is small with a witness $P$, $|P| \geq \mathfrak{m}$, then, by Lemma 3.1, $\{-g + B : g \in P\}$ is a point-countable family of Borel subsets of $2^\omega$ not in $I$. But under $\text{MA} + \neg\text{CH}$ this is again prevented by [10, Lemma 1.E(b)] of Fremlin, since then, by a theorem of Martin and Solovay (see, e.g., [18, Theorem 11.1]), no Borel set not in $I$ can be covered by fewer than $\mathfrak{m}_2$ members of $I$.

The first part of (3) is clear. To prove the second part, it suffices to take a dense $G_\delta$ null-set $B$. $\Box$

The following result generalizes a theorem of Cichoń, Kharazishvili and Weglorz (cf. [5, Lemma 4]) stating that a perfectly small Borel subset of $\mathbb{R}$ has Lebesgue measure zero.

**Theorem 3.5.** Every perfectly small Borel subset of $2^\omega$ is in $I_{ccc}$.

**Proof.** The idea of the proof is closely related to a reasoning described at the end of [4, Section 4] by Cichoń, Jasiński, Kamburelis and Szczepaniak.

Fix an invariant ccc $\sigma$-ideal $I$ in $2^\omega$. Assume that a Borel set $A \subseteq 2^\omega$ is small with a perfect witness $P \subseteq 2^\omega$.

Consider the set

$$B = \{(x, y) \in 2^\omega \times P : x + y \in A\}.$$ 

The set $B$ is Borel in $2^\omega \times P$ and $B_x = P \cap (A + x)$ is countable for each $x \in 2^\omega$.

By a result of Reclaw and Zakrzewski (see [20, Theorem 2.1]) there is $y \in P$ such...
that \( B^y \in I \) (actually, this holds for all but countably many points \( y \in P \)). But \( B^y = A + y \), so \( A \in I \) by the invariance of \( I \).

\[ \square \]

**Remark 3.6.** The proof above shows also that every perfectly small Borel subset of a Polish uncountable abelian group \( G \) is in the \( \sigma \)-ideal \( I_{\text{ccc}}(G) \) generated by the Borel subsets of \( G \) which belong to every invariant ccc \( \sigma \)-ideal on \( G \).

An immediate consequence of Theorem 3.5 is the fact that Borel perfectly small subsets of \( 2^N \) generate an invariant \( \sigma \)-ideal on \( 2^N \). This generalizes a result of Cichoń, Jasiński, Kamburelis and Szczepaniak (see [4, Proposition 4.4]) stating that if \( \mathbb{R} \) is the union of two uncountable Borel sets \( A \) and \( B \), then \( |(A + x) \cap B| = \mathfrak{c} \) for a certain \( x \in \mathbb{R} \).

**Corollary 3.7.** \( 2^N \) is not the union of any countable collection of Borel perfectly small sets. Equivalently, if \( A_n \) are Borel sets, \( n \in \mathbb{N} \), and \( 2^N = \bigcup_n A_n \), then there is \( n_0 \) such that every perfect set \( P \subseteq 2^N \) has a perfect subset \( Q \), a translate of which is contained in \( A_{n_0} \).

We do not know at the moment whether the \( \sigma \)-ideal \( I_{\text{ccc}} \) is generated by Borel perfectly small subsets of \( 2^N \). In fact, we do not even know if \( I_{\text{ccc}} \neq I_0 \), the \( \sigma \)-ideal generated by the collection \( \mathcal{F}_0 \) of Borel sets which have perfectly many pairwise disjoint translates. The next result yields a much simpler fact that \( I_{\text{ccc}} \neq \mathcal{F}_0 \).

**Theorem 3.8.** There exists a compact, perfectly 3-small subset \( C \) of \( 2^N \) such that \( C + C = 2^N \). In particular, \( C \in I_{\text{ccc}} \setminus \mathcal{F}_0 \).

**Proof.** First we shall make the following observation.

**Lemma 3.9.** There are sets \( A, S \subseteq \{0, 1\}^4 \) such that:

1. \( A + A = \{0, 1\}^4 \),
2. \( |S| = 3 \) and \( \bigcap_{s \in S} (s + A) = \emptyset \).

To see this, it suffices to let \( A = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2, 3, 4\}\} \) and \( S = \{\{1\}, \{1, 2\}, \{1, 2, 3, 4\}\} \), where we identify subsets of \( \{1, 2, 3, 4\} \) with the corresponding elements of \( \{0, 1\}^4 \) via their characteristic functions.

Now, to prove the theorem, let \( I_k = [4k, 4k + 4) \cap \mathbb{N}, k \in \mathbb{N} \), be consecutive 4-element intervals in \( \mathbb{N} \).

Applying Lemma 3.9 for each \( k \in \mathbb{N} \) choose sets \( A_k, S_k \subseteq \{0, 1\}^{I_k} \) such that

1. \( A_k + A_k = \{0, 1\}^{I_k} \),
2. \( |S_k| = 3 \) and \( \bigcap_{s \in S_k} (s + A_k) = \emptyset \).

Define \( C = \prod_{k \in \mathbb{N}} A_k \) or, more precisely,

3. \( C = \{x \in 2^\mathbb{N} : \forall k \in \mathbb{N} x|I_k \in A_k\} \).

Then, by (1), we have

4. \( C + C = 2^\mathbb{N} \).

Next consider the set

5. \( U = \{(x_1, x_2, x_3) \in (2^\mathbb{N})^3 : \exists k \in \mathbb{N} \{x_1|I_k, x_2|I_k, x_3|I_k\} = S_k\} \).

Note that \( U \) is a dense open subset of \( (2^\mathbb{N})^3 \). By the Mycielski partition theorem (see [13, Theorem 19.1]), there is a perfect set \( P \subseteq 2^\mathbb{N} \) such that

6. \( (x_1, x_2, x_3) \in U \) for any three different elements \( x_i \in P \).
In view of (4), to complete the proof it is enough to show that

(7) $P$ is a witness that $C$ is 3-small.

So let $x_1$, $x_2$, $x_3$ be three different elements of $P$. Then, by (6), $(x_1, x_2, x_3) \in U$, so there is $k \in \mathbb{N}$ with $\{x_1|I_k, x_2|I_k, x_3|I_k\} = S_k$ (see (5)).

This implies (see (2)) that $(x_1|I_k + A_k) \cap (x_2|I_k + A_k) \cap (x_3|I_k + A_k) = \emptyset$ and hence (see (3)) $(x_1 + C) \cap (x_2 + C) \cap (x_3 + C) = \emptyset$.

This, however, by Lemma 3.1, completes the proof of (7).

The last statement of Theorem 3.8 follows from Remark 3.3. □

Remark 3.10. A related example is provided by the standard middle 1/3 Cantor set $C \subseteq \mathbb{R}$. By [6, Theorem 2.5], $C$ is perfectly small, so by Theorem 3.5 $C$ is in the $\sigma$-ideal $I_{ccc}(\mathbb{R})$ (see Remark 3.6). On the other hand, $C$ is not perfectly 2-small since, by a well-known theorem of Steinhaus, $C - C = [-1, 1]$.

From Theorem 3.5 we may obtain a strengthening (for $2^\mathbb{N}$) of a theorem of Elekes and Steprans [8], who proved (see [8, Theorem 1.2]), answering a question of Darji and Keleti [6], that there exists a compact subset of $\mathbb{R}$ of Lebesgue measure zero which is not perfectly small.

Proposition 3.11. There exists a compact null-set $C \subseteq 2^\mathbb{N}$ which is not in $I_{ccc}$; hence, in particular, it is not perfectly small.

Proof. Any invariant ccc $\sigma$-ideal $I$ on $2^\mathbb{N}$ of the form $M_{C,W}$ constructed by Roslanowski and Shelah [21, Conclusion 4.7] has the property that there is a compact null-set not in $I$. □

If we do not require $A$ to be both compact and not in $I_{ccc}$, then we can get more.

Proposition 3.12. In every invariant, ccc $\sigma$-ideal in $2^\mathbb{N}$:

(1) there is a Borel set which is not in $I_{ccc}$,
(2) there is a compact set which is not perfectly small.

Proof. To prove (1), note that no invariant, ccc $\sigma$-ideal in $2^\mathbb{N}$ is contained in $I_{ccc}$, since the latter is not ccc by Theorem 2.1.

To prove (2), we shall use the following lemma due to Balcerzak, Roslanowski and Shelah [2, Lemma 2.2]: for every $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ and a set $A \subseteq \{0, 1\}^m$ such that every $n$ translates of $A$ have non-empty intersection and likewise for $A' = \{0, 1\}^m \setminus A$.

Applying the lemma inductively, choose a strictly increasing sequence $n_k \in \mathbb{N}$, $n_0 = 0$, and associated sequence of sets $A_k \subseteq \{0, 1\}^{I_k}$, where $I_k = [n_k, n_{k+1}) \cap \mathbb{N}$ for $k \in \mathbb{N}$, such that

(1) every $2^{k+1}$ translates of $A_k$ by elements of $\{0, 1\}^{I_k}$ have non-empty intersection and likewise for $A'_k$.

Next, following the proof of [2, Theorem 2.1], define $f : 2^\mathbb{N} \to 2^\mathbb{N}$ by $f(x)(k) = 1$ if $x \upharpoonright I_k \in A_k$ and $f(x)(i) = 0$ if $x \upharpoonright I_k \in A'_k$.

Fix an invariant, ccc $\sigma$-ideal $I$ on $2^\mathbb{N}$. Clearly, each of the sets $f^{-1}(y)$, where $y \in 2^\mathbb{N}$, is compact and at least one of them is in $I$. Without loss of generality
assume that the set
\[(2) \quad C = \{x \in 2^\mathbb{N} : \forall k \in \mathbb{N} \; x \upharpoonright I_k \in A_k\}\]
is in \(I\). We shall prove that \(C\) is not perfectly small.

To that end fix an arbitrary perfect set \(P \subseteq 2^\mathbb{N}\). We will show (cf. \[8\) Theorem 1.2]) that there exists a perfect set \(Q \subseteq P\) and \(y \in 2^\mathbb{N}\) such that \(y + Q \subseteq C\).

Let \(T_P \subseteq \{0, 1\}^{\mathbb{N}}\) be the perfect pruned tree on \(\mathbb{N}\) (see \[13\]) such that \(P = [T_P]\), the set of all infinite branches of \(T_P\). We shall find a perfect subtree \(T_Q\) of \(T_P\) and then define \(Q = [T_Q]\).

To construct \(T_Q\), inductively define integers \(0 = k_0 < k_1 < k_2 < \ldots\) and families \(R_i \subseteq \{s \in T_P : \text{length}(s) = n_{k_i+1}\}\) such that for each \(i \in \mathbb{N}\):
\[(3) \quad |R_0| = 2, \forall s \in R_i, |\{t \in R_{i+1} : s \subseteq t\}| = 2 \text{ and } \forall t \in R_{i+1}, t \upharpoonright n_{k_{i+1}} \in R_i.\]
Consequently, letting \(S_i = \{s \upharpoonright [n_{k_i}, n_{k_{i+1}}) : s \in R_i\}\), we have \(|S_i| \leq 2^{k+1}\); hence
\[(4) \quad |\{s \upharpoonright I_k : s \in S_i\}| \leq 2^{k+1} \text{ for each } i \in \mathbb{N} \text{ and } k \in [k_i, k_{i+1}).\]
It then follows from (1) and (4) that
\[(5) \quad \bigcap_{s \in S_i} (s \upharpoonright I_k + A_k) \neq \emptyset \text{ for each } i \in \mathbb{N} \text{ and } k \in [k_i, k_{i+1}).\]
With the help of (5) define \(y \in 2^\mathbb{N}\) in such a way that for each \(i \in \mathbb{N}\) and \(k \in [k_i, k_{i+1}),\)
\[(6) \quad y \upharpoonright I_k \in \bigcap_{s \in S_i} (s \upharpoonright I_k + A_k),\]
which in turn is equivalent to
\[(7) \quad y \upharpoonright I_k + s \upharpoonright I_k \in A_k \text{ for every } s \in S_i.\]
Letting \(T_Q\) be the smallest tree on \(\mathbb{N}\) containing \(\bigcup_{i \in \mathbb{N}} R_i\) and \(Q = [T_Q]\) we easily conclude from (2) and (7) that \(y + Q \subseteq C\).

As another immediate corollary of Theorem \[3.5\] we get the consistency of the existence of a small set (even one with a witness of cardinality \(\mathfrak{c}\)) which is not perfectly small (cf. Proposition \[3.13\(3))\).

**Corollary 3.13.** If \(I\) is an invariant ccc \(\sigma\)-ideal on \(2^\mathbb{N}\) and there exists an \(I\)-Lusin set \(S \subseteq 2^\mathbb{N}\), then every Borel set from \(I\) is small with a witness \(S\). In particular, there is one not in \(I_{ccc}\), hence not perfectly small.

With the help of Corollary \[3.13\] and Proposition \[3.11\] we get a result related to a theorem of Mátrai stating that it is consistent with ZFC that even for open sets \(C \subseteq (2^\mathbb{N})^{\mathbb{N}_0}\) the existence of a \(C\)-homogeneous set of cardinality \(\mathfrak{c}\) does not imply the existence of a perfect one (cf. \[16\) Theorem 1.1]).

**Proposition 3.14.** If there exists a Sierpiński set \(S \subseteq 2^\mathbb{N}\), then there exists an open subset \(C\) of \((2^\mathbb{N})^{\mathbb{N}_1}\) with the property that \(S\) is \(C\)-homogeneous but no perfect \(C\)-homogeneous set exists.

**Proof.** Let \(A \subseteq 2^\mathbb{N}\) be a compact null-set which is not perfectly small (cf. Proposition \[3.11\]) and \(C = (2^\mathbb{N})^{\mathbb{N}_1} \setminus F_{\mathbb{N}_1} [2^\mathbb{N} \times A^{\mathbb{N}_1}]\). The function
\[F_{\mathbb{N}_1} : 2^\mathbb{N} \times (2^\mathbb{N})^{\mathbb{N}_1} \to (2^\mathbb{N})^{\mathbb{N}_1}\]
being continuous and the set \(A\) being compact, \(C\) is an open subset of \((2^\mathbb{N})^{\mathbb{N}_1}\).

By Corollary \[3.13\] and Lemma \[3.13\(3), \(S\) is \(C\)-homogeneous, but no perfect \(C\)-homogeneous set exists since there is no perfect witness of smallness of \(A\). \(\square\)
4. Covering $2^\mathbb{N}$ by $\aleph_1$ Many Disjoint Translates of a Compact Set

The purpose of this section is to point out some links between the subject of this paper and that of covering $2^\mathbb{N}$ (or, more generally, an uncountable Polish group) by less than $\mathfrak{c}$ many translates of its compact subset. The latter has been recently dealt with by many authors (see [6], [7], [8], [9], [19], [15]).

Let us start with an obvious generalization of a simple observation of Darji and Keleti (see [6, Lemma 2]).

**Lemma 4.1.** Let $G$ be an uncountable abelian group and $\aleph_1 \leq \kappa \leq \mathfrak{c}$. Then less than $\kappa$ many translates of a small set with a witness of cardinality at least $\kappa$ do not cover $G$.

Note that when $\mathfrak{c} > \aleph_1$, Lemma 4.1 prevents $2^\mathbb{N}$ from being covered by $\aleph_1$ many translates of any Borel perfectly small subset $B$. At the same time Theorem 3.5 implies that $B \in I_{ccc}$ and, in fact, the smallness of a Borel set $B \subseteq 2^\mathbb{N}$ is, at least consistency-wise, the strongest among conditions considered in Section 3 (cf. Proposition 3.4) sufficient for this.

These remarks lead to the natural question whether in the absence of CH, at least for a compact set $A \subseteq 2^\mathbb{N}$, the properties of $2^\mathbb{N}$ not being covered by $\aleph_1$ many translates of $A$ on the one hand and $A$ being a member of $I_{ccc}$ on the other hand, are related.

We have the following observation.

**Proposition 4.2.** It is consistent with the negation of CH that there is a compact null-set $A \subseteq 2^\mathbb{N}$ such that $A \not\in I_{ccc}$ but no $\aleph_1$ many translates of $A$ cover $2^\mathbb{N}$. In fact, the latter is true for any null-set $A$ provided a Sierpiński set $S \subseteq 2^\mathbb{N}$ of size greater than $\aleph_1$ exists and CH is false.

**Proof.** This is an immediate consequence of Proposition [3.11] and Lemma [4.1].

Miller [17] proved that it is consistent with the negation of CH that $2^\mathbb{N}$ (equivalently, any uncountable Polish space) can be partitioned into $\aleph_1$ many disjoint non-empty closed sets. In contrast to this, we have the following result.

**Theorem 4.3.** CH is equivalent to each of the following statements:

1. $2^\mathbb{N}$ can be partitioned into $\aleph_1$ many disjoint translates of a closed set,
2. some uncountable locally compact Polish abelian group can be partitioned this way,
3. all uncountable locally compact Polish abelian groups can be partitioned this way.

Moreover, $\neg$CH implies that if an uncountable locally compact Polish abelian group is the union of a collection $A$ of $\aleph_1$ many translates of a closed set, then for every natural number $n$ there is $x \in G$ such that $x$ is a member of at least $n$ elements of $A$.

**Proof.** Let $G$ be an uncountable locally compact Polish abelian group.

It suffices to prove that $\neg$CH implies that $G$ cannot be covered by $\aleph_1$ many translates of a closed set in such a way that for some $n > 1$ no $x \in G$ is a member of $n$ elements of the covering.

So assume that $\mathfrak{c} > \aleph_1$.

Let $G$ be an uncountable locally compact Polish group. Suppose, towards a contradiction, that $A$ is a closed subset of $G$ and for some $n > 1$ there is a covering
of $G$ by $\aleph_1$ many translates of $A$ such that no $x \in G$ is a member of $n$ elements of the covering. Then, by Lemma 3.1, $A$ is $n$-small and by Proposition 3.2 it is perfectly small. But this implies, by Lemma 4.1, that $G$ cannot be covered by less than $\aleph$ many translates of $A$. □

5. Open problems

Let us conclude with a list of some natural open questions.

(1) **Problem 1.** What is the relation of $I_{ccc}$ to the intersection of all invariant ccc $\sigma$-ideals on $2^{\mathbb{N}}$? In particular, is there a non-Borel set $A \subseteq 2^{\mathbb{N}}$ which belongs to every invariant ccc $\sigma$-ideal but is not in $I_{ccc}$? (This question was pointed out to me by the referee, whom I wish to thank for all his/her valuable remarks.)

(2) **Problem 2.** What is the relation of $I_{ccc}$ to the $\sigma$-ideal generated by Borel perfectly small (perfectly $2$-small, respectively) subsets of $2^{\mathbb{N}}$? (Cf. Theorem 3.5.)

(3) **Problem 3.** Is every Borel $n$-small, $1 < n < \aleph_0$, subset of a Polish abelian group $G$ perfectly $n$-small? (Cf. Proposition 3.2; a closely related problem for $n = 2$ was formulated in [3] as Question 2.)

References


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