PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 141, Number 3, March 2013, Pages 943–958
S 0002-9939(2012)11467-4
Article electronically published on July 24, 2012

SOLUTIONS OF AN ADVECTED PHASE FIELD SYSTEM WITH LOW REGULARITY VELOCITY

BIANCA MORELLI RODOLFO CALSAVARA AND JOSÉ LUIZ BOLDRINI

(Communicated by Walter Craig)

Abstract. We present a result on existence of solutions for a system of highly nonlinear partial differential equations related to a phase field model for non-isothermal solidification/melting processes in the case of two possible crystallization states and flow of the molten material.

The flow is incompressible with a velocity which is assumed to be given, but with low regularity. We prove the existence of solutions for the associated system and also give estimates for the temperature and the phase fields related to each of the crystallization states in terms of the low regularity norms of the given flow velocity.

These results constitute a fundamental step in the proof of the existence of solutions of a complete model for solidification obtained by coupling the present equations with a singular Navier-Stokes system for the flow velocity. The analysis of this complete model is done in a forthcoming article.

1. Introduction

In this work we study a system of nonlinear partial differential equations subjected to boundary and initial conditions related to a phase field model for solidification and melting of certain metallic alloys that may solidify in two different kinds of crystallization, taking into account the possibility of flow of the molten material. The system is the following:

$$
\begin{aligned}
&\tau_t - b\Delta\tau + v.\nabla\tau = l_1\alpha_t + l_2\beta_t + l_3\gamma_t + f && \text{in } Q,\\
&\alpha_t - k\Delta\alpha + v.\nabla\alpha = g_1(\tau,\alpha,\beta,\gamma) + g_3(\tau,\alpha,\beta,\gamma) && \text{in } Q,\\
&\beta_t - k\Delta\beta + v.\nabla\beta = g_2(\tau,\alpha,\beta,\gamma) - g_3(\tau,\alpha,\beta,\gamma) && \text{in } Q,\\
&\gamma_t - k\Delta\gamma + v.\nabla\gamma = -g_1(\tau,\alpha,\beta,\gamma) - g_2(\tau,\alpha,\beta,\gamma) && \text{in } Q,
\end{aligned}
\tag{1.1}
$$

$$
\begin{aligned}
&\partial\tau/\partial n = \partial\alpha/\partial n = \partial\beta/\partial n = \partial\gamma/\partial n = 0 && \text{on } \partial\Omega\times(0,T),\\
&\tau = \tau_0,\ \alpha = \alpha_0,\ \beta = \beta_0,\ \gamma = \gamma_0 && \text{in } \Omega\times\{t=0\}.
\end{aligned}
$$

The notation used here is as follows. The set $\Omega \subset \mathbb{R}^2$ is a bounded C^2-domain, $0 < T < \infty$, $Q = \Omega\times(0,T)$ and n denotes the outer unitary normal to the boundary $\partial\Omega$.

The unknown phase field functions α and β represent respectively the solid fractions of the two possible different kinds of crystallization: the unknown phase

Received by the editors July 27, 2011.

2010 *Mathematics Subject Classification.* Primary 35K40, 80A22; Secondary 47H10, 76R10.

Key words and phrases. Parabolic partial differential equations, fixed point, solidification, phase field, advection, convection.

The first author was supported in part by FAPESP, Brazil, Grant 2010/10087-8.

The second author was supported in part by CNPq, Brazil, Grant 307833/2008-9.

field function γ represents the corresponding liquid fraction; thus, by their physical meanings, we must have α, β, $\gamma \geq 0$ and $\alpha + \beta + \gamma = 1$.

The unknown function τ is associated to the temperature.

The incompressible velocity field is denoted by v and it is assumed to be given, but it has low regularity. The reason for this will be explained later on, and the precise required regularity for the velocity will be described in the statement of our main result (Theorem 2.4).

In the equation for the temperature, $b > 0$ is a given constant related to the thermal diffusion coefficient, while l_1, l_2 and l_3 are given constants related to the latent heats associated to each kind of physical state; the function f is related to the given density of heat sources and sinks.

In the equations for the phase fields, $k > 0$ is a given constant related to the width of the transition layers; interactions among the phase fields are given by the functions

$$\begin{aligned} g_1 &= g_1(\tau, \alpha, \beta, \gamma) = -a_1\alpha\gamma(\gamma - \alpha + c_1\tau + d_1), \\ g_2 &= g_2(\tau, \alpha, \beta, \gamma) = -a_2\beta\gamma(\gamma - \beta + c_2\tau + d_2), \\ g_3 &= g_3(\tau, \alpha, \beta, \gamma) = -a_3\alpha\beta(\beta - \alpha + c_3\tau + d_3), \end{aligned} \tag{1.2}$$

where a_1, a_2, a_3, c_1, c_2, c_3, d_1, d_2, d_3 are given constants.

The initial conditions τ_0, α_0, β_0, and γ_0 are suitable given functions that according to their physical interpretation satisfy $\alpha_0 + \beta_0 + \gamma_0 = 1$.

The present model for alloy solidification involves more than one phase field and it can be seen as a generalization of the model treated by Hoffman and Jiang in [12] and of the one treated by Steinbach *et al.* in [21] and [22]. As in [12], the present model assumes that the energy stored in the transition layers is isotropic, but in [12] only one crystallization type is allowed; there is not the possibility of flow in the melt. As in [21] and [22] we have two possible crystallization states and the same interaction potentials for the phase fields, and thus the same functions g_1, g_2 and g_3; but in [21] and [22] the temperature is supposed to be known and there is no possibility of flow in the melt. Besides, rigorous mathematical analysis is not the main interest in [21] and [22].

To be fair, we should mention that there are many articles that deal with different types of questions concerning phase field models. Here, we just mention a few representative articles dealing with some of the mathematical aspects of the subject: Caginalp *et al.* [6]–[9], Hoffman and Jiang [12], Krejči, Sprekels and Stefanelli [13], Sprekels and Zheng [19], Planas [17], [18] and Boldrini *et al.* [3], [4], [5], [23]. For more information, the interested reader can consult the references of those papers.

We also call the attention of the reader to the fact that the equations of (1.1) could be rewritten just in terms of α and β since on physical grounds one must have $\gamma = 1 - \alpha - \beta$. Since in [21] and [22] it is presented using the three phase fields, just for comparison we also preferred to present it as in (1.1).

Next, to justify the present article, it is important to call the reader's attention to certain mathematical aspects of the problem.

From the point of view of the phase fields, the fact that there are two of them (α and β) in our system brings many more mathematical difficulties as compared to models with just one phase field. In fact, in this last case, the higher power nonlinearities have the right sign at least when one is obtaining the weaker estimates. On the other hand, with more than one phase field, in the equations we have higher powers nonlinearities that are products of different phase fields and thus we have

no a priori control of their signs. These difficulties demand a very careful treatment even in finding the weaker estimates. Moreover, due to their physical meaning, we will also have to prove that the phase fields are nonnegative and add up to one.

We also remark that, differently from what occurs in the usual phase field models, in the present one there are terms in which the temperature appears multiplying the phase fields, bringing in nonlinearities that are harder to handle than the ones in the usual models.

Another important aspect comes in when, as is the present case, one takes into account the possibility of the flow of the material due to thermal or concentration gradients. In this case, advection terms, whose coefficients are the components of the flow velocity, are added to the associated phase fields and temperature equations. A major difficulty then appears: in general, only rather low regularity can be assumed for the flow velocity. In fact, for the complete model for the solidification (see Calsavara and Boldrini [10]), the velocity is also an unknown variable and must be determined by extra equations for the flow; in many models these equations become a singular Navier-Stokes equation modified by a Carman-Koseny type term (see Calsavara and Boldrini [10], for instance). Since such a term goes to infinity as one goes from a molten region to an interface with a solid region, and such interface may have complex geometries (dendrites), in general only low regularity is expected for the flow velocity.

This brings in another mathematical difficulty to an already difficult highly nonlinear problem because it makes it harder to find suitable estimates for the unknown variables.

Therefore, besides our interest in system (1.1) for its own sake, one of the main motivations for the present article is to present results that will be fundamental in a forthcoming article [10], where we will prove the existence of solutions of the complete model for solidification in the case of two possible crystallization states and flow of molten material.

For this, due to the reasons already explained, we must not only prove the existence of solutions of system (1.1) but also obtain suitable estimates for α, β and τ in terms of a low regularity norm of the velocity, namely, in a norm implied by the lowest estimates that can usually be obtained for the velocity in several variants of the Navier-Stokes equations.

To obtain those results, we have to adapt some ideas used in Boldrini *et al.* [2] to handle the fact that we have more than one phase field, to the case with advection terms with low regularity coefficients. Our results are stated in Theorem 2.4 at the end of the next section.

Finally, this paper is organized as follows. In Section 2 we fix the notation, recall certain results that will be used in the paper and state our main result concerning the existence of a solution for problem (1.1). Section 3 is devoted to the analysis of an auxiliary problem and we prove the existence of the original problem.

2. Preliminaries, hypotheses and main result

We will use standard notation for Sobolev spaces. For convenience of referencing, here we recall certain facts that will be useful in the paper. Given $1 \leq p \leq +\infty$ and $k \in \mathbb{N}$, we denote the usual Sobolev space by

$$W_p^k(\Omega) = \{f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega), |\alpha| \leq k\}.$$

The properties for such spaces can be found for instance in Adams [1]; here we only mention the following result, which is a consequence of the Sobolev Embedding Theorem given in Adams [1] (Th. 5.4, p. 97):

Lemma 2.1. *If $\Omega \subset \mathbb{R}^2$ satisfies the cone property and p satisfies $1 \leq p < \infty$, then*

(i) $W_p^2(\Omega) \hookrightarrow W_q^{2-\frac{2}{q}}(\Omega)$, *for all* $p \leq q \leq 2p$;
(ii) $W_p^k(\Omega) \hookrightarrow L^\infty(\Omega)$, *for all* $k.p > 2$ *with* $k \in \mathbb{N}$;
(iii) $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$, *for all* $q \in \mathbb{N}$,

with continuous embedding.

The first through fourth equations of problem (1.1) will be studied in functional spaces denoted by

$$W_q^{2,1}(Q) \ = \{f \in L^q(Q) : D^\alpha f \in L^q(Q), \forall\, 1 \leq |\alpha| \leq 2, f_t \in L^q(Q)\},$$

and in functional spaces

$$L^p(0,T;B) = \left\{f : (0,T) \to B : \| \, \|f(t)\|_B \, \|_{L^p((0,T))} < +\infty\right\}.$$

For results concerning these spaces, we refer for instance to Ladyženskaja [14] and Mikhaylov [16]. Here we recall a result that sometimes is called the Lions-Peetre embedding theorem (see [15], p. 15); it is also a consequence of Lemma 3.3, p. 80, in Ladyženskaja [14]:

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded C^2-domain and let $Q = \Omega \times (0,T)$, with $0 < T < \infty$. Then, $W_q^{2,1}(Q) \subset L^p(Q)$, with compact and continuous embedding for*

$$p = \begin{cases} \text{any positive number} & \text{if} \quad q = 2, \\ \infty & \text{if} \quad q > 2. \end{cases}$$

Finally, let us see a result about existence, uniqueness and regularity for parabolic differential partial equations given by Ladyženskaja [14] (Theorem 9.1, p. 341).

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain. Suppose that $f \in L^q(Q)$ and $c \in L^q(Q)$, with $q > 1$, $\phi_0 \in W_q^{2-2/q}(\Omega)$ satisfy the compatibility condition $\partial\phi_0/\partial n|_{\partial\Omega} = 0$ and $v \in L^r(Q)$, with $r = \max\{q, 4\}$ if $q \neq 4$ and $r = 4+\varepsilon$ for any $\varepsilon \geq 0$ if $q = 4$. Then, the problem*

$$\begin{array}{ll} \phi_t - k\Delta\phi + v.\nabla\phi + c\phi = f & \text{in } Q, \\ \partial\phi/\partial n = 0 & \text{on } \partial\Omega \times (0,T), \\ \phi = \phi_0 & \text{in } \Omega \times \{t = 0\} \end{array}$$

has a unique solution $\phi \in W_q^{2,1}(Q)$. This solution satisfies the estimate

$$\|\phi\|_{W_q^{2,1}(Q)} \leq C\left[\|f\|_{L^q(Q)} + \|\phi_0\|_{W_q^{2-2/q}(\Omega)}\right],$$

where C depends on Ω, T, k, $||v||_{L^r(Q)}$ and $||c||_{L^q(Q)}$.

Next, for easy reference, we collect some hypotheses that will be assumed for the rest of this work.

Main hypotheses.

$$
\begin{aligned}
&\textbf{(i)}\ \Omega \subset \mathbb{R}^2 \text{ is a bounded } C^2\text{-domain;}\\
&\textbf{(ii)}\ 0 < T < \infty \text{ and } Q = \Omega \times (0,T);\\
&\textbf{(iii)}\ l_1 \text{ (or } l_1'), l_2 \text{ (or } l_2'), l_3, c_1, c_2, c_3, d_1, d_2, d_3 \text{ are real constants;}\\
&\qquad b, k, a_1, a_2, a_3 \text{ are positive constants;}\\
&\textbf{(iv)}\ \tau_0,\ \alpha_0,\ \beta_0,\ \gamma_0 \in W_2^2(\Omega);\\
&\textbf{(v)}\ \alpha_0,\ \beta_0,\ \gamma_0 \geq 0, \text{ and } \alpha_0 + \beta_0 + \gamma_0 = 1;\\
&\textbf{(vi)}\ \partial\tau_0/\partial n = \partial\alpha_0/\partial n = \partial\beta_0/\partial n = \partial\gamma_0/\partial n = 0.
\end{aligned}
\tag{2.1}
$$

Let us observe that $\alpha_0 + \beta_0 = 1$ implies $\gamma_0 = 0$. Then the last condition means that the initial velocity is null in the initial solid region.

To deal with the regularity of v, we need to consider the following functional space: V is the closure of $\mathcal{V} := \{w \in (C^\infty(\Omega))^2 : \text{ div } w = 0\}$ in $(H_0^1(\Omega))^2$.

Our objective in this work is to show the following existence result of a solution for problem (1.1) under the previous hypotheses.

Theorem 2.4. *Assume that hypotheses* (2.1) *hold. Let* $f \in L^q(Q)$, *with* $2 < q < 4$, *and* $v \in L^4(Q) \cap L^2(0,T;V)$. *Then problem* (1.1) *has a unique solution* $(\tau, \alpha, \beta, \gamma) \in \left[W_q^{2,1}(Q)\right]^4$. *Besides, this solution satisfies* $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$, $\alpha + \beta + \gamma = 1$ *and*

$$
\begin{aligned}
&\|\tau\|_{W_q^{2,1}(Q)} + \|\alpha\|_{W_q^{2,1}(Q)} + \|\beta\|_{W_q^{2,1}(Q)} + \|\gamma\|_{W_q^{2,1}(Q)}\\
&\leq C(\|\tau_0\|_{W_2^2(\Omega)} + \|\alpha_0\|_{W_2^2(\Omega)} + \|\beta_0\|_{W_2^2(\Omega)} + \|\gamma_0\|_{W_2^2(\Omega)} + \|f\|_{L^q(Q)}),
\end{aligned}
\tag{2.2}
$$

where the constant C *depends on* Ω, T, $||v||_{L^4(Q)}$ *and the constants of problem* (1.1).

3. Existence of solution

To obtain the existence of solutions of problem (1.1), firstly we consider an equivalent problem. For this, we observe that by adding the second, third and fourth equations in (1.1), we obtain that

$$(\alpha + \beta + \gamma)_t = 0 \quad \text{in} \quad Q.$$

Thus, from hypothesis (2.1) $\alpha_0 + \beta_0 + \gamma_0 = 1$, we conclude that if $(\tau, \alpha, \beta, \gamma) \in [W_2^{2,1}(Q)]^4$ is a solution of problem (1.1), then $\alpha + \beta + \gamma = 1$ in Q. So, by replacing $1 - \alpha - \beta$ by γ in (1.1), we get the following equivalent problem:

$$
\begin{aligned}
&\tau_t - b\Delta\tau + v.\nabla\tau = l_1'\alpha_t + l_2'\beta_t + f && \text{in } Q,\\
&\alpha_t - k\Delta\alpha + v.\nabla\alpha = \widetilde{g}_1(\tau,\alpha,\beta) + \widetilde{g}_3(\tau,\alpha,\beta) && \text{in } Q,\\
&\beta_t - k\Delta\beta + v.\nabla\beta = \widetilde{g}_2(\tau,\alpha,\beta) - \widetilde{g}_3(\tau,\alpha,\beta) && \text{in } Q,\\
&\partial\tau/\partial n = \partial\alpha/\partial n = \partial\beta/\partial n = 0 && \text{on } \partial\Omega \times (0,T),\\
&\tau = \tau_0,\ \alpha = \alpha_0,\ \beta = \beta_0 && \text{in } \Omega \times \{t = 0\},
\end{aligned}
\tag{3.1}
$$

where

$$l_1' = l_1 - l_3, \qquad l_2' = l_2 - l_3 \tag{3.2}$$

and

$$\widetilde{g}_i(\tau,\alpha,\beta) = g_i(\tau,\alpha,\beta,1-\alpha-\beta), \tag{3.3}$$

with g_i given by (1.2), for $i = 1, 2, 3$.

For this problem, the following result holds.

Proposition 3.1. *Assume that hypotheses* (2.1) *hold. Let* $f \in L^q(Q)$, *with* $2 < q < 4$, *and* $v \in L^4(Q) \cap L^2(0,T;V)$. *Then problem* (3.1) *has a unique solution* $(\tau, \alpha, \beta) \in W_q^{2,1}(Q) \times W_q^{2,1}(Q) \times W_q^{2,1}(Q)$ *such that* $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta \leq 1$ *and satisfies the following estimate:*

$$\begin{aligned} &\|\tau\|_{W_q^{2,1}(Q)} + \|\alpha\|_{W_q^{2,1}(Q)} + \|\beta\|_{W_q^{2,1}(Q)} \\ &\leq C(\|\tau_0\|_{W_2^2(\Omega)} + \|\alpha_0\|_{W_2^2(\Omega)} + \|\beta_0\|_{W_2^2(\Omega)} + \|f\|_{L^q(Q)}), \end{aligned} \tag{3.4}$$

where the constant C *depends on* Ω, T, $||v||_{L^4(Q)}$ *and the constants of problem* (3.1).

To prove this proposition, let us apply the Leray-Schauder fixed point theorem (see Friedman [11]) in the Banach space

$$\mathcal{B} := L^\infty(Q) \times L^\infty(Q) \times L^\infty(Q).$$

For this, consider the family of operators $\mathcal{T}_\lambda : \mathcal{B} \to \mathcal{B}$, for $0 \leq \lambda \leq 1$, defined by any $(\theta, \widetilde{\alpha}, \widetilde{\beta}) \in \mathcal{B}$ as

$$\mathcal{T}_\lambda(\theta, \widetilde{\alpha}, \widetilde{\beta}) = (\tau, \alpha, \beta),$$

where (τ, α, β) is defined as the solution of the following problem:

$$\begin{aligned} &\tau_t - b\Delta\tau + v.\nabla\tau = l_1'\alpha_t + l_2'\beta_t + f && \text{in } Q, \\ &\alpha_t - k\Delta\alpha + v.\nabla\alpha = \lambda\widetilde{g}_1(\theta, \widetilde{\alpha}, \widetilde{\beta}) + \lambda\widetilde{g}_3(\theta, \widetilde{\alpha}, \widetilde{\beta}) && \text{in } Q, \\ &\beta_t - k\Delta\beta + v.\nabla\beta = \lambda\widetilde{g}_2(\theta, \widetilde{\alpha}, \widetilde{\beta}) - \lambda\widetilde{g}_3(\theta, \widetilde{\alpha}, \widetilde{\beta}) && \text{in } Q, \\ &\partial\tau/\partial n = \partial\alpha/\partial n = \partial\beta/\partial n = 0 && \text{on } \partial\Omega \times (0,T), \\ &\tau = \tau_0,\ \alpha = \alpha_0,\ \beta = \beta_0 && \text{in } \Omega \times \{t = 0\}, \end{aligned} \tag{3.5}$$

where l_1', l_2', $\widetilde{g}_1$, $\widetilde{g}_2$ and $\widetilde{g}_3$ are given by (3.2) and (3.3).

Thus, to prove Proposition 3.1, that is, to obtain the existence of solutions of problem (3.1) we must verify that for each λ the operator $\mathcal{T}_\lambda$ is well defined and that the hypotheses of Leray-Schauder's fixed point theorem are satisfied. Such conditions will be checked in the sequence of lemmas that follow. However, before starting to do these tasks, let us consider an auxiliary lemma.

Lemma 3.2. *Assume that hypotheses* (2.1) *hold. Let* $f \in L^q(Q)$, *with* $2 < q < 4$, *and* $v \in L^4(Q) \cap L^2(0,T;V)$. *Then any possible solution* $(\tau, \alpha, \beta) \in W_q^{2,1}(Q) \times W_q^{2,1}(Q) \times W_q^{2,1}(Q)$ *of problem* (3.1) *satisfies:*

$$\alpha \geq 0, \quad \beta \geq 0 \quad \textit{and} \quad \alpha + \beta \leq 1.$$

Proof of Lemma 3.2. Let $(\tau, \alpha, \beta) \in [W_q^{2,1}(Q)]^3 \subset [L^\infty(Q)]^3$ be a solution of problem (3.1).

Firstly let us prove that $\alpha \geq 0$. Multiplying the second equation of problem (3.1) by $-(\alpha_-) = \min\{\alpha, 0\}$ and integrating in $\Omega \times (0,t)$, with $0 \leq t \leq T$, we have

$$\begin{aligned} \frac{1}{2}\int_\Omega (\alpha_-(t))^2\ dx + k\int_0^t\int_\Omega |\nabla(\alpha_-)|^2\ dx\ dt &\leq \frac{1}{2}\int_\Omega [(\alpha_0)_-]^2\ dx \\ &+ C\int_0^t\int_\Omega (\alpha_-)^2\ dx\ dt - \int_0^t\int_\Omega [v.\nabla(\alpha_-)](\alpha_-)\ dx\ dt, \end{aligned}$$

because τ, α, $\beta \in L^\infty(Q)$. Since $\alpha_0 \geq 0$ and $v \in L^2(0,T;V)$, we have $\int_\Omega [(\alpha_0)_-]^2\ dx = 0$ and $\int_0^t\int_\Omega [v.\nabla(\alpha_-)](\alpha_-)\ dx\ dt = 0$, respectively. Then,

$$\frac{1}{2}\int_\Omega (\alpha_-(t))^2\ dx + k\int_0^t\int_\Omega |\nabla(\alpha_-)|^2\ dx\ dt \leq C\int_0^t\int_\Omega (\alpha_-)^2\ dx\ dt.$$

By Gronwall's lemma,

$$\int_\Omega (\alpha_-(t))^2\, dx = 0, \quad \text{for all } t \in [0,T],$$

and thus $\alpha_- = 0$ a.e. in Q. So $\alpha \geq 0$ a.e. in Q.

By multiplying the third equation of problem (3.1) by $-(\beta_-) = \min\{\beta, 0\}$ and proceeding in the same way as in the previous case, we obtain that $\beta \geq 0$ a.e. in Q.

Finally, let us prove that $\alpha + \beta \leq 1$. For this, consider $\gamma = 1 - \alpha - \beta \in L^\infty(Q)$. Observe that γ satisfies

$$\begin{array}{ll} \gamma_t - k\Delta\gamma + v.\nabla\gamma = -g_1 - g_2 & \text{in } Q, \\ \partial\gamma/\partial n = 0 & \text{on } \partial\Omega \times (0,T), \\ \gamma = \gamma_0 & \text{in } \Omega \times \{t = 0\}, \end{array}$$

where g_1 and g_2 are given by (1.2) respectively, and $\gamma_0 := 1 - \alpha_0 - \beta_0$.

Since $\alpha_0 + \beta_0 \leq 1$ we have $\gamma_0 \geq 0$. Then, by multiplying the equation of the problem above by $-(\gamma_-) = \min\{\gamma, 0\}$ and proceeding similarly as in the previous cases, we obtain that $\gamma \geq 0$ a.e. in Q. Thus, $\alpha + \beta = 1 - \gamma \leq 1$. □

In the next sequence of lemmas, we will check that in fact the hypotheses of Leray-Schauder's fixed point theorem are satisfied.

Lemma 3.3. *Assume that hypotheses* (2.1) *hold. Let* $f \in L^q(Q)$, *with* $2 < q < 4$, *and* $v \in L^4(Q)$. *Then* $\mathcal{T}_\lambda(\theta, \widetilde{\alpha}, \widetilde{\beta})$ *is well defined for each* $(\theta, \widetilde{\alpha}, \widetilde{\beta}) \in \mathcal{B}$ *and each* $0 \leq \lambda \leq 1$.

Proof of Lemma 3.3. Indeed, since $(\theta, \widetilde{\alpha}, \widetilde{\beta}) \in \mathcal{B} := [L^\infty(Q)]^3$, we have $\lambda\widetilde{g}_1(\theta, \widetilde{\alpha}, \widetilde{\beta})$, $\lambda\widetilde{g}_3(\theta, \widetilde{\alpha}, \widetilde{\beta}) \in L^\infty(Q)$, for all $\lambda \in [0,1]$. Besides, $v \in L^4(Q)$. So, by Proposition 2.3 applied to the second equation of (3.5), there exists a unique solution $\alpha \in W_q^{2,1}(Q)$ of this equation. Since $q > 2$, $\alpha \in W_q^{2,1}(Q) \subset L^\infty(Q)$. Analogously, by applying Proposition 2.3 to the third equation of (3.5), we obtain that it has a unique solution $\beta \in W_q^{2,1}(Q) \subset L^\infty(Q)$.

Finally, we have $(\alpha, \beta) \in W_q^{2,1}(Q) \times W_q^{2,1}(Q)$ and $f \in L^q(Q)$, so the second member of the first equation in (3.5) belongs to $L^q(Q)$. By Lemma 2.1, $\tau_0 \in W_2^2(Q) \subset W_q^{2-2/q}(Q)$, we have that there exists a unique $\tau \in W_q^{2,1}(Q)$ that is a solution of the first equation of (3.5), by Proposition 2.3. Since $q > 2$, we have $W_q^{2,1}(Q) \subset L^\infty(Q)$, and thus $\tau \in L^\infty(Q)$.

In this way, we conclude that there exists a unique $(\tau, \widetilde{\alpha}, \widetilde{\beta}) \in \mathcal{B}$ that is a solution of (3.5), and thus $\mathcal{T}_\lambda : \mathcal{B} \to \mathcal{B}$ is well defined for all $\lambda \in [0,1]$. Moreover, $\mathcal{T}_\lambda(\mathcal{B}) \subset [W_q^{2,1}(Q)]^3$, for all $\lambda \in [0,1]$. □

Lemma 3.4. *Assume that hypotheses* (2.1) *hold. Let* $f \in L^q(Q)$, *with* $2 < q < 4$, *and* $v \in L^4(Q)$. *Then, for each fixed* $\lambda \in [0,1]$, *the operator* $\mathcal{T}_\lambda : \mathcal{B} \to \mathcal{B}$ *is continuous and compact.*

Proof of Lemma 3.4. For this, we fix $\lambda \in [0,1]$, consider $(\theta_1, \widetilde{\alpha}_1, \widetilde{\beta}_1)$, $(\theta_2, \widetilde{\alpha}_2, \widetilde{\beta}_2) \in \mathcal{B}$, with $(\tau_i, \alpha_i, \beta_i) = \mathcal{T}_\lambda(\theta_i, \widetilde{\alpha}_i, \widetilde{\beta}_i)$, for $i = 1, 2$, and define $(\tau, \alpha, \beta) = (\tau_1, \alpha_1, \beta_1) - (\tau_2, \alpha_2, \beta_2)$.

By using Proposition 2.3 for the problem satisfied by α, we have that, for all $2 < q < 4$,

$$\begin{aligned}\|\alpha\|_{W_q^{2,1}(Q)} &\leq C\|\lambda g_1(\theta_1,\widetilde{\alpha}_1,\widetilde{\beta}_1) - \lambda g_1(\theta_2,\widetilde{\alpha}_2,\widetilde{\beta}_2)\|_{L^q(Q)} \\ &\quad + C\|\lambda g_3(\theta_1,\widetilde{\alpha}_1,\widetilde{\beta}_1) - \lambda g_3(\theta_2,\widetilde{\alpha}_2,\widetilde{\beta}_2)\|_{L^q(Q)} \\ &\leq C\left[\|\theta_1-\theta_2\|_{L^\infty(Q)} + \|\widetilde{\alpha}_1-\widetilde{\alpha}_2\|_{L^\infty(Q)} + \|\widetilde{\beta}_1-\widetilde{\beta}_2\|_{L^\infty(Q)}\right] \\ &\leq C\left\|(\theta_1,\widetilde{\alpha}_1,\widetilde{\beta}_1)-(\theta_2,\widetilde{\alpha}_2,\widetilde{\beta}_2)\right\|_{\mathcal{B}},\end{aligned}$$

by Hölder's inequality since θ_i, $\widetilde{\alpha}_i$, $\widetilde{\beta}_i \in L^\infty(Q)$, for $i = 1, 2$. Here C depends on $\|(\theta_1,\widetilde{\alpha}_1,\widetilde{\beta}_1)\|_{\mathcal{B}}$ and $\|(\theta_2,\widetilde{\alpha}_2,\widetilde{\beta}_2)\|_{\mathcal{B}}$.

Similarly, by using again Proposition 2.3 for the equations satisfied by β and τ, we also obtain

$$\|\beta\|_{W_q^{2,1}(Q)} \leq C\left\|(\theta_1,\widetilde{\alpha}_1,\widetilde{\beta}_1)-(\theta_2,\widetilde{\alpha}_2,\widetilde{\beta}_2)\right\|_{\mathcal{B}} \quad \text{and}$$

$$\begin{aligned}\|\tau\|_{W_q^{2,1}(Q)} &\leq C\,\|l_1'\alpha_t + l_2'\beta_t\|_{L^q(Q)} \leq C(\|\alpha\|_{W_q^{2,1}(Q)} + \|\beta\|_{W_q^{2,1}(Q)}) \\ &\leq C\left\|(\theta_1,\widetilde{\alpha}_1,\widetilde{\beta}_1)-(\theta_2,\widetilde{\alpha}_2,\widetilde{\beta}_2)\right\|_{\mathcal{B}},\end{aligned}$$

for all $2 < q < 4$, with C depending on $\|(\theta_1,\widetilde{\alpha}_1,\widetilde{\beta}_1)\|_{\mathcal{B}}$ and $\|(\theta_2,\widetilde{\alpha}_2,\widetilde{\beta}_2)\|_{\mathcal{B}}$.

Hence, $\mathcal{T}_\lambda : \mathcal{B} \to [W_q^{2,1}(Q)]^3$ is continuous, and by using the compact embedding given in Lemma 2.2, we get that $\mathcal{T}_\lambda : \mathcal{B} \to \mathcal{B}$ is also a compact operator for each fixed $\lambda \in [0,1]$. □

Lemma 3.5. *Assume that hypotheses* (2.1) *hold. Let $f \in L^q(Q)$, with $2 < q < 4$, and $v \in L^4(Q)$. Then, given any bounded subset $\mathcal{A} \subset \mathcal{B}$ the operator $\mathcal{T}_\lambda(\theta,\widetilde{\alpha},\widetilde{\beta})$ is continuous in λ, uniformly with respect to $(\theta,\widetilde{\alpha},\widetilde{\beta}) \in \mathcal{A}$.*

Proof of Lemma 3.5. Indeed, take $(\theta,\widetilde{\alpha},\widetilde{\beta}) \in \mathcal{A}$ and consider $\lambda_1, \lambda_2 \in [0,1]$ and $(\tau_i,\alpha_i,\beta_i) = \mathcal{T}_{\lambda_i}(\theta,\widetilde{\alpha},\widetilde{\beta})$, for $i = 1, 2$. Proceeding as before, we have

$$\|\alpha_1-\alpha_2\|_{W_q^{2,1}(Q)} + \|\beta_1-\beta_2\|_{W_q^{2,1}(Q)} + \|\tau_1-\tau_2\|_{W_q^{2,1}(Q)} \leq C|\lambda_1-\lambda_2|,$$

where C depends on $\|(\theta_1,\widetilde{\alpha}_1,\widetilde{\beta}_1)\|_{\mathcal{B}}$ and $\|(\theta_2,\widetilde{\alpha}_2,\widetilde{\beta}_2)\|_{\mathcal{B}}$.

From this, since $\mathcal{A}$ is bounded in $\mathcal{B}$, we obtain

$$\|(\tau_1,\alpha_1,\beta_1)-(\tau_2,\alpha_2,\beta_2)\|_{\mathcal{B}} \leq C|\lambda_1-\lambda_2|,$$

with C depending only on $\mathcal{A}$, and thus $\mathcal{T}_{(\cdot)}(\theta,\widetilde{\alpha},\widetilde{\beta}) : [0,1] \to \mathcal{B}$ is continuous in λ, uniformly with respect to $(\theta,\widetilde{\alpha},\widetilde{\beta}) \in \mathcal{A}$. □

Lemma 3.6. *Assume that hypotheses* (2.1) *hold. Let $f \in L^q(Q)$, with $2 < q < 4$, and $v \in L^4(Q)$. Then the operator $\mathcal{T}_0$ has a unique fixed point.*

Proof of Lemma 3.6. For $\lambda = 0$ problem (3.5) becomes

$$\begin{array}{ll}\tau_t - b\Delta\tau + v.\nabla\tau = l_1'\alpha_t + l_2'\beta_t + f & \text{in } Q, \\ \alpha_t - k\Delta\alpha + v.\nabla\alpha = 0 & \text{in } Q, \\ \beta_t - k\Delta\beta + v.\nabla\beta = 0 & \text{in } Q, \\ \partial\tau/\partial n = \partial\alpha/\partial n = \partial\beta/\partial n = 0 & \text{on } \partial\Omega\times(0,T), \\ \tau = \tau_0,\ \alpha = \alpha_0,\ \beta = \beta_0 & \text{in } \Omega\times\{t=0\},\end{array}$$

where l_1' and l_2' are given by (3.2).

By applying Proposition 2.3 to each of the equations satisfied by α and β, we obtain that they have unique solutions α and $\beta \in W_q^{2,1}(Q)$. Once such α and β are obtained, we obtain that the equation τ, for which we can apply Proposition 2.3 again, also has a unique solution $\tau \in W_q^{2,1}(Q)$.

We conclude that there exists a unique $(\tau, \alpha, \beta) \in W_q^{2,1}(Q) \times W_q^{2,1}(Q) \times W_q^{2,1}(Q)$ solution of problem (3.5) with $\lambda = 0$, and thus the operator $\mathcal{T}_0$ has a unique fixed point $(\tau, \alpha, \beta) \in \mathcal{B}$. □

Lemma 3.7. *Assume that hypotheses* (2.1) *hold. Let* $f \in L^q(Q)$, *with* $2 < q < 4$, *and* $v \in L^4(Q) \cap L^2(0,T;V)$. *Then there exists a constant* $K > 0$ *independent of* λ *so that every possible fixed point* (τ, α, β) *of* $\mathcal{T}_\lambda$ *satisfies* $\|(\tau, \alpha, \beta)\|_{\mathcal{B}} \leq K$ *for all* $\lambda \in [0,1]$.

Proof of Lemma 3.7. For this, consider $(\tau, \alpha, \beta) \in \mathcal{B}$, a fixed point of $\mathcal{T}_\lambda$ for some $\lambda \in [0,1]$, i.e. $(\tau, \alpha, \beta) = \mathcal{T}_\lambda(\tau, \alpha, \beta)$. Then, (τ, α, β) satisfies

$$\begin{aligned}
&\tau_t - b\Delta\tau + v.\nabla\tau = l_1'\alpha_t + l_2'\beta_t + f && \text{in } Q,\\
&\alpha_t - k\Delta\alpha + v.\nabla\alpha = \lambda\widetilde{g}_1(\tau,\alpha,\beta) + \lambda\widetilde{g}_3(\tau,\alpha,\beta) && \text{in } Q,\\
&\beta_t - k\Delta\beta + v.\nabla\beta = \lambda\widetilde{g}_2(\tau,\alpha,\beta) - \lambda\widetilde{g}_3(\tau,\alpha,\beta) && \text{in } Q,\\
&\partial\tau/\partial n = \partial\alpha/\partial n = \partial\beta/\partial n = 0 && \text{on } \partial\Omega \times (0,T),\\
&\tau = \tau_0,\ \alpha = \alpha_0,\ \beta = \beta_0 && \text{in } \Omega \times \{t = 0\},
\end{aligned} \tag{3.6}$$

where l_1', l_2', $\widetilde{g}_1$, $\widetilde{g}_2$ and $\widetilde{g}_3$ are given by (3.2) and (3.3).

Next, we observe that by applying Lemma 3.2 to problem (3.6) we obtain that (τ, α, β) satisfy $0 \leq \alpha,\ \beta \leq 1$.

By multiplying the second equation of problem (3.6) by α and integrating the obtained equality on $\Omega \times (0,t)$, with any $0 \leq t \leq T$, we obtain

$$\begin{aligned}
\frac{1}{2}\int_\Omega \alpha(t)^2 dx + \int_0^t\int_\Omega |\nabla\alpha|^2 dxdt &= \frac{1}{2}\|\alpha_0\|^2_{L^2(\Omega)} - \int_0^t\int_\Omega (v.\nabla\alpha)\alpha\ dxdt\\
&+ \int_0^t\int_\Omega (-a_1 - a_1d_1 - a_3d_3 + 2a_1\beta + a_1d_1\beta - a_3\beta - a_1\beta^2)\alpha^2 dxdt\\
&+ \int_0^t\int_\Omega (3a_1 + a_1d_1 - 3a_1\beta + a_3\beta)\alpha^3 dxdt - 2a_1\int_0^t\int_\Omega \alpha^4 dxdt\\
&+ \int_0^t\int_\Omega (-a_1c_1\alpha + a_1c_1\alpha^2 + a_1c_1\alpha\beta - a_3c_3\alpha\beta)\alpha\tau\ dxdt.
\end{aligned}$$

Since $v \in V$ a.e. in $[0,T]$, we have that $\int_0^t\int_\Omega (v.\nabla\alpha)\alpha\ dxdt = 0$. By using that $0 \leq \alpha, \beta \leq 1$, by Lemma 3.2, and by using Young's inequality in the last term, we have

$$\begin{aligned}
&\int_\Omega \alpha(t)^2 dx + \int_0^t\int_\Omega |\nabla\alpha|^2 dxdt + \int_0^t\int_\Omega \alpha^4 dxdt\\
&\leq C\left[\|\alpha_0\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega \left(\tau^2 + \alpha^2 + \beta^2\right) dxdt\right].
\end{aligned} \tag{3.7}$$

By multiplying the third and first equations of problem (3.6) respectively by β and $\tau - l_1'\alpha - l_2'\beta$ and proceeding as in the previous case, we obtain

$$\begin{aligned}
&\int_\Omega \beta(t)^2 dx + \int_0^t\int_\Omega |\nabla\beta|^2 dxdt + \int_0^t\int_\Omega \beta^4 dxdt\\
&\leq C\left[\|\beta_0\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega \left(\tau^2 + \alpha^2 + \beta^2\right) dxdt\right]
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\int_\Omega \left(\tau(t) - l_1'\alpha(t) - l_2'\beta(t)\right)^2 dx + \int_0^t \int_\Omega \left(b|\nabla\tau|^2 - b.l_1'\nabla\tau.\nabla\alpha - b.l_2'\nabla\tau.\nabla\beta\right) dxdt \\ \leq C\Big[\|\tau_0\|^2_{L^2(\Omega)} + \|\alpha_0\|^2_{L^2(\Omega)} + \|\beta_0\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(Q)} \\ + \int_0^t \int_\Omega (v.\nabla\tau)(l_1'\alpha + l_2'\beta) dxdt + \int_0^t \int_\Omega \left(\tau^2 + \alpha^2 + \beta^2\right) dxdt\Big].\end{aligned}$$

By multiplying inequality (3.7) by a suitable positive constant, inequality (3.8) by another suitable constant, and adding the results to the last inequality, we obtain

$$\begin{aligned}\int_\Omega \left(\tau^2(t) + \alpha^2(t) + \beta^2(t)\right) dx \\ + \int_0^t \int_\Omega \left(|\nabla\tau|^2 + |\nabla\alpha|^2 + |\nabla\beta|^2\right) dxdt + \int_0^t \int_\Omega \left(\alpha^4 + \beta^4\right) dxdt \\ \leq C\left(\|\tau_0\|^2_{L^2(\Omega)} + \|\alpha_0\|^2_{L^2(\Omega)} + \|\beta_0\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(Q)}\right) \\ + C\int_0^t \int_\Omega |v.\nabla\tau|(\alpha+\beta) dxdt + C\int_0^t \int_\Omega \left(\tau^2 + \alpha^2 + \beta^2\right) dxdt.\end{aligned} \tag{3.9}$$

Next, by using Hölder's, Gagliardo-Nirenberg's and Young's inequalities and since $\alpha, \beta \geq 0$, by Lemma 3.2, we have

$$\begin{aligned}\int_\Omega |v.\nabla\tau|\alpha \; dx &\leq C\|v\|_{L^4(\Omega)}\|\nabla\tau\|_{L^4(\Omega)}\|\alpha\|_{L^2(\Omega)} \\ &\leq C\|v\|_{L^4(\Omega)}\|\nabla\tau\|^{1/2}_{L^2(\Omega)}\|\Delta\tau\|^{1/2}_{L^2(\Omega)}\|\alpha\|_{L^4(\Omega)} \\ &\leq C_\varepsilon\|v\|^4_{L^4(\Omega)}\|\nabla\tau\|^2_{L^2(\Omega)} + \varepsilon\|\Delta\tau\|^2_{L^2(\Omega)} + \varepsilon\|\alpha\|^2_{L^2(\Omega)}.\end{aligned}$$

In the same way,

$$\int_\Omega |v.\nabla\tau|\beta \; dx \leq C_\varepsilon\|v\|^4_{L^4(\Omega)}\|\nabla\tau\|^2_{L^2(\Omega)} + \varepsilon\|\Delta\tau\|^2_{L^2(\Omega)} + \varepsilon\|\beta\|^2_{L^2(\Omega)}.$$

By replacing the last two inequalities in the previous one, we obtain

$$\begin{aligned}\int_\Omega \left(\tau^2(t) + \alpha^2(t) + \beta^2(t)\right) dx + \int_0^t \int_\Omega \left(|\nabla\tau|^2 + |\nabla\alpha|^2 + |\nabla\beta|^2\right) dxdt \\ \leq C\left(\|\tau_0\|^2_{L^2(\Omega)} + \|\alpha_0\|^2_{L^2(\Omega)} + \|\beta_0\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(Q)}\right) \\ + C_\varepsilon\int_0^t \|v\|^4_{L^4(\Omega)}\|\nabla\tau\|^2_{L^2(\Omega)} dt + C\int_0^t \int_\Omega \left(\tau^2 + \alpha^2 + \beta^2\right) dxdt \\ + \varepsilon\int_0^t \int_\Omega \Delta\tau^2 \; dxdt,\end{aligned} \tag{3.10}$$

for all $0 \leq t \leq T$.

Next, by multiplying the first, second, and third equations of problem (3.6) respectively by τ_t, α_t, and β_t and proceeding similarly as we did for the last case,

we obtain

$$\begin{aligned}\int_0^t\int_\Omega \left(\tau_t^2+\alpha_t^2+\beta_t^2\right)dxdt+\int_\Omega \left(|\nabla\tau(t)|^2+|\nabla\alpha(t)|^2+|\nabla\beta(t)|^2\right)dx\\ \leq C\left(\|\tau_0\|^2_{W_2^2(\Omega)}+\|\alpha_0\|^2_{W_2^2(\Omega)}+\|\beta_0\|^2_{W_2^2(\Omega)}+\|f\|^2_{L^2(Q)}\right)\\ +C\int_0^t\int_\Omega\left[|v.\nabla\tau|.|\tau_t|+|v.\nabla\alpha|.|\alpha_t|+|v.\nabla\beta|.|\beta_t|\right]dxdt\\ +C_\varepsilon\int_0^t\int_\Omega\left(\tau^2+\alpha^2+\beta^2\right)dxdt+\varepsilon\int_0^t\int_\Omega\left(\tau_t^2+\alpha_t^2+\beta_t^2\right)dxdt.\end{aligned}$$

Now observe that by using Hölder's, Gagliardo-Nirenberg's and Young's inequalities, we have

$$\begin{aligned}\int_\Omega (v.\nabla\tau)(\tau_t)\ dx &\leq C\|v\|_{L^4(\Omega)}\|\nabla\tau\|_{L^4(\Omega)}\|\tau_t\|_{L^2(\Omega)}\\ &\leq C\|v\|_{L^4(\Omega)}\|\nabla\tau\|^{1/2}_{L^2(\Omega)}\|\Delta\tau\|^{1/2}_{L^2(\Omega)}\|\tau_t\|_{L^2(\Omega)}\\ &\leq C_\varepsilon\|v\|^4_{L^4(\Omega)}\|\nabla\tau\|^2_{L^2(\Omega)}+\varepsilon\|\Delta\tau\|^2_{L^2(\Omega)}+\varepsilon\|\tau_t\|^2_{L^2(\Omega)}.\end{aligned}$$

In the same way, we obtain

$$\begin{aligned}\int_\Omega |v.\nabla\alpha|.|\alpha_t|\ dx &\leq C_\varepsilon\|v\|^4_{L^4(\Omega)}\|\nabla\alpha\|^2_{L^2(\Omega)}+\varepsilon\|\Delta\alpha\|^2_{L^2(\Omega)}+\varepsilon\|\alpha_t\|^2_{L^2(\Omega)},\\ \int_\Omega |v.\nabla\beta|.|\beta_t|\ dx &\leq C_\varepsilon\|v\|^4_{L^4(\Omega)}\|\nabla\beta\|^2_{L^2(\Omega)}+\varepsilon\|\Delta\beta\|^2_{L^2(\Omega)}+\varepsilon\|\beta_t\|^2_{L^2(\Omega)}.\end{aligned}$$

By replacing the last two inequalities in the previous one, we get

$$\begin{aligned}\int_0^t\int_\Omega \left(\tau_t^2+\alpha_t^2+\beta_t^2\right)dxdt+\int_\Omega \left(|\nabla\tau(t)|^2+|\nabla\alpha(t)|^2+|\nabla\beta(t)|^2\right)dx\\ \leq C\left(\|\tau_0\|^2_{W_2^2(\Omega)}+\|\alpha_0\|^2_{W_2^2(\Omega)}+\|\beta_0\|^2_{W_2^2(\Omega)}+\|f\|^2_{L^2(Q)}\right)\\ +C\int_0^t\|v\|^4_{L^4(\Omega)}\left(\|\nabla\tau\|^2_{L^2(\Omega)}+\|\nabla\alpha\|^2_{L^2(\Omega)}+\|\nabla\beta\|^2_{L^2(\Omega)}\right)dt\\ +C\int_0^t\int_\Omega\left(\tau^2+\alpha^2+\beta^2\right)dxdt\\ +\epsilon\int_0^t\int_\Omega\left(\tau_t^2+\alpha_t^2+\beta_t^2+\Delta\tau^2+\Delta\alpha^2+\Delta\beta^2\right)dxdt,\end{aligned}\tag{3.11}$$

for all $0\leq t\leq T$.

Finally, by multiplying the first, second, and third equations of problem (3.6) respectively by $-\Delta\tau$, $-\Delta\alpha$, and $-\Delta\beta$ and proceeding similarly as was done for the

previous cases, we obtain
(3.12)

$$\begin{aligned}\int_\Omega \left(|\nabla\tau(t)|^2+|\nabla\alpha(t)|^2+|\nabla\beta(t)|^2\right)dx+\int_0^t\int_\Omega\left(|\Delta\tau|^2+|\Delta\alpha|^2+|\Delta\beta|^2\right)dxdt\\ \leq C\left(\|\tau_0\|^2_{W_2^2(\Omega)}+\|\alpha_0\|^2_{W_2^2(\Omega)}+\|\beta_0\|^2_{W_2^2(\Omega)}+\|f\|^2_{L^2(Q)}\right)\\ +C\int_0^t\|v\|^4_{L^4(\Omega)}\left(\|\nabla\tau\|^2_{L^2(\Omega)}+\|\nabla\alpha\|^2_{L^2(\Omega)}+\|\nabla\beta\|^2_{L^2(\Omega)}\right)dt\\ +C\int_0^t\int_\Omega\left(\tau^2+\alpha^2+\beta^2\right)dxdt\\ +\epsilon\int_0^t\int_\Omega\left(\tau_t^2+\alpha_t^2+\beta_t^2+\Delta\tau^2+\Delta\alpha^2+\Delta\beta^2\right)dxdt,\end{aligned}$$

for all $0\leq t\leq T$.

By adding inequalities (3.10) until (3.12) and taking $\epsilon>0$ small enough, we have

$$\begin{aligned}\int_\Omega\left[\tau^2(t)+\alpha^2(t)+\beta^2(t)\right]dx+\int_0^t\int_\Omega\left(\tau_t^2+\alpha_t^2+\beta_t^2\right)dxdt\\ +\int_0^t\int_\Omega\left(|\nabla\tau|^2+|\nabla\alpha|^2+|\nabla\beta|^2+|\Delta\tau|^2+|\Delta\alpha|^2+|\Delta\beta|^2\right)dxdt\\ \leq C\left(\|\tau_0\|^2_{W_2^2(\Omega)}+\|\alpha_0\|^2_{W_2^2(\Omega)}+\|\beta_0\|^2_{W_2^2(\Omega)}+\|f\|^2_{L^2(Q)}\right)\\ +C\int_0^t\|v\|^4_{L^4(\Omega)}\left(\|\nabla\tau\|^2_{L^2(\Omega)}+\|\nabla\alpha\|^2_{L^2(\Omega)}+\|\nabla\beta\|^2_{L^2(\Omega)}\right)dt\\ +C\int_0^t\int_\Omega\left(\tau^2+\alpha^2+\beta^2\right)dxdt,\end{aligned}$$

for all $0\leq t\leq T$.

By using Gronwall's lemma, we obtain

$$\begin{aligned}\int_\Omega\left[\tau^2(t)+\alpha^2(t)+\beta^2(t)\right]dx+\int_0^t\int_\Omega\left(\tau_t^2+\alpha_t^2+\beta_t^2\right)dxdt\\ +\int_0^t\int_\Omega\left(|\nabla\tau|^2+|\nabla\alpha|^2+|\nabla\beta|^2+|\Delta\tau|^2+|\Delta\alpha|^2+|\Delta\beta|^2\right)dxdt\\ \leq C\left(\|\tau_0\|^2_{W_2^2(\Omega)}+\|\alpha_0\|^2_{W_2^2(\Omega)}+\|\beta_0\|^2_{W_2^2(\Omega)}+\|f\|^2_{L^2(Q)}\right),\end{aligned}$$

for all $0\leq t\leq T$, where C depends on T, Ω and $\|v\|_{L^4(Q)}$.

Then,

$$\begin{aligned}\|\tau\|_{W_2^{2,1}(Q)}+\|\alpha\|_{W_2^{2,1}(Q)}+\|\beta\|_{W_2^{2,1}(Q)}\\ \leq C\left(\|\tau_0\|_{W_2^2(\Omega)}+\|\alpha_0\|_{W_2^2(\Omega)}+\|\beta_0\|_{W_2^2(\Omega)}+\|f\|_{L^2(Q)}\right):=K_1.\end{aligned}$$

Since $W_2^{2,1}(Q)\subset L^q(Q)$ with continuous embedding, for any $2\leq p<\infty$, we have τ, α, $\beta\in L^q(Q)$ and

$$\begin{aligned}\|\tau\|_{L^q(Q)}+\|\alpha\|_{L^q(Q)}+\|\beta\|_{L^q(Q)}\\ \leq C\left(\|\tau\|_{W_2^{2,1}(Q)}+\|\alpha\|_{W_2^{2,1}(Q)}+\|\beta\|_{W_2^{2,1}(Q)}\right)\leq C.K_1.\end{aligned}$$

Next, by observing that τ, α, $\beta \in L^q(Q)$ and $0 \leq \alpha$, $\beta \leq 1$, we conclude that the second member of the second equation of problem (3.6) belongs to $L^q(Q)$. Since $\alpha_0 \in W_2^2(\Omega) \subset W_q^{2-2/q}(\Omega)$, by applying Proposition 2.3 we then obtain $\alpha \in W_q^{2,1}(Q) \subset L^\infty(Q)$ and

$$\begin{aligned}\|\alpha\|_{L^\infty(Q)} &\leq C\|\alpha\|_{W_q^{2,1}(Q)} \\ &\leq C\left(\|\lambda\widetilde{g}_1(\tau,\alpha,\beta)+\lambda\widetilde{g}_3(\tau,\alpha,\beta)\|_{L^q(Q)}+\|\alpha_0\|_{W_2^2(\Omega)}\right) \\ &\leq C\left(\|\alpha\|_{L^q(Q)}+\|\tau\|_{L^q(Q)}+\|\alpha_0\|_{W_2^2(\Omega)}\right) \leq C.K_1.\end{aligned}$$

Similarly, we obtain $\beta \in W_q^{2,1}(Q) \subset L^\infty(Q)$ and

$$\|\beta\|_{L^\infty(Q)} \leq C\|\beta\|_{W_q^{2,1}(Q)} \leq C.K_1.$$

Finally, since α, $\beta \in W_q^{2,1}(Q)$, it follows that α_t, $\beta_t \in L^q(Q)$ and the second member of the first equation of problem (3.6) belongs to $L^q(Q)$. By applying Proposition 2.3, we obtain $\tau \in W_q^{2,1}(Q) \subset L^\infty(Q)$ and

$$\|\tau\|_{L^\infty(Q)} \leq C\|\tau\|_{W_q^{2,1}(Q)} \leq C.K_1 + \|f\|_{L^q(Q)}.$$

Thus, by using the previous estimates we get

(3.13)
$$\begin{aligned}\|(\tau,\alpha,\beta)\|_{\mathcal{B}} &\leq C\left(\|\tau\|_{L^\infty(Q)}+\|\alpha\|_{L^\infty(Q)}+\|\beta\|_{L^\infty(Q)}\right) \\ &\leq C\left(\|\tau\|_{W_q^{2,1}(Q)}+\|\alpha\|_{W_q^{2,1}(Q)}+\|\beta\|_{W_q^{2,1}(Q)}\right) \\ &\leq C\left(\|\tau_0\|_{W_2^2(\Omega)}+\|\alpha_0\|_{W_2^2(\Omega)}+\|\beta_0\|_{W_2^2(\Omega)}+\|f\|_{L^q(Q)}\right) := K,\end{aligned}$$

where K is the constant referred to in the statement of Lemma 3.7, and the positive constant C does not depend on $\lambda \in [0,1]$, and in fact only depends on Ω, T, $||v||_{L^4(Q)}$ and the constants of problem (3.1). □

Remark 3.8. By using the fact that $0 \leq \alpha$, $\beta \leq 1$, one could derive simpler alternative forms of several of the previous estimates. Following these alternatives, however, one would obtain, instead of (2.2), a final estimate with an extra additive term depending on the given function v in problem (1.1). To obtain the stated form of (2.2), that is, an estimate with just a constant depending on the problem parameters multiplying certain norms of the initial conditions and of the forcing term f, we must proceed as we did in this article.

Remark 3.9. For use in a forthcoming paper [10], in the following we present an estimate derived as suggested above, that is, by using the fact that $0 \leq \alpha$, $\beta \leq 1$.

It follows from the proof of Lemma 3.7 that, under the same hypothesis of Theorem 2.4, the solution $(\tau,\alpha,\beta,\gamma)$ of problem (1.1) satisfies

$$\begin{aligned}&\int_\Omega \left(\tau^2(t)+\alpha^2+\beta^2+\gamma^2\right)dx + \int_0^t\int_\Omega \left(|\nabla\tau|^2+|\nabla\alpha|^2+|\nabla\beta|^2+|\nabla\gamma|^2\right)dxdt \\ &\quad \leq C\left(\|\tau_0\|^2_{L^2(\Omega)}+\|\alpha_0\|^2_{L^2(\Omega)}+\|\beta_0\|^2_{L^2(\Omega)}+\|f\|^2_{L^2(Q)}+\|v\|^2_{L^2(Q)}\right),\end{aligned}$$

for all $0 \leq t \leq T$, where the constant C depends on Ω, T and the constants of problem (1.1).

Proof. Indeed, let us consider inequality (3.9). By using Hölder's and Young's inequalities and since $0 \leq \alpha, \beta \leq 1$, by Lemma 3.2, we have

$$\int_\Omega |v.\nabla\tau|\alpha \; dx \leq \|v\|_{L^2(\Omega)}\|\nabla\tau\|_{L^2(\Omega)}\|\alpha\|_{L^\infty(\Omega)} \leq C_\varepsilon\|v\|^2_{L^2(\Omega)} + \varepsilon\|\nabla\tau\|^2_{L^2(\Omega)},$$

and in the same way,

$$\int_\Omega |v.\nabla\tau|\beta \; dx \leq C_\varepsilon\|v\|^2_{L^2(\Omega)} + \varepsilon\|\nabla\tau\|^2_{L^2(\Omega)}.$$

By choosing $\varepsilon > 0$ small enough and replacing the last two inequalities in the previous one, we obtain, for all $0 \leq t \leq T$,

$$\begin{aligned}\int_\Omega \left(\tau^2(t) + \alpha^2(t) + \beta^2(t)\right) dx + \int_0^t \int_\Omega \left(|\nabla\tau|^2 + |\nabla\alpha|^2 + |\nabla\beta|^2\right) dxdt \\ \leq C\left(\|\tau_0\|^2_{L^2(\Omega)} + \|\alpha_0\|^2_{L^2(\Omega)} + \|\beta_0\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(Q)}\right) \\ + C\int_0^t \int_\Omega |v|^2 dxdt + C\int_0^t \int_\Omega \left(\tau^2 + \alpha^2 + \beta^2\right) dxdt.\end{aligned}$$

Gronwall's lemma implies that, for all $0 \leq t \leq T$ and C depending on T and Ω,

$$\begin{aligned}\int_\Omega \left(\tau^2(t) + \alpha^2(t) + \beta^2(t)\right) dx + \int_0^t \int_\Omega \left(|\nabla\tau|^2 + |\nabla\alpha|^2 + |\nabla\beta|^2\right) dxdt \\ \leq C\left(\|\tau_0\|^2_{L^2(\Omega)} + \|\alpha_0\|^2_{L^2(\Omega)} + \|\beta_0\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(Q)} + \|v\|^2_{L^2(Q)}\right).\end{aligned}$$

By proceeding similarly in the equation satisfied by $\gamma = 1 - \alpha - \beta$, with the help of the last estimate, we obtain

$$\begin{aligned}&\int_\Omega \gamma^2(t) \; dx + \int_0^t \int_\Omega |\nabla\gamma|^2 dxdt \\ &\quad \leq C\left(\|\tau_0\|^2_{L^2(\Omega)} + \|\alpha_0\|^2_{L^2(\Omega)} + \|\beta_0\|^2_{L^2(\Omega)} + \|\gamma_0\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(Q)} + \|v\|^2_{L^2(Q)}\right),\end{aligned}$$

for all $0 \leq t \leq T$, where C depends on T and Ω. □

Proof of Proposition 3.1. By the previous lemmas we can apply the Leray-Schauder fixed point theorem to the family of operators $\mathcal{T}_\lambda$. We conclude that there exists $(\tau, \alpha, \beta) \in \left[W^{2,1}_q(Q)\right]^3 \subset \mathcal{B}$, which is a fixed point of $\mathcal{T}_1$ that also satisfies (3.13); in other words, there exists a solution of problem (3.1) satisfying estimate (3.4). By standard arguments, we prove that the solution $(\tau, \alpha, \beta) \in \mathcal{B} \cap [W^{2,1}_2(Q)]^3$ of problem (3.1) is unique. Since $q > 2$, it follows that the solution $(\tau, \alpha, \beta) \in \left[W^{2,1}_q(Q)\right]^3 \subset \mathcal{B}$ of problem (3.1) is unique. □

Finally, Theorem 2.4 follows from Proposition 3.1.

Proof of Theorem 2.4. Consider problem (1.1). By Proposition 3.1 there exists a unique solution $(\tau, \alpha, \beta) \in \left[W^{2,1}_q(Q)\right]^3$ of the corresponding problem (3.1). Let us consider $(\tau, \alpha, \beta, \gamma) \in \left[W^{2,1}_q(Q)\right]^4$, where (τ, α, β) is the solution of problem (3.1) and $\gamma = 1-\alpha-\beta$. So we have that $(\tau, \alpha, \beta, \gamma)$ is the unique solution of problem (1.1) and satisfies $\alpha + \beta + \gamma = 1$, $\alpha \geq 0$, $\beta \geq 0$ and estimate (3.4). By the proof of

Lemma 3.2 we also have that $\gamma \geq 0$. In addition, by considering the problem satisfied by γ, i.e.,

$$\begin{aligned} &\gamma_t - k\Delta\gamma + v.\nabla\gamma = -g_1 - g_2 && \text{in } Q, \\ &\partial\gamma/\partial n = 0 && \text{on } \partial\Omega \times (0,T), \\ &\gamma = \gamma_0 && \text{in } \Omega \times \{t = 0\}, \end{aligned}$$

where g_1 and g_2 are given by (1.2), and proceeding similarly as was done to obtain the estimates for α and β in the proof of Lemma 3.7, we obtain that

$$\|\gamma\|_{W_q^{2,1}(Q)} \leq C(\|\tau_0\|_{W_2^2(\Omega)} + \|\alpha_0\|_{W_2^2(\Omega)} + \|\beta_0\|_{W_2^2(\Omega)} + \|\gamma_0\|_{W_2^2(\Omega)} + \|f\|_{L^2(Q)}),$$

where the constant C depends on Ω, T, v and the constants of problem (1.1). Thus (2.2) holds. □

References

1. R. Adams, *Sobolev Spaces*. Academic Press, 1975. MR0450957 (56:9247)
2. J. L. Boldrini, B. M. C. Caretta, E. Fernández-Cara, *Analysis of a two-phase field model for the solidification of an alloy*, J. Math. Anal. Appl. **357** (2009), 25–44. MR2526803 (2010e:35138)
3. J.L. Boldrini, G. Planas, *Weak solutions of a phase-field model for phase change of an alloy with thermal properties*, Math. Meth. in Appl. Sci. **25**, no. 14 (2002), 1177–1193. MR1925439 (2003i:35279)
4. J.L. Boldrini, G. Planas, *A tridimensional phase-field model with convection for phase-change of an alloy*, Disc. and Cont. Dyn. Syst. **13**, no. 2 (2005), 429–450. MR2152398 (2006d:35105)
5. J.L. Boldrini, C.L.D. Vaz, *Existence and regularity of solutions of a phase field model for solidification with convection of pure materials in two dimensions*, Elect. J. Diff. Equations **2003**, no. 109 (2003), 1–25. MR2011582 (2005b:35302)
6. G. Caginalp, J. Jones, *A derivation and analysis of phase field models of thermal alloys*, Annal. Phys. **237** (1995), 66–107.
7. G. Caginalp, *Phase field computations of single-needle crystals, crystal growth and motion by mean curvature*, SIAM J. Sci. Comput. **15 (1)** (1994), 106–126. MR1257157 (94k:65122)
8. G. Caginalp, *Stefan and Hele-Shaw type models as asymptotic limits of the phase-field equations*, Phys. Rev. A **39 (11)** (1989), 5887–5896. MR998924 (90c:80004)
9. G. Caginalp, *An analysis of a phase field model of a free boundary*, Arch. Rat. Mech. Anal. **92** (1986), 205–245. MR816623 (87c:80011)
10. B.M.R. Calsavara, J.L. Boldrini, *On a system coupling the two-crystallizations Allen-Cahn equations and a singular Navier-Stokes system*, in preparation (2011).
11. A. Friedman, *Partial Differential Equations of Parabolic Type*. Prentice Hall, 1964. MR0181836 (31:6062)
12. K. Hoffman, L. Jiang, *Optimal control of a phase field model for solidification*, Numer. Funct. Anal. and Optimiz. **13** (1992), 11–27. MR1163315 (93h:35218)
13. P. Krejči, J. Sprekels, U. Stefanelli, *Phase-field models with hysteresis in one-dimensional thermoviscoplasticity*, SIAM J. Math. Anal. **34**, no. 2 (2002), 409–434. MR1951781 (2005d:74031)
14. O. A. Ladyženskaja, V. A. Solonnikov, N. N. Ural′ceva, *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs, vol. 23, Amer. Math. Soc., Providence, RI, 1968. MR0241822 (39:3159b)
15. J. L. Lions, *Control of Singular Distributed Systems*, Gauthier-Villars, 1983. MR0712486 (85c:93002)
16. V. P. Mikhaylov, *Partial Differential Equations*. Mir Publishers, Moscow, 1978. MR498162 (80b:35002)
17. G. Planas, J. L. Boldrini, *A bidimensional phase-field model with convection for change phase of an alloy*, J. Math. Anal. and Appl. **303** (2005), 669–687. MR2122569 (2005k:35407)
18. G. Planas, *Existence of solutions to a phase-field model with phase-dependent heat absorption*, Elect. J. Diff. Equations **2007**, no. 28 (2007), 1–12. MR2299582 (2007k:35240)
19. P. Sprekels, S. Zheng, *Global existence and asymptotic behaviour for a nonlocal phase-field model for non-isothermal phase transitions*, J. Math. Anal. **279** (2003), 97–110. MR1970493 (2004c:45015)

20. J. P. Simon, *Compact sets in the space* $L^p(0,T;B)$, Annali di Matematica Pura ed Applicata, serie quarta, tomo **146** (1987), 65–96. MR916688 (89c:46055)
21. I. Steinbach, F. Pezzolla, B. Nestler, M. Seesselberg, R. Prieler, G.J. Schimitz, J.L.L. Rezende, *A phase field concept for multiphase systems*. Physica D **94** (1996), 135–147.
22. I. Steinbach, F. Pezzolla, *A generalized field method for multiphase transformations using interface fields*, Physica D **134** (1999), 385–393. MR1725913 (2000h:80008)
23. C.L.D. Vaz, J.L. Boldrini, *A semidiscretization scheme for a phase-field type model for solidification*, Portugaliae Mathematica **63**, no. 3 (2006), 261–292. MR2254930 (2007e:35303)

University of Campinas – UNICAMP, School of Applied Sciences, Pedro Zaccaria Street, 1300, CEP 13484-350, Limeira, SP, Brazil
E-mail address: biancamrc@yahoo.com

University of Campinas – UNICAMP, IMECC, Sergio Buarque de Holanda Street, 651, CEP 13083-859, Campinas, SP, Brazil
E-mail address: bodrini@ime.unicamp.br