TORIC HIRZEBRUCH-RIEMANN-ROCH
VIA ISHIDA’S THEOREM ON THE TODD GENUS

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Abstract. We give a simple proof of the Hirzebruch-Riemann-Roch theorem for smooth complete toric varieties, based on Ishida’s result that the Todd genus of a smooth complete toric variety is one.

1. Introduction

The Hirzebruch-Riemann-Roch theorem relates the Euler characteristic of a coherent sheaf $F$ on a smooth complete $n$-dimensional variety $X$ to intersection theory via the formula

$$\chi(F) = \int ch(F)Td(T_X).$$

In [2], Brion and Vergne prove an equivariant Hirzebruch-Riemann-Roch theorem for complete simplicial toric varieties. If the toric variety is actually smooth, it is possible to derive (1) from their result. In this note, we give a simple direct proof of (1) when $X$ is a smooth complete toric variety. Such a variety is determined by a smooth complete rational polyhedral fan $\Sigma \subseteq \mathbb{R}^n$, where $N \simeq \mathbb{Z}^n$ is a lattice; we write $X$ for the associated toric variety $X_{\Sigma}$. We will make use of the following standard facts about toric varieties. First,

$$Td(X_{\Sigma}) = \prod_{\rho \in \Sigma(1)} \frac{D_\rho}{1 - e^{-D_\rho}},$$

where $\Sigma(k)$ denotes the set of $k$-dimensional faces of $\Sigma$. For $\tau \in \Sigma(k)$ there is an associated torus invariant orbit $O(\tau)$, and we use $V(\tau)$ to denote the orbit closure $\overline{O(\tau)}$, which has dimension $n - k$. A key fact is that (see [4], Proposition 3.2.7)

$$V(\tau) = \overline{O(\tau)} \simeq X_{\text{Star}(\tau)}.$$

Since $\Sigma$ is smooth, all orbits are also smooth, and if $\rho_i, \rho_j$ are distinct elements of $\Sigma(1)$, then (see [4], Lemma 12.5.7)

$$[D_{\rho_i}, V(\rho_j)] = \begin{cases} V(\tau) & \tau = \rho_i + \rho_j \in \Sigma \\ 0 & \rho_i, \rho_j \text{ are not both in any cone in } \Sigma. \end{cases}$$
The final ingredient we need is a result of Ishida: building on work of Brion \cite{5}, in \cite{1} Ishida shows that (1) holds for the structure sheaf of a smooth complete toric variety $X$:

\begin{equation}
1 = \int Td(T_X) = \left[ \prod_{\rho \in \Sigma(1)} \frac{D_{\rho}}{1 - e^{-D_{\rho}}} \right]_n.
\end{equation}

2. The proof

For a smooth complete toric variety, any coherent sheaf has a resolution by line bundles \cite{3}, so it suffices to consider the case $F = \mathcal{O}_X(D)$. Let $X = X_\Sigma$ and recall that Pic($X$) is generated by the classes of the divisors $D_{\rho}$, $\rho \in \Sigma(1)$. We will show that if (1) holds for a divisor $D$, then it also holds for $D + D_{\rho}$ and $D - D_{\rho}$, for any $\rho \in \Sigma(1)$. We begin with the case $D - D_{\rho}$ and induct on the dimension of $X$.

A smooth complete toric variety of dimension one is simply $\mathbb{P}^1$, so the base case holds by Riemann-Roch for curves. Suppose the theorem holds for all smooth complete fans of dimension $< n$ and let $\Sigma$ be a smooth complete fan of dimension $n$. When $D = 0$ the result holds by Ishida’s theorem. Let $\rho \in \Sigma(1)$ and partition the rays of $\Sigma$ as

\[ \Sigma(1) = \rho \cup \Sigma'(1) \cup \Sigma''(1), \]

where the rays in $\Sigma'(1)$ are in one-to-one correspondence with the rays of the fan $\text{Star}(\rho)$. Let $X' = X_{\text{Star}(\rho)} \simeq V(\rho)$. Tensoring the standard exact sequence

\[ 0 \rightarrow \mathcal{O}_X(-D_{\rho}) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{X'} \rightarrow 0 \]

with $\mathcal{O}_X(D)$ yields the sequence

\[ 0 \rightarrow \mathcal{O}_X(D - D_{\rho}) \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{O}_{X'}(D) \rightarrow 0. \]

From the additivity of the Euler characteristic, we have

\[ \chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X(D - D_{\rho})) = \chi(\mathcal{O}_{X'}(D)). \]

Our hypotheses imply that

\[ \int_{X'} e^D Td(T_{X'}) = \chi(\mathcal{O}_{X'}(D)), \]

\[ \int_X e^D Td(T_X) = \chi(\mathcal{O}_X(D)), \]

so it suffices to show that

\begin{equation}
\int_{X'} \text{ch}(D) Td(T_{X'}) = \int_X (e^D - e^{D - D_{\rho}}) Td(T_X)
\end{equation}

\[ = \int_X e^D \left( \frac{1 - e^{-D_{\rho}}}{D_{\rho}} \right) D_{\rho} Td(T_X). \]

Break the Todd class of $X$ into two parts:

\[ Td(T_X) = \prod_{\gamma \in \Sigma'(1) \cup \rho} \frac{D_\gamma}{1 - e^{-D_\gamma}} \cdot \prod_{\gamma \in \Sigma''(1)} \frac{D_\gamma}{1 - e^{-D_\gamma}}. \]
In (5), the term $\frac{1 - e^{-D_{\rho}}}{D_{\rho}}$ cancels the corresponding term in $Td(T_X)$, so that

$$\int_X e^D \left( 1 - e^{-D_{\rho}} \right) D_{\rho} Td(T_X) = \int_X e^D D_{\rho} \prod_{\gamma \in \Sigma'(1) \cup \Sigma''(1)} \frac{D_{\gamma}}{1 - e^{-D_{\gamma}}},$$

(6)

The second equality follows since $D_{\rho} \cdot D_{\gamma} = 0$ if $\gamma \in \Sigma''(1)$. By smoothness, all intersections are either zero or one, and thus

$$\int_X e^D D_{\rho} \prod_{\gamma \in \Sigma'(1)} \frac{D_{\gamma}}{1 - e^{-D_{\gamma}}} = \left[ e^D \right]_{V(\rho)} \prod_{\gamma \in \Sigma'(1)} \frac{D_{\gamma}}{1 - e^{-D_{\gamma}}}.$$

$$= \left[ e^D \right]_{V(\rho)} \prod_{\gamma \in \Sigma'(1)} \frac{D_{\gamma}}{1 - e^{-D_{\gamma}}} \cdot$$

$$= \int_{X'} e^D \cdot Td(T_{X'}).$$

This proves the result for $D - D_{\rho}$. For $D + D_{\rho}$, the result follows using the substitution $e^{D_{\rho}} - 1 = e^{D_{\rho}} (1 - e^{-D_{\rho}})$.

**Question.** Ishida’s proof (4) is not easy. Does there exist a simple proof of (4)?

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**References**


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