POSITIVE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS WITH SMALL PERTURBATIONS

RYUJI KAJIKIYA

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Abstract. In this paper, we study the semilinear elliptic equation with a small perturbation. We assume the main term in the equation to have a mountain pass structure but do not suppose any condition for the perturbation term. Then we prove the existence of a positive solution. Moreover, we prove the existence of at least two positive solutions if the perturbation term is nonnegative.

1. Introduction and main results

We prove the existence of positive solutions for the semilinear elliptic equation

\begin{align*}
-\Delta u &= f(x, u) + \lambda g(x, u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \), \( N \geq 1 \) and \( f(x, u) \), \( g(x, u) \) are continuous on \( \overline{\Omega} \times [0, \infty) \) and \( \lambda \) is a real parameter whose absolute value is small. We assume a condition on \( f(x, u) \) such that \((1.1), (1.2)\) with \( \lambda = 0 \) has a mountain pass structure, and therefore it has a positive solution when \( \lambda = 0 \). The most typical nonlinear term is \( f(x, u) = a(x)u^p \) or \( f(x, u) = a(x)u^p + b(x)u^q \), where \( a, b \in C(\Omega) \) and \( a(x) \) or \( b(x) \) may change its sign. The purpose of this paper is to prove the existence of a positive solution for \( |\lambda| \) small enough under the mountain pass assumption on \( f(x, u) \) only without any conditions on \( g(x, u) \). The nonlinear term \( f(x, u) = a(x)u^p \) was studied by Afrouzi and Brown [1], Alama and Tarantello [2], Brown and Zhang, [4], Li and Wang [6] and the author [5]. However the assumptions in this paper are more general than those of the papers above. We emphasize that our theorem does not need any assumptions on \( g(x, u) \). We assume the conditions below.

(f1) There exist positive constants \( p, C \) such that \( 1 < p < \infty \) if \( N = 1, 2 \) and \( 1 < p < (N + 2)/(N - 2) \) if \( N \geq 3 \) and

\[ |f(x, s)| \leq C(s^p + 1) \quad \text{for } s \geq 0, \quad x \in \Omega. \]

(f2) There exist constants \( \alpha > 2, \theta \in [0, 2), C > 0 \) such that

\[ \alpha F(x, s) - sf(x, s) \leq C|s|^\theta + C \quad \text{for } s \geq 0, \quad x \in \Omega, \]

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Let $g$ and (f5) ensure that a mountain pass solution is strictly positive. For any Assumption (f3) holds if
(f1)–(f5) hold. Then the following assertions hold.

(i) There exists a $\lambda_0 > 0$ such that (1.1), (1.2) have a positive solution $u_\lambda$ when $|\lambda| \leq \lambda_0$. Furthermore, for any sequence $\lambda_j$ converging to zero, along a subsequence $u_{\lambda_j}$, converges to $u_0$ in $W^{2,q}(\Omega)$ for all $q \in [1, \infty)$, where $u_0$ is a mountain pass solution of (1.1), (1.2) with $\lambda = 0$ and where $W^{2,q}(\Omega)$ denotes the Sobolev space.

(ii) If $g(x,0) \geq 0$, $\neq 0$ in $\Omega$, then (1.1), (1.2) have another nonnegative solution $v_\lambda$ for $\lambda > 0$ small enough such that $0 \leq v_\lambda(x) < u_\lambda(x)$ and $v_\lambda \to 0$ in $W^{2,q}(\Omega)$ as $\lambda \to 0$ for all $q \in [1, \infty)$. Moreover, if

\begin{equation}
\liminf_{s \to 0^+} \left( \min_{x \in \Omega} (g(x,s) - g(x,0))/s \right) > -\infty,
\end{equation}

then each $v_\lambda$ is strictly positive.

We give sufficient conditions for (f3)–(f5). Assumptions (f4) and (f5) are fulfilled if

\begin{equation}
\lim_{s \to 0^+} f(x,s)/s = 0 \quad \text{uniformly on } \Omega.
\end{equation}

Assumption (f3) holds if $f(x,s)$ is superlinear at $s = \infty$ in a small neighborhood of $x_0$, i.e.,

\begin{equation}
\lim_{s \to \infty} \left( \min_{|x-x_0| \leq \delta_0} f(x,s)/s \right) = \infty.
\end{equation}

There are many examples of $f(x,s)$ satisfying our assumptions. An easy example of the sign-changing nonlinear term is $f(x,s) = a(x)s^p + b(x)s^q$, where $a, b \in C(\overline{\Omega})$, $1 < q < p$ if $N = 1, 2$ and $1 < q < p < (N + 2)/(N - 2)$ if $N \geq 3$. The function $f(x,s)$ satisfies (f1)–(f5) if either (i) or (ii) below holds:

(i) $a(x)$ may change its sign, but $a(x_0) > 0$ at some $x_0 \in \Omega$ and $b(x) \leq 0$ in $\Omega$.

(ii) $a(x) \geq 0$, $\neq 0$ in $\Omega$ and $b(x)$ is any function.
Indeed, it is easy to verify (f1), (1.2) and (1.3). Let us check (f2). In Case (i), we choose $\alpha = p + 1$ so that
\[
(p + 1)F(x, s) - sf(x, s) = \frac{p - q}{q + 1} b(x) s^{q+1} \leq 0.
\]
In Case (ii), we choose $\alpha = q + 1$ so that
\[
(q + 1)F(x, s) - sf(x, s) = \frac{q - p}{p + 1} a(x) s^{p+1} \leq 0.
\]
Thus (f2) holds.

2. Proof of the Theorem

We shall prove Theorem 1.1. Our approach is based on the mountain pass lemma and the maximum principle. We always assume (f1)–(f5). Assumptions (f4) and (f5) imply $f(x, 0) = 0$. Throughout the paper, we put $f(x, s) = 0$ for $s < 0$, and hence $f(x, s)$ is defined on $\overline{\Omega} \times \mathbb{R}$ and continuous. Moreover, (f4) and (f5) are still valid as $s \to 0$ instead of $s \to 0^+$ and (f2) holds for all $s \in \mathbb{R}$.

We call $u$ a solution of (1.1), (1.2) if it belongs to $H^1_0(\Omega) \cap L^\infty(\Omega)$ and satisfies (1.1) in the distribution sense. By the bootstrap argument with the elliptic regularity theorem, $u$ belongs to $W^{2,q}(\Omega)$ for all $q \in [1, \infty)$ and satisfies (1.1) a.e. in $\Omega$. Especially, $u$ lies in $C^1(\Omega)$.

Lemma 2.1. Any nontrivial solution $u$ of (1.1), (1.2) with $\lambda = 0$ is strictly positive and $\partial u/\partial \nu < 0$ on $\partial \Omega$. Here $\partial/\partial \nu$ denotes the outward normal derivative.

Proof. Let $u$ be a nontrivial solution of (1.1), (1.2) with $\lambda = 0$. Put
\[
D := \{ x \in \Omega : u(x) < 0 \}.
\]
Assume that $D \neq \emptyset$. By the extension of $f(x, s)$ on $s \leq 0$,
\[
-\Delta u = f(x, u) = 0 \text{ in } D, \quad u = 0 \text{ on } \partial D.
\]
Thus $u \equiv 0$ in $D$, a contradiction. Therefore $D$ must be empty; i.e., $u \geq 0$ in $\Omega$. Put $A := \|u\|_\infty$. By (f5), there exists a $C > 0$ such that $f(x, s) \geq -Cs$ for $0 \leq s \leq A$ and $x \in \Omega$. This inequality gives us
\[
(C - \Delta)u = Cu + f(x, u) \geq 0 \text{ in } \Omega.
\]
By the Hopf maximum principle, $u$ is strictly positive and $\partial u/\partial \nu < 0$ on $\partial \Omega$. □

For (1.1) with $\lambda = 0$, we define the Lagrangian functional $I_0(u)$ by
\[
I_0(u) := \int_\Omega \left( \frac{1}{2} |\nabla u|^2 - F(x, u) \right) dx,
\]
where $F(x, u)$ is defined in (f2). In what follows, $\|\cdot\|_p$ denotes the $L^p(\Omega)$ norm. $H^1_0(\Omega)$ stands for the usual Sobolev space equipped with the norm $\|u\|_{H^1_0(\Omega)} := \|\nabla u\|_2$. Because of (f1), $I_0$ is well defined in $H^1_0(\Omega)$ and becomes a $C^1$ functional.

Lemma 2.2. $I_0$ satisfies the Palais-Smale condition.

Proof. Let $u_n$ be any sequence in $H^1_0(\Omega)$ such that $I_0(u_n)$ is bounded and $\|I_0'(u_n)\|$ converges to zero. From an easy calculation, we see that
\[
I_0'(u)u = \int_\Omega (|\nabla u|^2 - uf(x, u)) \, dx,
\]
which shows that
\[ \alpha I_0(u_n) - I'_0(u_n)u_n = \frac{\alpha - 2}{2} \| \nabla u_n \|_2^2 - \int_{\Omega} (\alpha F(x, u_n) - u_n f(x, u_n)) \, dx. \] (2.1)

Hereafter we assume \( \theta \geq 1 \) in (f2) because in case \( \theta < 1 \) we replace \( \theta \) by \( 1 \) and \( C \) by a larger constant. Then the norm \( \| \cdot \|_\theta \) makes sense. Since \( |I_0(u_n)| \) and \( \| I'_0(u_n) \| \) are bounded, we use (f2) to get a constant \( C > 0 \) such that
\[ \frac{\alpha - 2}{2} \| \nabla u_n \|_2^2 = \alpha I_0(u_n) - I'_0(u_n)u_n + \int_{\Omega} (\alpha F(x, u_n) - u_n f(x, u_n)) \, dx \]
\[ \leq C + C \| \nabla u_n \|_2 + C \| u_n \|_\theta \]
\[ \leq C + C \| \nabla u_n \|_2 + C' \| \nabla u_n \|_\theta, \]
where we have used the Sobolev embedding. Since \( \theta < 2 \), \( \| \nabla u_n \|_2 \) is bounded. Then a subsequence of \( u_n \) weakly converges in \( H^1_0(\Omega) \). This convergence becomes a strong one, which can be proved in the standard method. See [3, 7, 8, 9] for the details. The proof is complete.

**Lemma 2.3.** \( I_0 \) has a mountain pass geometry; i.e., there exist \( u_1 \in H^1_0(\Omega) \) and constants \( r, \rho > 0 \) such that \( I_0(u_1) < 0, \| \nabla u_1 \|_2 > r \) and
\[ I_0(u) \geq \rho \quad \text{when} \quad \| \nabla u \|_2 = r. \] (2.2)

**Proof.** Recall that (f4) is still valid as \( s \to 0 \) instead of \( s \to 0^+ \). Then we have \( s_0 > 0 \) and \( \mu \in (0, \mu_1) \) such that
\[ f(x, s)/s < \mu \quad \text{for} \quad |s| < s_0, \]
which implies that
\[ F(x, s) \leq (\mu/2)s^2 \quad \text{for} \quad |s| \leq s_0. \]

This inequality with (f1) shows that
\[ F(x, s) \leq (\mu/2)s^2 + C|s|^{p+1} \quad \text{for} \quad s \in \mathbb{R}, \]
with some \( C > 0 \). Since \( \mu_1 \) is the first eigenvalue of \(-\Delta\), it follows that \( \| \nabla u \|_2^2 \geq \mu_1 \| u \|_2^2 \) for \( u \in H^1_0(\Omega) \). Then \( I_0 \) is estimated as
\[ I_0(u) \geq \frac{1}{2} \| \nabla u \|_2^2 - \frac{\mu}{2} \| u \|_2^2 - C\| u \|_{p+1}^{p+1} \geq \frac{\mu_1 - \mu}{2\mu_1} \| \nabla u \|_2^2 - C' \| \nabla u \|_2^{p+1}. \]

This shows the existence of \( r \) and \( \rho \) satisfying (2.2). Let \( x_0, \delta_0 \) be as in (f3). Let \( \phi \) be a function such that \( \phi \in C^1_0(\Omega), \phi \geq 0, \phi \not\equiv 0 \) and the support of \( \phi \) is in \( B(x_0, \delta_0) \). Here \( B(x_0, \delta_0) \) is a ball centered at \( x_0 \) with radius \( \delta_0 \). By (f3),
\[ \min\{F(x, s)/s^2 : x \in B(x_0, \delta_0)\} \to \infty \quad \text{as} \quad s \to \infty. \]
Put \( a := \| \phi \|_\infty/2 \) and
\[ D := \{ x \in B(x_0, \delta_0) : \phi(x) \geq a \}. \]

For \( t \geq 0 \), we compute
\[ I_0(t\phi) = (t^2/2)\| \nabla \phi \|_2^2 - \int_{\Omega} F(x, t\phi) \, dx \]
\[ \leq \left( \frac{t^2}{2} \right) \| \nabla \phi \|_2^2 - \int_D \frac{F(x, t\phi)}{t^2 \phi^2} \phi^2 \, dx \to -\infty \quad \text{as} \quad t \to \infty. \]
We fix $t > 0$ so large that $I_0(t\phi) < 0$ and $t\|\nabla \phi\|_2 > r$. Then $u_1 := t\phi$ satisfies the assertion of the lemma.

For $u_1$ in Lemma 2.3 we define
\[ \Gamma := \{ \gamma \in C([0, 1], H^1_0(\Omega)) : \gamma(0) = 0, \gamma(1) = u_1 \}, \]
\[ c_0 := \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I_0(\gamma(t)). \]

**Lemma 2.4.** $c_0$ is a critical value of $I_0$.

**Proof.** This is a well-known mountain pass lemma. For the proof, we refer the reader to [3, 7, 8, 9].

We call $u$ a mountain pass solution of $I_0$ if $I_0'(u) = 0$ and $I_0(u) = c_0$. In general, a mountain pass solution is not necessarily unique but we have an a priori estimate for all mountain pass solutions in the next lemma.

**Lemma 2.5.** There exists a constant $C > 0$ such that $\|u\|_{C^1(\overline{\Omega})} \leq C$ for any mountain pass solution $u$ of $I_0$.

**Proof.** Let $u$ be any mountain pass solution of $I_0$. Since $I_0'(u) = 0$ and $I_0(u) = c_0$, we use (2.1) with (f2) to get
\[ \frac{\alpha - 2}{2} \|\nabla u\|_2^2 \leq \alpha c_0 + C \|u\|_\theta^2 + C \leq \alpha c_0 + C' \|\nabla u\|_\theta^2 + C. \]
This gives an a priori bound of the $H^1_0(\Omega)$ norm of $u$; i.e., $\|\nabla u\|_2 \leq C$ with a $C > 0$ independent of $u$. By the bootstrap argument with (f1) and the elliptic regularity theorem, we get the upper bound of the $W^{2,q}(\Omega)$ norm of $u$ for all $q \in [1, \infty)$. Especially, an a priori $C^1(\bar{\Omega})$ estimate of $u$ follows.

By Lemma 2.5, we have an $M > 0$ such that
\[ \|u\|_{\infty} \leq M \quad \text{for any mountain pass solution } u \text{ of } I_0. \]

Now, we define
\[ \tilde{g}(x, s) = \begin{cases} g(x, 0) & \text{if } s \leq 0, \\ g(x, s) & \text{if } 0 \leq s \leq 2M, \\ g(x, 2M) & \text{if } s \geq 2M. \end{cases} \]

Then $\tilde{g}(x, s)$ is continuous and bounded on $\overline{\Omega} \times \mathbb{R}$. We choose a function $h \in C^\infty_0(\mathbb{R})$ such that $0 \leq h \leq 1$ in $\mathbb{R}$, $h(s) = 1$ for $|s| \leq 2M$ and $h(s) = 0$ for $|s| \geq 4M$. We define
\[ I_\lambda(u) := \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 - F(x, u) - \lambda h(u)\tilde{G}(x, u) \right) dx, \]
\[ \tilde{G}(x, u) := \int_0^u \tilde{g}(x, s) ds. \]

A critical point of $I_\lambda$ is a solution of
\[ -\Delta u = f(x, u) + \lambda h(u)\tilde{g}(x, u) + \lambda h'(u)\tilde{G}(x, u) \quad \text{in } \Omega, \]
with $u = 0$ on $\partial \Omega$. Our plan to prove Theorem 1.1 is as follows. First, we find a mountain pass solution $u_\lambda$ of $I_\lambda$. Next, we prove that $0 < u_\lambda(x) \leq 2M$ for $|\lambda|$ small enough. Then $h'(u_\lambda) = 0$, $h(u_\lambda) = 1$, $\tilde{g}(x, u_\lambda) = g(x, u_\lambda)$ and therefore $u_\lambda$ becomes a solution of (1.1), (1.2).
Using the same argument as in Lemma 2.6 with the fact that $h(s)\tilde{G}(x,s)$ and its partial derivative on $s$ are bounded, we get the next lemma.

**Lemma 2.6.** For each $\lambda \in \mathbb{R}$, $I_\lambda$ satisfies the Palais-Smale condition.

**Lemma 2.7.** There exists a $\lambda_0 > 0$ such that $I_\lambda$ has a mountain pass geometry when $|\lambda| \leq \lambda_0$.

**Proof.** Since $h(s)\tilde{G}(x,s)$ is bounded on $\overline{\Omega} \times \mathbb{R}$, we have
\begin{equation}
I_0(u) - |\lambda|C \leq I_\lambda(u) \leq I_0(u) + |\lambda|C \quad \text{for } u \in H^1_0(\Omega),
\end{equation}
where $C > 0$ is independent of $\lambda$ and $u$. Let $r, \rho$ and $u_1$ be as in Lemma 2.5. For $|\lambda|$ small enough, it follows that
\begin{equation}
I_\lambda(u_1) \leq I_0(u_1) + |\lambda|C < 0,
\end{equation}
\begin{equation}
I_\lambda(u) \geq \rho - |\lambda|C \geq \rho/2 \quad \text{when } \|\nabla u\|_2 = r.
\end{equation}
The proof is complete. $\square$

We define the mountain pass value $c_\lambda$ of $I_\lambda$ by
\begin{equation*}
c_\lambda := \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I_\lambda(\gamma(t)).
\end{equation*}
Then $c_\lambda \to c_0$ as $\lambda \to 0$ by (2.6).

**Lemma 2.8.** Let $\lambda_n \in \mathbb{R}$ be a sequence converging to zero and $u_n$ a mountain pass solution of $I_{\lambda_n}$. Then a subsequence of $u_n$ converges to a limit $u_0$ in $W^{2,q}(\Omega)$ for all $q \in [1,\infty)$, where $u_0$ is a mountain pass solution of $I_0$.

**Proof.** By definition, $I_{\lambda_n}(u_n) = c_{\lambda_n}$, $I'_{\lambda_n}(u_n) = 0$ and hence $u_n$ satisfies (2.4) with $\lambda$ replaced by $\lambda_n$. Using the same argument as in Lemma 2.6 with the boundedness of $c_{\lambda_n}$, we can prove that the $W^{2,q}(\Omega)$ norm of $u_n$ is bounded for any $q \in [1,\infty)$. By the compact embedding, a subsequence of $u_n$ converges to a limit $u_0$ in $C^1(\overline{\Omega})$. Then $u_0$ satisfies that $I_0(u_0) = c_0$ and $I'_0(u_0) = 0$, i.e., that $u_0$ is a mountain pass solution of $I_0$. The right-hand side of (2.4) with $u = u_n$ and $\lambda = \lambda_n$ converges to that with $u = u_0$ and $\lambda = 0$ uniformly on $x \in \Omega$. The elliptic regularity theorem again ensures that $u_n$ converges to $u_0$ strongly in $W^{2,q}(\Omega)$ for all $q \in [1,\infty)$. $\square$

We shall prove the positivity and a priori estimate of mountain pass solutions for $I_\lambda$. To this end, for $\delta > 0$, we put
\begin{equation*}
\Omega_\delta := \{x \in \Omega : \text{dist}(x, \partial \Omega) < \delta\},
\end{equation*}
where $\text{dist}(x, \partial \Omega)$ denotes the distance from $x$ to $\partial \Omega$.

**Lemma 2.9.** There exist constants $\lambda_0, \delta, a, b > 0$ such that any mountain pass solution $u$ of $I_\lambda$ with $|\lambda| \leq \lambda_0$ satisfies (i) and (ii) below.
\begin{enumerate}
  \item $0 < u(x) \leq 2M$ in $\Omega$, where $M$ has been defined by (2.3).
  \item $\partial u/\partial \nu < -a$ in $\Omega_\delta$ and $u(x) > b$ in $\Omega \setminus \Omega_\delta$. Here $\partial/\partial \nu$ is well defined at each point in $\Omega_\delta$ for $\delta > 0$ small because $\partial \Omega$ is smooth.
\end{enumerate}

**Proof.** First, we shall prove $|u(x)| \leq 2M$ for $|\lambda| > 0$ small enough. Suppose that our claim is false. Then there exist sequences $\lambda_n \in \mathbb{R}$ and $u_n$ such that $\lambda_n$ converges to zero, $u_n$ is a mountain pass solution of $I_{\lambda_n}$ and $\|u_n\|_\infty > 2M$. By Lemma 2.8, a subsequence of $u_n$ converges to a mountain pass solution $u_0$ of $I_0$ in $C^1(\overline{\Omega})$. Since $\|u_0\|_\infty \leq M$ by (2.7), it follows that $\|u_n\|_\infty < 2M$ for $n$ large enough. A
contradiction occurs. Thus we have \( \|u\|_{\infty} \leq 2M \). The positivity of \( u \) in (i) follows from (ii).

Next, we shall prove that \( \partial u/\partial \nu < -a \) in \( \Omega_{\delta} \) with some \( a, \delta > 0 \) independent of \( u \). Suppose on the contrary that there exist \( \lambda_n, x_n, u_n \) such that \( \lambda_n \to 0 \), \( \text{dist}(x_n, \partial \Omega) \to 0 \), \( u_n \) is a mountain pass solution of \( I_{\lambda_n} \) and

\[
\liminf_{n \to \infty} \partial u_n(x_n)/\partial \nu \geq 0.
\]

We choose a subsequence of \( x_n \) which converges to a limit \( x_0 \in \partial \Omega \). By Lemma 2.8, a subsequence of \( u_n \) converges to a mountain pass solution \( u_0 \) of \( I_0 \) in \( C^1(\Omega) \). Then \( \partial u_0/\partial \nu(x_0) \geq 0 \), a contradiction to Lemma 2.7. Thus \( \partial u/\partial \nu < -a \) in \( \Omega_{\delta} \) with some \( a, \delta > 0 \). Fix such \( a, \delta > 0 \). Then by the same method as above, we can prove that \( u(x) > b \) in \( \Omega \setminus \Omega_{\delta} \) with some \( b > 0 \). \( \square \)

In Lemma 2.3 we replace \( r \) by any positive constant smaller than \( r \). Then (2.2) is still valid after \( \rho \) is replaced by a smaller positive constant. Hence (2.6) still holds if \( |\lambda| \) is replaced by a small one. Thus the next lemma follows.

**Lemma 2.10.** There exists an \( r_0 > 0 \) such that for any \( r \in (0, r_0) \), there exist constants \( \rho, \lambda' > 0 \) which satisfy

\[
I_{\lambda}(u) \geq \rho \quad \text{when} \quad \|\nabla u\|_2 = r, \quad |\lambda| < \lambda'.
\]

The lemma above will be used to find a small positive solution of (1.1), (1.2).

We are now in a position to prove Theorem 1.1.

**Proof of Theorem 1.1** Choose \( \lambda_0 > 0 \) which satisfies Lemmas 2.7 and 2.9. Let \( u_\lambda \) be a mountain pass solution of \( I_\lambda \) with \( |\lambda| \leq \lambda_0 \). Then \( 0 < u_\lambda(x) \leq 2M \) by Lemma 2.9. Thus \( h'(u_\lambda) = 0 \), \( h(u_\lambda) = 1 \), \( \tilde{g}(x, u_\lambda) = g(x, u_\lambda) \) and therefore \( u_\lambda \) becomes a solution of (1.1), (1.2). Let \( \lambda_j \) be any sequence converging to zero. By Lemma 2.3, a subsequence \( u_{\lambda_j} \) converges to a mountain pass solution \( u_0 \) of \( I_0 \) in \( W^{2,q}(\Omega) \) for all \( q \in [1, \infty) \).

We now suppose that \( g(x, 0) \geq 0 \), \( g(x, 0) \not\equiv 0 \) in \( \Omega \). By (2.6), we have

\[
\inf_{\|\nabla u\|_2 = r} I_{\lambda}(u) \geq \rho/2 > 0 = I_0(0).
\]

Let \( B \) be the set of \( u \in H^1_0(\Omega) \) such that \( \|\nabla u\|_2 \leq r \). Then the minimum of \( I_\lambda \) in \( B \) is achieved at an interior point \( v_\lambda \). Indeed, choose a sequence \( u_n \) in \( B \) such that \( I_\lambda(u_n) \) converges to the infimum of \( I_\lambda \) in \( B \). A subsequence of \( u_n \) weakly converges in \( H^1_0(\Omega) \) to a point \( v_\lambda \) in \( B \). By the weakly lower semicontinuity of \( I_\lambda \), we have

\[
I_\lambda(v_\lambda) \leq \liminf_{n \to \infty} I_\lambda(u_n),
\]

which means that \( v_\lambda \) is a minimum point of \( I_\lambda \) in \( B \). Since \( I_\lambda(0) = 0 \), we have \( I_\lambda(v_\lambda) \leq 0 < I_\lambda(u_\lambda) \), where \( u_\lambda \) is a mountain pass solution of \( I_\lambda \). Therefore \( v_\lambda \neq u_\lambda \). In the same way as in Lemma 2.5 with \( |\lambda| \) and \( r > 0 \) small enough, we can prove that \( \|v_\lambda\|_{\infty} \leq M \). Hence \( \tilde{g}(x, v_\lambda) = g(x, v_\lambda) \) and \( v_\lambda \) is a solution of (1.1). Moreover, \( v_\lambda \neq 0 \) because \( g(x, 0) \not\equiv 0 \). Thus \( v_\lambda \) is a nontrivial solution. We shall show that \( v_\lambda(x) \geq 0 \) for \( \lambda > 0 \). Let \( D \) be the set of \( x \in \Omega \) such that \( v_\lambda(x) < 0 \). Since \( f(x, s) = f(x, 0) = 0 \) and \( g(x, s) = g(x, 0) \geq 0 \) for \( s < 0 \), we see that for \( \lambda > 0 \),

\[
-\Delta v_\lambda = f(x, v_\lambda) + \lambda g(x, v_\lambda) \geq 0 \quad \text{in} \quad D, \quad v_\lambda = 0 \quad \text{on} \quad \partial D,
\]

which shows that \( v_\lambda \geq 0 \) in \( D \), a contradiction to the definition of \( D \). Thus \( D \) must be empty, and \( v_\lambda(x) \geq 0 \) in \( \Omega \). By Lemma 2.10 \( \|\nabla v_\lambda\|_2 \to 0 \) as \( \lambda \to 0 \). By the
bootstrap argument, the $W^{2,q}(\Omega)$ norm of $v_\lambda$ converges to zero for all $q \in [1, \infty)$, and hence $v_\lambda \to 0$ in $C^1(\Omega)$. Then Lemma 2.9 (ii) shows that $v_\lambda(x) < u_\lambda(x)$ in $\Omega$ for $\lambda > 0$ small enough.

We suppose that (1.3) holds. Put $A := \|v_\lambda\|_{\infty}$. By (1.3), there is a $C > 0$ such that

$$g(x, s) - g(x, 0) \geq -Cs \quad \text{for } 0 \leq s \leq A, \ x \in \Omega.$$  

Moreover, $f(x, s) \geq -Cs$ for $0 \leq s \leq A$ in the proof of Lemma 2.1. Then we have

$$((1 + \lambda)C - \Delta)v_\lambda = f(x, v_\lambda) + Cv_\lambda + \lambda(g(x, v_\lambda) - g(x, 0) + Cv_\lambda) + \lambda g(x, 0) \geq 0.$$  

By the strong maximum principle, $v_\lambda$ is strictly positive. The proof is complete. □

References


Department of Mathematics, Faculty of Science and Engineering, Saga University, Saga, 840-8502, Japan

E-mail address: kajikiya@ms.saga-u.ac.jp