A MONOMIAL BASIS
FOR THE HOLOMORPHIC FUNCTIONS ON $c_0$

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Abstract. For over thirty years it has been known that the monomials form a basis for the $n$-homogeneous polynomials on certain infinite dimensional Banach spaces. Recently, Defant and Kalton have shown that these are never unconditional. In this article we show that the monomials form a basis for both the holomorphic functions and the holomorphic functions of bounded type on $c_0$, both with their natural topologies.

INTRODUCTION

Various authors have considered monomial expansions of polynomials defined on infinite dimensional Banach spaces ([1, 2, 4, 8, 10, 13]), and although it has been shown that they form a Schauder basis in some spaces, so far we do not have analogous results for spaces of holomorphic functions. To show that the monomials form a basis for various spaces of holomorphic functions on $c_0$ we use three different decompositions: $S_\ast$-absolute decompositions of locally convex spaces (see section 3.3 in [6]), finite dimensional monotone decompositions of a Banach space [4], and a Schauder basis (p. 32 in [3]). We discuss these in section 1. In section 2 we recall the definitions of the different spaces of holomorphic functions and discuss the square order on the monomials. We prove our main result in section 3.

1. LINEAR DECOMPOSITIONS

A sequence of subspaces $\{E_n\}_{n=1}^\infty$ of a locally convex space $E$ is called a decomposition for $E$ if for each $x \in E$ there exists a unique sequence $(x_n)_{n=1}^\infty$, $x_n \in E_n$ for all $n$, such that

$$x = \sum_{n=1}^\infty x_n := \lim_{n \to \infty} \sum_{j=1}^n x_j.$$  

We say that $\{E_n\}_{n=1}^\infty$ is an $S_\ast$-absolute decomposition of $E$ if $E$ admits a fundamental system $\mathcal{N}$ of semi-norms $p$ such that for any sequence of scalars $(\alpha_n)_{n=1}^\infty$ satisfying $\limsup_{n \to \infty} |\alpha_n|^{1/n} < \infty$ the semi-norm $q$,

$$q\left(\sum_{n=1}^\infty x_n\right) := \sum_{n=1}^\infty |\alpha_n|p(x_n),$$  

is continuous.

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This concept coincides with the notion of a global Schauder decomposition given in [7] and [14] and is a variation on $S$-absolute decomposition discussed in [6], section 3.3.

If each $E_n$ is a finite dimensional space, then the decomposition is called finite dimensional. If $E$ is a normed linear space with norm $\| \cdot \|$, then the decomposition constant is the constant defined as the infimum of all $c$ such that
\[
\| \sum_{j=1}^{n} x_j \| \leq c \| \sum_{j=1}^{m} x_j \|
\]
for all positive integers $m$ and $n$, $n < m$, and all $x_j \in E_j$. The decomposition constant is always greater than or equal to 1, and if it equals 1 we say that the decomposition is monotone. A renorming generally changes the decomposition constant.

If each $E_n$ is one dimensional and $e_n$ spans $E_n$, we say that $(e_n)_{n=1}^{\infty}$ is a Schauder basis for $E$. In this case there exists for each $x$ a sequence of scalars $(x_n)_{n=1}^{\infty}$ such that $x = \sum_{n=1}^{\infty} x_n e_n$, and we use the term basis constant in place of decomposition constant. The linear functional $x \mapsto \sum_{n=1}^{\infty} x_n e_n$ is called the $n$th coefficient functional and is denoted by $e_n^*$. If $E$ has an $S_*$-decomposition, $(E_n)_{n=1}^{\infty}$, then it admits, by (1), a fundamental system $N$ of semi-norms $p$ such that
\[
p(\sum_{n=1}^{\infty} x_n) = \sum_{n=1}^{\infty} p(x_n)
\]
for all $\sum_{n=1}^{\infty} x_n \in E$, $x_n \in E_n$ for all $n$. If each $E_n$ has a Schauder basis $(e_{n,m})_{m=1}^{\infty}$, then an ordering of $(e_{n,m})_{n,m=1}^{\infty}$ into a sequence is given by a bijective mapping $\phi : N^2 \to N$. We say that the ordering is compatible if
\[
m < \overline{m} \implies \phi(n,m) < \phi(n,\overline{m})
\]
for all $n$, that is, if it induces on each $E_n$ its original order (see Proposition 4.1 in [6]).

**Lemma 1.** Let $\phi : N^2 \to N$ denote a compatible ordering. Then for every positive integer $j$, there exists a finite subset $S_j$ of positive integers and a finite set of positive integers $(k_n(j))_{n \in S_j}$ such that
\[
\{k : 1 \leq k \leq j\} = \bigcup_{n \in S_j} \{\phi(n,m) : 1 \leq m \leq k_n(j)\}.
\]
Moreover, if $l$ is a positive integer, then $S_{j+l} = S_j \cup S$ for some finite subset $S$ of $N$, disjoint from $S_j$, and $k_n(j) \leq k_n(j+l)$ for all $n \in S_j$.

**Proof.** We prove this result by induction on $j$. Since $\phi$ is surjective there exists a pair of integers $(n_1, m_1)$ such that $\phi(n_1, m_1) = 1$. Let $S_1 = \{n_1\}$. If $m_1 > 1$, then, since $\phi$ is injective, we have $\phi(n_1, 1) > 1 = \phi(n_1, m_1)$ and this contradicts (4). Hence $m_1 = 1$ and (5) holds when $j = 1$.  

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Now suppose (5) holds for the positive integer \( j \). This implies, in particular, that

\[
\{1, 2, \ldots, j\} \subset \bigcup_{n \in S_j} \{\phi(n, m) : m \in \mathbb{N}\}.
\]

By surjectivity of \( \phi \) we can find a pair of positive integers \( (n_{j+1}, m_{j+1}) \) such that \( \phi(n_{j+1}, m_{j+1}) = j + 1 \). We consider two cases.

If \( n_{j+1} \notin S_j \), then, since \( \phi \) is injective, (6) implies that \( \phi(n_{j+1}, m) \geq j + 1 \) for all \( m \in \mathbb{N} \). If \( m_{j+1} > 1 \), then, since \( \phi(n_{j+1}, m_{j+1}) = j + 1 \), the injectivity of \( \phi \) implies \( \phi(n_{j+1}, 1) > j + 1 \). We then have \( \phi(n_{j+1}, 1) > j + 1 = \phi(n_{j+1}, m_{j+1}) \) and this contradicts (4). Hence \( m_{j+1} = 1 \). Letting \( S_{j+1} = S_j \cup \{n_{j+1}\} \), \( k_n(j) = k_n(j + 1) \) for \( n \in S_j \) and \( k_{n_{j+1}}(j + 1) = 1 \) we obtain (5).

If \( n_{j+1} \in S_j \), let \( S_{j+1} = S_j \). If \( m_{j+1} > k_{n_{j+1}}(j) + 1 \), then, by (6) and since \( \phi \) is injective,

\[
\phi(n_{j+1}, k_{n_{j+1}}(j) + 1) > j + 1 = \phi(n_{j+1}, m_{j+1})
\]

and this contradicts (4). Hence \( m_{j+1} \leq k_{n_{j+1}}(j) + 1 \). By (6), \( \phi(n_{j+1}, l) \leq j \) for all \( l \leq k_{n_{j+1}}(j) \) and, as \( \phi \) is injective, this implies \( m_{j+1} > k_{n_{j+1}}(j) \). Hence \( m_{j+1} = k_{n_{j+1}}(j) + 1 \). If \( n \in S_j, n \neq n_{j+1} \), let \( k_n(j + 1) = k_n(j) \) and let \( k_{n_{j+1}}(j + 1) = k_{n_{j+1}}(j) + 1 \). This implies that (5) holds for \( j + 1 \). By induction this completes the proof of (5) and the remainder of the proof follows easily. \( \blacksquare \)

**Theorem 1.** Suppose \( E \) has an \( S_\ast \)-decomposition, \( \{E_n\}_{n=1}^\infty \), with fundamental system \( \mathcal{N} \) of semi-norms \( p \) satisfying (3) and that each \( E_n \) has a Schauder basis \( (e_{n,m})_{m=1}^\infty \). Then \( (e_{n,m})_{n,m=1}^\infty \), with any compatible ordering, is a basis for \( E \) if the basis constants \( c_{p,n} \) of \( (E_n, p) \) for \( (e_{n,m})_{n,m=1}^\infty \) have at most exponential growth, i.e.,

\[
\limsup_{n \to \infty} c_{p,n}^{1/n} < \infty
\]

for \( p \in \mathcal{N} \).

**Proof.** Let \( \phi : \mathbb{N}^2 \to \mathbb{N} \) denote a fixed compatible order on \( \mathbb{N}^2 \). By the definitions of decomposition and basis we see that \( (e_{n,m})_{n,m=1}^\infty \) spans a dense subspace of \( E \) and hence it suffices to show it is a basic sequence in \( E \). To show this we apply Theorem 6, p. 298, in [9]. Let \( p \) denote a semi-norm on \( E \) satisfying (3). By (1), the semi-norm

\[
q\left( \sum_{k=1}^\infty x_k \right) := \sum_{k=1}^\infty c_{p,n} p(x_n), x_n \in E_n, \sum_{n=1}^\infty x_n \in E
\]

is continuous on \( E \).

Let \( (\alpha_k)_{k \in \mathbb{N}} \) denote an arbitrary set of scalars. We now use the notation employed in Lemma 1. If \( j \) and \( l \) are positive integers, then \( S_{j+l} = S_j \cup S \) for some finite subset \( S \subset \mathbb{N} \) disjoint from \( S_j \), and \( k_n(j) \leq k_n(j + l) \) for all \( n \in S_j \). We then have
for all positive integers \( j \) and \( l \),

\[
P\left(\sum_{k=1}^{j} \alpha_k e_{\phi^{-1}(k)}\right) = P\left(\sum_{n \in S_j} \left\{ \sum_{m=1}^{k_n(j)} \alpha_{\phi(n,m)} e_{n,m} \right\}\right)
\]

\[
= \sum_{n \in S_j} p\left(\sum_{m=1}^{k_n(j)} \alpha_{\phi(n,m)} e_{n,m}\right)
\]

\[
\leq \sum_{n \in S_j} c_{p,n} p\left(\sum_{m=1}^{k_n(j+l)} \alpha_{\phi(n,m)} e_{n,m}\right)
\]

\[
\leq \sum_{n \in S_j} c_{p,n} p\left(\sum_{m=1}^{k_n(j+l)} \alpha_{\phi(n,m)} e_{n,m}\right) + c_{n,p} p\left(\sum_{n \in S} \left\{ \sum_{m=1}^{k_n(j+l)} \alpha_{\phi(n,m)} e_{n,m} \right\}\right)
\]

\[
= \sum_{n \in S_{j+l}} c_{p,n} p\left(\sum_{m=1}^{k_n(j+l)} \alpha_{\phi(n,m)} e_{n,m}\right)
\]

\[
= q\left(\sum_{k=1}^{j} \alpha_k e_{\phi^{-1}(k)}\right).
\]

This completes the proof. \( \square \)

Let \( c_0 = \{(z_j)_{j=1}^{\infty} : z_j \in \mathbb{C} \text{ all } j, \lim_{j \to \infty} z_j = 0\} \) and let \( c_0^+ = \{(z_j)_{j=1}^{\infty} \in c_0 : z_j \geq 0 \text{ for all } j\} \). We denote by \((e_j)_{j=1}^{\infty}\) the standard unit vector basis for \( c_0 \) and let \((e_j^*)_{j=1}^{\infty}\) denote the dual unit vector basis for \( \ell_1 = c_0' \). The polydiscs

\[
P_\beta := \{(z_j)_{j=1}^{\infty} \in c_0 : |z_j| \leq \beta_j \text{ all } j\}
\]

form a fundamental system for the compact subsets of \( c_0 \) when \( \beta = (\beta_j)_{j=1}^{\infty} \) ranges over \( c_0^+ \).

### 2. Polynomials and holomorphic functions

In this section we discuss concepts from infinite dimensional holomorphy and refer to \cite{6} and \cite{11} for details. Our main result concerns holomorphic functions on \( c_0 \).

For each positive integer \( n \) and each Banach space \( X \), let \( \mathcal{P}(^nX) \) denote the space of continuous \( n \)-homogeneous polynomials on \( X \). Endowed with the supremum norm of uniform convergence over the unit ball \( B \) of \( X \), \( \mathcal{P}(^nX) \) is a Banach space.

Let \( \mathbb{N}^{(\mathbb{N})} \) denote the set of all sequences of nonnegative integers which are eventually zero. If \((m_i)_{i=1}^{\infty} \in \mathbb{N}^{(\mathbb{N})}\), we call \( |m| := \sum_i m_i \) and \( l(m) := \text{sup}\{i : m_i \neq 0\} \) the modulus and length of \( m \), respectively, and call the mapping

\[
(z_j)_{j=1}^{\infty} \in c_0 \longrightarrow z^m := z_1^{m_1} \cdots z_n^{m_n} \cdots
\]

a monomial (we use the convention \( 0^0 = 1 \)). For positive integers \( n \) and \( k \) let \( \mathcal{P}_k(^n) \) denote the subspace of \( \mathcal{P}(^n) \) spanned by \( \{z^m : l(m) = k, |m| = n\} \). For all \( n \) the sequence \( \{\mathcal{P}_k(^n)\}_{k=1}^{\infty} \) is a finite dimensional decomposition of \( \mathcal{P}(^n) \).
(see [4], and [6], section 4.1). This follows, for \( P \in \mathcal{P}^{(n+1)c_0} \), from the identities

\[
P(\sum_{j=1}^{\infty} z_j e_j) = P(z_1 e_1) + \sum_{k=1}^{\infty} \{ P(\sum_{j=1}^{k-1} z_j e_j) - P(\sum_{j=1}^{k} z_j e_j) \}
\]

\[
= a_1 z_1^{n+1} + \sum_{s=1}^{n+1} a_s z_1^{n+1-s} z_s + \sum_{s=1, t \geq 0, s+t < n+1} a_{s,t} z_1^{n-s-t} z_2^s z_3^t + \cdots
\]

and

\[
Q_{k+1}(\sum_{j=1}^{\infty} z_j e_j) := P(\sum_{j=1}^{k} z_j e_j) - P(\sum_{j=1}^{k+1} z_j e_j) =: R_{k+1}(\sum_{j=1}^{\infty} z_j e_j) \cdot z_{k+1},
\]

where \( Q_{k+1} \in \mathcal{P}_{k+1}^{(n+1)c_0} \) and \( R_{k+1} \in \bigoplus_{j=1}^{k+1} \mathcal{P}_j^{(n)c_0} \). Hence

\[
Q_{k+1} = R_{k+1} \cdot e_{k+1}^*.
\]

We consider different equivalent norms on \( P \in \mathcal{P}^{(n)c_0} \), generated by uniform convergence over bounded polydiscs in \( c_0 \). If \( \lambda_j \in c_0^+ \),

\[
A := \{ (z_j)_{j=1}^{\infty} \in c_0 : |z_j| \leq \lambda_j, j = 1, 2, \ldots \}
\]

is a bounded polydisc in \( c_0 \), and \( Q_k \in \mathcal{P}_k^{(n)c_0} \) for all \( k \), then for all \( s, t \) with \( s < t \), we have, since \( Q_t(\sum_{j=1}^{k} z_j e_j) = 0 \) for all \( l > k \),

\[
\| \sum_{k=1}^{s} Q_k \|_A \leq \| \sum_{k=1}^{t} Q_k \|_A,
\]

and this implies that \( \{ \mathcal{P}_k^{(n)c_0} \}_{k=1}^{\infty} \) is a finite dimensional monotone decomposition of \( \mathcal{P}^{(n)c_0} \) for all \( n \). Using the identity \( Q_{k+1} = R_{k+1} \cdot e_{k+1}^* \) and the fact that holomorphic functions on polydiscs achieve their absolute maxima on the distinguished boundary we see that

\[
\| Q_{k+1} \|_A = \lambda_{k+1} \| R_{k+1} \|_A.
\]

We now define the square order on the monomials in \( \mathcal{P}^{(n)c_0} \). On \( c_0' = \ell_1 \) we use the sequential order inherited from the standard unit vector basis \( (e_j^*)_{j=1}^{\infty} \). The square order on the monomials in \( \mathcal{P}^{(n+1)c_0} \) is defined as follows: if \( m = (m_i)_{i=1}^{\infty} \) and \( m' = (m'_i)_{i=1}^{\infty} \) are in \( \mathbb{N}^{(n)} \) and \( |m| = |m'| \), then \( m < m' \) if either \( l(m) < l(m') \) or \( l(m) = l(m') \) and for some positive integer \( s \leq l(m), m_s < m'_s \) and \( m_t = m'_t \) for all \( t > s \).

The square order on the monomials appears naturally when we use the finite dimensional decomposition \( \mathcal{P}_k^{(n+1)c_0} \). Clearly, if \( k < k' \), then the monomials in \( \mathcal{P}_k^{(n+1)c_0} \) precead those in \( \mathcal{P}_{k'}^{(n+1)c_0} \) and the order within \( \mathcal{P}_k^{(n+1)c_0} \) is determined by the order inherited from \( \mathcal{P}^{(n)c_0} \). If \( m \in \mathbb{N}^{(n)} \) and \( l(m) = s \), then there is a unique \( m \in \mathbb{N}^{(n)} \) such that \( z^m = z_2^{m_s} z_3^{m_t} \) for all \( \sum_{j=1}^{\infty} z_j e_j \in c_0 \). Note that \( |m| = |m| - 1 \) and \( l(m) \leq l(m) \). If \( m, m' \in \mathbb{N}^{(n)} \), then \( m < m' \) if either \( l(m) < l(m') \) or \( l(m) = l(m') \) and \( m < m' \).

The square order was introduced by Ryan ([13]), and various authors have shown that the monomials of degree \( n \) with the square order are a Schauder basis for \( \mathcal{P}^{(n)c_0} \). Theorem 2 contains within it yet another proof of this fact, modulo the result of W. Bogdanowicz and A. Pelczyński in 1957 (see [8], p. 81) that polynomials on \( c_0 \) are weakly continuous on bounded sets. The order is important as Defant and
Kalton have shown in [2] that when the monomials of degree \( n \) for any \( n \geq 2 \) form an unconditional basis for the space of \( n \)-homogeneous polynomials on a Banach space \( X \), endowed with the norm of uniform convergence over the unit ball of \( X \), then \( X \) is finite dimensional.

We let \( \mathcal{H}(X) \) denote the space of holomorphic functions on the Banach space \( X \) and let \( \mathcal{H}_b(X) \) denote the subspace of \( \mathcal{H}(X) \) consisting of all \( f \) bounded on bounded subsets of \( X \). Let \( \mathcal{K}(X) \) denote the set of all compact subsets of \( X \). By Proposition 3.18 in [5] the semi-norms 

\[
p_K \left( \sum_{n=0}^{\infty} P_n \right) := \sum_{n=0}^{\infty} \|P_n\|_K,
\]

where \( K \in \mathcal{K}(X) \), generate the compact open topology \( \tau_0 \) on \( \mathcal{H}(X) \). When \( X = c_0 \), Ex. 6, p. 15, in [3] implies that the semi-norms 

\[
p_\beta \left( \sum_{n=0}^{\infty} P_n \right) := \sum_{n=0}^{\infty} \sup \{ |P_n(z)| : z = (z_j)_{j=1}^\infty \in c_0, |z_j| \leq \beta_j \text{ for all } j \},
\]

where \( \beta = (\beta_j)_{j=1}^\infty \), ranges over \( c_0^+ \), generates \( (\mathcal{H}(c_0), \tau_0) \).

For an arbitrary Banach space \( X \), Proposition 4.39 in [5] shows that the \( \tau_\omega \) topology of Nachbin is generated by 

\[
p_{K,\alpha} \left( \sum_{n=0}^{\infty} P_n \right) := \sum_{n=0}^{\infty} \|P_n\|_{K+\alpha B_X},
\]

where \( K \in \mathcal{K}(X) \) and \( (\alpha_n)_{n=0}^\infty \in c_0^+ \) are arbitrary.

Similarly, the \( \tau_b \) topology on \( \mathcal{H}_b(X) \) is generated by the semi-norms 

\[
p(\sum_{n=0}^{\infty} P_n) := \sum_{n=0}^{\infty} \|P_n\|_{\alpha B_{c_0}},
\]

where \( \sum_{n=0}^{\infty} P_n \in \mathcal{H}_b(c_0) \) and \( \alpha \) ranges over \( R^+ \).

If \( X = c_0 \), \( P \) is a monomial, and \( K = \{(z_j)_{j=1}^\infty : |z_j| \leq \beta_j \text{ for all } j\} \), where \( (\beta_j)_{j=1}^\infty \in c_0^+ \), then 

\[
\|P\|_{K+\alpha B_{c_0}} = \{|P(z)| : z = (z_j)_{j=1}^\infty, |z_j| \leq \beta_j + \alpha\}.
\]

Note that if \( [\beta] := \{(z_j)_{j=1}^\infty : |z_j| \leq \beta_j \text{ for all } j\} \), where \( (\beta_j)_{j=1}^\infty \in c_0^+ \), then for any \( \sum_{j=1}^{n+1} \alpha_j e_j^* \) we have 

\[
(9) \quad \| \sum_{j=1}^{n} \alpha_j e_j^* \|_{[\beta]} = \sum_{j=1}^{n} |\alpha_j \beta_j| \leq \sum_{j=1}^{n+1} |\alpha_j \beta_j| = \| \sum_{j=1}^{n+1} \alpha_j e_j^* \|_{[\beta]}.
\]

This shows that for any space \( X \) with basis \( (e_j)_{j=1}^n \) and closed unit ball \( \{\sum_{j=1}^{n} \alpha_j e_j : |\alpha_j| \leq \beta_j \text{ for all } j\} \) the basis constant for \( (e_j^*)_{j=1}^n \) is 1.

3. A Schauder basis for \( (\mathcal{H}(c_0), \tau_\omega) \) and \( (\mathcal{H}_b(c_0), \tau_b) \)

In this section we let \( (P_{n,m})_{m=1}^\infty \) denote the monomials of degree \( n \) on \( c_0 \) endowed with the square order and we suppose that the set of all monomials is given a compatible order.

**Theorem 2.** The monomials with a compatible order are a Schauder basis for \( (\mathcal{H}(c_0), \tau_\omega) \), \( (\mathcal{H}(c_0), \tau_0) \), and \( (\mathcal{H}_b(c_0), \tau_b) \).
Proof. In view of the fundamental systems of semi-norms described in the previous section it suffices, by Theorem 1, to take an arbitrary bounded polydisc

\[ A := \{(z_m)_{m=1}^\infty \in c_0 : |z_m| \leq \lambda_m \text{ all } m\}, \]

where \( \lambda_m \geq 0 \) for all \( m \), and to show that the basis constant, \( c_n \), for \( (P^n(c_0), \| \cdot \|_A) \) satisfies \( c_n \leq 3^n \) for all \( n \). We prove this by induction. Different choices of \( A \) then prove the required result for the different spaces of holomorphic functions.

The square ordering on \( P^1(c_0) = c_0^{+1} = \ell_1 \) is just the standard ordering of the positive integers and, by (9), the basis constant is 1. We now suppose that \( c_n \leq 3^n \) and aim to show \( c_{n+1} \leq 3^{n+1} \).

Let \( (\alpha_m)_{m=1}^\infty \) denote an arbitrary sequence of scalars. Fix positive integers \( s \) and \( t \), \( s < t \). For some nonnegative integer \( k \) we have the expansion

\[ \sum_{m=1}^s \alpha_m P_{n+1,m} = \sum_{k=1}^{k+1} \sum_{u=1}^{Q_{n+1,u}} \alpha_m P_{n+1,m}, \]

where \( Q_{n+1,u} := \sum_{1 \leq m \leq s, l(P_{n+1,m})=u} \alpha_m P_{n+1,m} \in P_u(n c_0) \) for \( 1 \leq u \leq k+1 \).

\[ \text{Note that each } P_{n+1,m} \text{ is a monomial of degree } n+1 \text{ and that if } m_1 < m_2, \text{ then } l(P_{n+1,m_1}) \leq l(P_{n+1,m_2}). \]

If \( s < m \leq t \), then \( l(P_{n+1,m}) \geq k+1 \) and for some integer \( k^* \geq k+1 \),

\[ \sum_{m=s+1}^t \alpha_m P_{n+1,m} = \sum_{u=k+1}^{k^*} \sum_{s \leq u \leq t, l(P_{n+1,m})=u} \alpha_m P_{n+1,m}. \]

If

\[ Q_{n+1,u}^{*} = \sum_{s \leq u \leq t, l(P_{n+1,m})=u} \alpha_m P_{n+1,m} \]

for \( k+1 < u \leq k^* \), then with the convention \( \sum_{u=k+1}^{k^*} = 0 \) when \( k+1 = k^* \), we have

\[ \sum_{m=s+1}^t \alpha_m P_{n+1,m} = \sum_{m=s+1}^t \alpha_m P_{n+1,m} + \sum_{u=k+1}^{k^*} Q_{n+1,u}^{*}. \]

If we let

\[ Q_{n+1,k+1}^{*} = Q_{n+1,k+1} + \sum_{s \leq u \leq t, l(P_{n+1,m})=k+1} \alpha_m P_{n+1,m}, \]

then

\[ \sum_{m=1}^t \alpha_m P_{n+1,m} = \sum_{m=1}^s \alpha_m P_{n+1,m} + \sum_{m=s+1}^t \alpha_m P_{n+1,m} \]

\[ = \sum_{u=1}^{k+1} Q_{n+1,u} + \sum_{s \leq u \leq t, l(P_{n+1,m})=k+1} \alpha_m P_{n+1,m} + \sum_{u=k+1}^{k^*} Q_{n+1,u}^{*} \]

\[ = \sum_{u=1}^{k} Q_{n+1,u} + Q_{n+1,k+1}^{*} + \sum_{u=k+1}^{k^*} Q_{n+1,u}^{*} \]

\[ = \sum_{u=1}^{k} Q_{n+1,u} + \sum_{u=k+1}^{k^*} Q_{n+1,u}^{*}. \]
This identity and (7) imply that

\[(13) \quad \| \sum_{u=1}^{k} Q_{n+1,u} \|_A \leq \| \sum_{m=1}^{t} \alpha_m P_{n+1,m} \|_A. \]

If \(l(P_{n+1,m}) \leq k\) for \(m < m_0\) and \(l(P_{n+1,m_0}) = k + 1\), then, by (10) and (12),

\[(14) \quad Q_{n+1,k+1} = \sum_{m=m_0}^{s} \alpha_m P_{n+1,m} = e_{k+1}^* \cdot \sum_{m=m_0}^{s} \alpha_m P_{n,m-m_0+1} \]

and

\[(15) \quad Q^*_{n+1,k+1} = \sum_{m=m_0}^{s^*} \alpha_m P_{n+1,m} = e_{k+1}^* \cdot \sum_{m=m_0}^{s^*} \alpha_m P_{n,m-m_0+1} \]

for some integer \(s^*, s \leq s^* \leq t\). Applying in turn (14), induction, (15), and (13), we obtain

\[
\|Q_{n+1,k+1}\|_A = \|e_{k+1}^* \cdot \sum_{m=m_0}^{s} \alpha_m P_{n,m-m_0+1}\|_A \\
= \|e_{k+1}^*\|_A \cdot \| \sum_{m=m_0}^{s} \alpha_m P_{n,m-m_0+1}\|_A \\
\leq 3^n \|e_{k+1}^*\|_A \cdot \| \sum_{m=m_0}^{s^*} \alpha_m P_{n,m-m_0+1}\|_A \\
= 3^n \|e_{k+1}^*\|_A \cdot \sum_{m=m_0}^{s^*} \alpha_m P_{n,m-m_0+1}\|_A \\
= 3^n \| \sum_{m=m_0}^{s^*} \alpha_m P_{n+1,m}\|_A \\
= 3^n \|Q^*_{n+1,k+1}\|_A \\
\leq 3^n \| \sum_{u=k+1}^{k^*} Q^*_{n+1,u}\|_A \\
\leq 3^n \{ \| \sum_{u=1}^{k} Q_{n+1,u} + \sum_{u=k+1}^{k^*} Q^*_{n+1,u}\|_A + \| \sum_{u=1}^{k} Q_{n+1,u}\|_A \} \\
\leq 2 \cdot 3^n \| \sum_{m=1}^{t} \alpha_m P_{n+1,m}\|_A.
\]
This estimate, together with (7) and (13), implies that
\[
\| \sum_{m=1}^{s} \alpha_m P_{n+1,m} \|_A = \| \sum_{u=1}^{k+1} Q_{n+1,u} \|_A \\
\leq \| \sum_{u=1}^{k} Q_{n+1,u} \|_A + \| Q_{n+1,k+1} \|_A \\
\leq (1 + 2 \cdot 3^n) \| \sum_{m=m_0}^{t} \alpha_m P_{n+1,m} \|_A
\]
and hence
\[
c_{n+1} \leq 1 + 2 \cdot 3^n \leq 3^{n+1}.
\]
This completes the proof. \(\square\)

**Added in proof.** The authors should mention that Ryan has shown that the monomials on \(l_1\), with the square order, form an unconditional basis for \(H(l_1)\) with the compact open topology. See [12].

Our result depended heavily on special features possessed by \(c_0\). Nevertheless, the existence of a monomial basis for the space of holomorphic functions on \(c_0\) facilitates the search for improved estimates, and this should, in turn, make \(c_0\) a more attractive and flexible space in which to find positive results and on which to construct counterexamples.

**References**


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