MINIMAL $C^1$-DIFFEOMORPHISMS OF THE CIRCLE WHICH ADMIT MEASURABLE FUNDAMENTAL DOMAINS

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Abstract. We construct, for each irrational number $\alpha$, a minimal $C^1$-diffeomorphism of the circle with rotation number $\alpha$ which admits a measurable fundamental domain with respect to the Lebesgue measure.

1. Introduction

The concept of ergodicity is important not only for measure-preserving dynamical systems but also for systems which admit a natural quasi-invariant measure. Given a probability space $(X, \mu)$ and a transformation $T$ of $X$, $\mu$ is said to be quasi-invariant if the push forward $T_\ast \mu$ is equivalent to $\mu$. In this case $T$ is called ergodic with respect to $\mu$, if a $T$-invariant Borel subset in $X$ is either null or conull.

A diffeomorphism of a differentiable manifold always leaves the Riemannian volume (also called the Lebesgue measure) quasi-invariant, and one can ask if a given diffeomorphism is ergodic with respect to the Lebesgue measure (below ergodic for short) or not. Answering a question of A. Denjoy [D], A. Katok (see for instance Chap. 12.7, p. 419, [KH]) and independently M. Herman (Chap. VII, p. 86, [H]) showed that a $C^1$-diffeomorphism of the circle with derivative of bounded variation is ergodic provided its rotation number is irrational. Contrarily Oliveira and da Rocha [OR] constructed a minimal $C^1$-diffeomorphism of the circle which is not ergodic.

At the opposite extreme of the ergodicity lies the concept of measurable fundamental domains. Given a transformation $T$ of a standard probability space $(X, \mu)$ leaving $\mu$ quasi-invariant, a Borel subset $C$ of $X$ is called a measurable fundamental domain if $T^n C$ $(n \in \mathbb{Z})$ is mutually disjoint and the union $\bigcup_{n \in \mathbb{Z}} T^n C$ is conull. In this case any Borel function on $C$ can be extended to a $T$-invariant measurable function on $X$, and an ergodic component of $T$ is just a single orbit. The purpose of this paper is to show the following theorem.
Theorem 1.1. For any irrational number $\alpha$, there is a minimal $C^1$-diffeomorphism of the circle with rotation number $\alpha$ which admits a measurable fundamental domain with respect to the Lebesgue measure.

Sections 2, 3 and 4 are devoted to the proof of Theorem 1.1. Let us mention an important remark and a further question.

Remark 1.2. In 2.1 of [DKN2], it is indicated how to construct examples of $C^1$-actions of the $n$-adic Thompson groups ($n \geq 10$) which are minimal but not ergodic. According to the referee, these actions admit measurable fundamental domains.

Question 1.3. Does there exist a minimal nonergodic $C^{1+\tau}$-diffeomorphism ($0 < \tau < 1$)? More generally for any $d \geq 2$ and $\tau > d^{-1}$, any free $\mathbb{Z}^d$-action by $C^{1+\tau}$-diffeomorphisms on $S^1$ is known to be minimal [DKN1]. Do there exist nonergodic actions? The method of this paper does not seem to be applicable to these problems.

2. A MEASURABLE FUNDAMENTAL DOMAIN FOR A LIPSCHITZ HOMEOMORPHISM

We regard the circle $S^1$ as $\mathbb{R}/\mathbb{Z}$. Suppose $R$ denotes the rotation by $\alpha$.

Claim 2.1. For any irrational number $\alpha$, we can construct a Cantor set $C \in S^1$ so that $R^n C \cap R^m C = \emptyset$ for any integers $n \neq m$.

We will give a proof for the claim in Section 4. Here we show how to construct such a Cantor set for an easy case, namely, $\alpha = (\sqrt{5} - 1)/2$.

Define a Cantor set $C$ in the circle by

$$C = \{ \sum_{k=1}^{\infty} \frac{\varepsilon_k}{2^k} \mid \varepsilon_k = 0 \text{ or } 1 \} \pmod{\mathbb{Z}}.$$ 

Note that all numbers in $C$ are well approximable by rational numbers.

Suppose $x \in R^n C \cap R^m C$; then $x - n\alpha, x - m\alpha \in C$. Therefore

$$(-n + m)\alpha \in C + (-C) = \{ \sum_{k \geq 3} \frac{\varepsilon'_k}{2^k} \mid \varepsilon'_k = 0 \text{ or } \pm 1 \} \pmod{\mathbb{Z}}.$$ 

$(-n + m)\alpha$ is badly approximable, while $C + (-C)$ consists of well approximable numbers, which is a contradiction. Therefore, this Cantor set $C$ satisfies the condition for Claim 2.1.

Fix a probability measure $\mu_0$ on $C$ without atoms such that $\text{supp}(\mu_0) = C$. We also choose a sequence $(a_i)_{i \in \mathbb{Z}}$ of positive numbers satisfying $\sum_{i \in \mathbb{Z}} a_i = 1$. Now we can define a probability measure $\mu$ on $S^1$ by

$$\mu := \sum_{i \in \mathbb{Z}} a_i R^i \mu_0.$$ 

The Radon-Nikodým derivative $\frac{dR^{-1} \mu}{d\mu}$ is equal to $\frac{a_{i+1}}{a_i}$ on the set $R^i C$. Now we assume that $\frac{a_{i+1}}{a_i} \in \left[\frac{1}{D}, D\right]$ for some $D > 1$. Then it follows that $\frac{dR^{-1} \mu}{d\mu} \in L^\infty(S^1, \mu)$. 

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We define a homeomorphism \( h \) of \( S^1 \) by \( h(0) = 0 \) and \( h(x) = y \) if and only if \( \text{Leb}[0, x] = \mu[0, y] \), where \( \text{Leb} \) denotes the Lebesgue measure on \( S^1 \) or, more briefly, \( h_* \text{Leb} = \mu \). Finally define a homeomorphism \( F \) of \( S^1 \) by \( F := h^{-1} \circ R \circ h \). Then
\[
\frac{dF_*^{-1} \text{Leb}}{d \text{Leb}} = \frac{dR_*^{-1} \mu}{d\mu} \circ h \in L^\infty(S^1, \text{Leb});
\]
i.e. the map \( F \) is a Lipschitz homeomorphism. The set \( C' = h^{-1}C \) is a measurable fundamental domain of \( F \).

3. Make it \( C^1 \)

3.1. \textbf{What shall we do?} To prove the theorem, we are trying to make the Radon-Nikodým derivative \( g = \frac{dR_*^{-1} \mu}{d\mu} \) continuous on \( S^1 \).

Fix an arbitrary point \( x_0 \in C \). For a positive integer \( i \),
\[
a_i = (a_i/a_{i-1}) \cdots (a_2/a_1)(a_1/a_0)a_0
\]
\[
= g(R^{i-1}x_0) \cdots g(Rx_0)g(x_0)a_0,
\]
\[
a_{-i} = (a_{-i+1}/a_{-i})^{-1} \cdots (a_{-1}/a_{-2})^{-1}(a_0/a_{-1})^{-1}a_0
\]
\[
= g(R^{-1}x_0)^{-1} \cdots g(R^{-2}x_0)^{-1}g(R^{-1}x_0)^{-1}a_0.
\]

Set \( \phi = \log g \) and define a map \( \Phi : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R} \) by \( \Phi(x, y) = (Rx, y + \phi(x)) \). A simple calculation shows that \( \Phi^n(x, y) = (R^nx, y + \phi^n(x)) \), where
\[
\phi^{(m)}(x) = \sum_{i=0}^{m-1} \phi(R^ix) \quad (m > 0),
\]
\[
\phi^{(-m)}(x) = -\sum_{i=1}^{m} \phi(R^{-i}x) \quad (m > 0),
\]
\[
\phi^{(0)}(x) = 0.
\]

Therefore \( a_i = \exp(\phi^{(i)}(x_0))a_0 \). To satisfy \( \sum_{i \in \mathbb{Z}} a_i = 1 \), it suffices to find \( \phi \) so that \( \sum_{i \in \mathbb{Z}} \exp(\phi^{(i)}(x_0)) < \infty \).

3.2. \textbf{Construction step I.} Now we forget about the \( a_i \)'s and the Cantor set \( C \). As a first step, we construct a function \( \phi \in C(S^1) \) satisfying \( \sum_{i \in \mathbb{Z}} \exp(\phi^{(i)}(x_0)) < \infty \) for a single point \( x_0 \), where \( \phi^{(i)} \) are defined by (3.1).

We will define continuous functions \( \phi_n \in C(S^1) \) \( (n \in \mathbb{N}) \) in such a way that \( \sum_{i=1}^{\infty} \|\phi_n\| < \infty \). Then \( \phi = \sum_{i=1}^{\infty} \phi_n \) converges uniformly; thus \( \phi \) is also continuous.

Fix an integer \( n \in \mathbb{N} \). Choose a sufficiently small neighbourhood \( J \) of \( x_0 \) so that \( R^{-2n}J, \ldots, R^{-1}J, R\cdot J, \ldots, R^{2n-1}J \) are disjoint. Consider a bump function \( f \) on \( J \) so that \( \text{supp } f \subset J \), \( f(x_0) = (3/4)^n \) and \( 0 \leq f(x) < (3/4)^n \) on \( J \setminus \{x_0\} \). Define \( \phi_n : S^1 \to \mathbb{R} \) by
\[
\phi_n(x) = \begin{cases} 
-f(R^{-i}x) & x \in R^i J, \; i = 0, 1, \ldots, 2^n - 1, \\
f(R^{-i}x) & x \in R^i J, \; i = -2^n, -2^n + 1, \ldots, -1, \\
0 & \text{otherwise.}
\end{cases}
\]

\textbf{Lemma 3.1.} \( \phi_n^{(i)}(x_0) \begin{cases} 
-|i|(3/4)^n & \text{for } -2^n \leq i \leq 2^n, \\
\leq 0 & \text{for any } i \in \mathbb{Z}.
\end{cases} \)
Proof. The equality for the first case is trivial. Define an increasing sequence
\((m_k)_{k \in \mathbb{Z}}\) by \(m_0 = 0\) and \(\{m_k | k \in \mathbb{Z}\} = \{m \in \mathbb{Z} | R^m x_0 \in J\}\). Since \(R^{-2^2} J, \ldots, R^{2^n-1} J\) are disjoint, \(m_{k+1} - m_k \geq 2^{n+1}\) for any \(k \in \mathbb{Z}\). Using this sequence, \(R^m x_0 \in R^i J\) if and only if \(m = m_k + i\) for some \(k\). Therefore,

\[
\phi^{(i+1)}(x_0) = \begin{cases} 
\phi^{(i)}(x_0) & m_{k-1} + 2^n \leq i < m_k - 2^n, \\
\phi^{(i)}(x_0) + f(R^{m_k} x_0) & m_k - 2^n \leq i < m_k, \\
\phi^{(i)}(x_0) - f(R^{m_k} x_0) & m_k \leq i < m_k + 2^n
\end{cases}
\]

for some \(k\); see also Figure 1. Induction for \(|k|\) shows that

\[
\phi^{(i)}(x_0) = \begin{cases} 
-2^n \cdot (3/4)^n & m_{k-1} + 2^n \leq i \leq m_k - 2^n, \\
-2^n \cdot (3/4)^n + (i - (m_k - 2^n)) f(R^{m_k} x_0) & m_k - 2^n \leq i \leq m_k, \\
-2^n \cdot (3/4)^n + ((m_k + 2^n) - i) f(R^{m_k} x_0) & m_k \leq i \leq m_k + 2^n.
\end{cases}
\]
Since \( f(R^m x_0) \leq (3/4)^n \), the inequality \( \phi^{(i)}(x_0) \leq 0 \) also holds. \(\Box\)

Therefore, if \( 2^n \leq |i| < 2^{n+1} \), \( \phi^{(i)}(x_0) \leq \phi^{(i)}_{n+1}(x_0) = -|i|(3/4)^{n+1} \leq -2^n \cdot (3/4)^{n+1} = -3/4 \cdot (3/2)^n \). Finally, \( \sum_{i \in \mathbb{Z}} \exp(\phi^{(i)}(x_0)) \leq 1 + \sum_{n=0}^{\infty} 2^{n+1} \exp(-3/4 \cdot (3/2)^n) = M < \infty. \)

3.3. Construction step II. We will execute the same argument for the Cantor set \( C \) instead of the single point \( x_0 \). Since \( R^{-2^n} C, \ldots, R^{-2^n-1} C \) are disjoint compact sets, there exists an \( \varepsilon \)-neighbourhood \( N \) of \( C \) such that \( R^{-2^n} N, \ldots, N, \ldots, R^{-2^n-1} N \) are disjoint. Define a bump function \( f \) so that \( \text{supp} f \subset N \), \( f(x) = (3/4)^n \) on \( C \) and \( 0 \leq f(x) < (3/4)^n \) on \( N \setminus C \). Now we apply the same argument as in the previous subsection to obtain the function \( \phi \in C(S^1) \) such that \( \sum_{i \in \mathbb{Z}} \exp(\phi^{(i)}(x)) < M < \infty \) for any \( x \in C \).

We define a finite measure \( \tilde{\mu} \) on \( S^1 \) by

\[
\tilde{\mu} := \sum_{i \in \mathbb{Z}} (\exp \circ \phi^{(i)} \circ R^{-1}) R_i^* \mu_0.
\]

Normalize \( \tilde{\mu} \) to obtain a probability measure \( \mu \), namely \( \mu := \frac{\tilde{\mu}}{\int_{S^1} d\mu} \).

Define \( h \) and \( F \) as in section 2. Then

\[
\frac{dF^{-1}_s \text{Leb}}{d \text{Leb}} = \frac{dR^{-1}_s \mu}{d\mu} \circ h = g \circ h
\]
is a continuous function because \( g(x) = \exp(\phi(x)) \). We have proved Theorem 1.1 up to Claim 2.1.

4. Construction of Cantor set for general \( \alpha \)

We are going to prove Claim 2.1 for a general irrational number \( \alpha \). For a real number \( x \) and a function \( p: \mathbb{N} \to \mathbb{N} \), define the approximation constant \( c_p(x) \) by

\[
c_p(x) := \liminf_{q \to \infty} \left( p(q) \cdot \text{dist}(x, \frac{1}{q} \mathbb{Z}) \right).
\]

A real number \( x \) is said to be \( p \)-approximable if \( c_p(x) = 0 \). Note that \( x \) is well approximable if \( x \) is \( p \)-approximable for \( p(q) = q^2 \), so this is a generalization of well-approximability.

It is clear that \( c_p(x) = 0 \) if \( x \) is a rational number. On the other hand, for any irrational number \( x \) we can find a function \( p \) satisfying \( c_p(x) > 0 \). Moreover, we will show the following lemma.

Lemma 4.1. For a given irrational number \( \alpha \), we can find a function \( p \) such that \( c_p(m \alpha) \geq 1 \) for any nonzero integer \( m \).

Proof. Since \( c_p(-m \alpha) = c_p(m \alpha) \), it is enough to show the lemma for the case \( m \in \mathbb{N} \). Let us start for any natural numbers \( n \) and \( q \) by taking a natural number \( p_n(q) \) so that \( p_n(q) \cdot \text{dist}(m \alpha, \frac{1}{q} \mathbb{Z}) \geq 1 \). Then define a function \( p \) by

\[
p(q) = \max_{1 \leq n \leq q} p_n(q).
\]
By this construction \( p(q) \geq p_m(q) \) for any \( q \geq m \). Therefore
\[
c_p(m\alpha) = \liminf_{q \to \infty} \left( p(q) \cdot \text{dist}(m\alpha, \frac{1}{q} \mathbb{Z}) \right)
\geq \liminf_{q \to \infty} \left( p_m(q) \cdot \text{dist}(m\alpha, \frac{1}{q} \mathbb{Z}) \right)
\geq 1.
\]
\[
\quad \square
\]

For this function \( p \), we inductively take an increasing sequence \( q_0, q_1, \ldots \) of natural numbers satisfying the following conditions: \( q_0 = 1 \), \( q_n|q_{n+1} \), \( q_n/q_{n+1} \leq 1/3 \) and \( p(q_n)/q_{n+1} \leq 2^{-n} \). Define a Cantor set \( C \) by
\[
C := \left\{ \sum_{n=1}^{\infty} \varepsilon_n \frac{\varepsilon_n}{q_n} \mid \varepsilon_n = 0 \text{ or } 1 \right\}.
\]
This Cantor set \( C \) consists of \( p \)-approximable numbers. We can also show the following lemma.

**Lemma 4.2.** For any \( \beta \in C - C \), the approximation constant \( c_p(\beta) \) is equal to 0, where
\[
C - C = \left\{ \sum_{n=1}^{\infty} \varepsilon'_n \frac{\varepsilon'_n}{q_n} \mid \varepsilon'_n = 0 \text{ or } \pm 1 \right\}.
\]

**Proof.**
\[
p(q_i) \cdot \text{dist}(\beta, \frac{1}{q_i} \mathbb{Z}) \leq p(q_i) \left| \sum_{n=i+1}^{\infty} \frac{\varepsilon'_n}{q_n} \right|
\leq p(q_i) \sum_{n=i+1}^{\infty} \frac{1}{q_n} = p(q_i) \sum_{n=i+1}^{\infty} \frac{q_{i+1}}{q_n} \leq \frac{1}{2^i} \sum_{k=0}^{\infty} \left( \frac{1}{3} \right)^k = \frac{3}{2^{i+1}}.
\]
Thus
\[
c_p(\beta) = \liminf_{q \to \infty} \left( p(q) \cdot \text{dist}(\beta, \frac{1}{q} \mathbb{Z}) \right)
\leq \liminf_{i \to \infty} \left( p(q_i) \cdot \text{dist}(\beta, \frac{1}{q_i} \mathbb{Z}) \right)
\leq \liminf_{i \to \infty} \frac{3}{2^{i+1}}
= 0.
\]
Therefore \( c_p(\beta) = 0 \). \( \square \)

Claim 2.1 follows from Lemma 4.1 and Lemma 4.2 so we have proved Theorem 1.1 for the general case.

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