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MINIMAL C^1-DIFFEOMORPHISMS OF THE CIRCLE WHICH ADMIT MEASURABLE FUNDAMENTAL DOMAINS

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Abstract. We construct, for each irrational number α, a minimal C^1-diffeomorphism of the circle with rotation number α which admits a measurable fundamental domain with respect to the Lebesgue measure.

1. Introduction

The concept of ergodicity is important not only for measure-preserving dynamical systems but also for systems which admit a natural quasi-invariant measure. Given a probability space (X, μ) and a transformation T of X, μ is said to be *quasi-invariant* if the push forward $T_*\mu$ is equivalent to μ. In this case T is called *ergodic* with respect to μ, if a T-invariant Borel subset in X is either null or conull.

A diffeomorphism of a differentiable manifold always leaves the Riemannian volume (also called the Lebesgue measure) quasi-invariant, and one can ask if a given diffeomorphism is ergodic with respect to the Lebesgue measure (below *ergodic* for short) or not. Answering a question of A. Denjoy [D], A. Katok (see for instance Chap. 12.7, p. 419, [KH]) and independently M. Herman (Chap. VII, p. 86, [H]) showed that a C^1-diffeomorphism of the circle with derivative of bounded variation is ergodic provided its rotation number is irrational. Contrarily Oliveira and da Rocha [OR] constructed a minimal C^1-diffeomorphism of the circle which is not ergodic.

At the opposite extreme of the ergodicity lies the concept of measurable fundamental domains. Given a transformation T of a standard probability space (X, μ) leaving μ quasi-invariant, a Borel subset C of X is called a *measurable fundamental domain* if T^nC $(n \in \mathbb{Z})$ is mutually disjoint and the union $\bigcup_{n\in\mathbb{Z}} T^nC$ is conull. In this case any Borel function on C can be extended to a T-invariant measurable function on X, and an ergodic component of T is just a single orbit. The purpose of this paper is to show the following theorem.

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Theorem 1.1. *For any irrational number α, there is a minimal C^1-diffeomorphism of the circle with rotation number α which admits a measurable fundamental domain with respect to the Lebesgue measure.*

Sections 2, 3 and 4 are devoted to the proof of Theorem 1.1.

Let us mention an important remark and a further question.

Remark 1.2. In 2.1 of [DKN2], it is indicated how to construct examples of C^1-actions of the n-adic Thompson groups ($n \geq 10$) which are minimal but not ergodic. According to the referee, these actions admit measurable fundamental domains.

Question 1.3. Does there exist a minimal nonergodic $C^{1+\tau}$-diffeomorphism ($0 < \tau < 1$)? More generally for any $d \geq 2$ and $\tau > d^{-1}$, any free $\mathbb{Z}^d$-action by $C^{1+\tau}$-diffeomorphisms on S^1 is known to be minimal [DKN1]. Do there exist nonergodic actions? The method of this paper does not seem to be applicable to these problems.

2. A measurable fundamental domain for a Lipschitz homeomorphism

We regard the circle S^1 as $\mathbb{R}/\mathbb{Z}$. Suppose R denotes the rotation by α.

Claim 2.1. For any irrational number α, we can construct a Cantor set $C \in S^1$ so that $R^n C \cap R^m C = \emptyset$ for any integers $n \neq m$.

We will give a proof for the claim in Section 4. Here we show how to construct such a Cantor set for an easy case, namely, $\alpha = (\sqrt{5}-1)/2$.

Define a Cantor set C in the circle by

$$C = \Big\{ \sum_{k=1}^{\infty} \frac{\varepsilon_k}{2^{3^k}} \ \Big| \ \varepsilon_k = 0 \text{ or } 1 \Big\} \pmod{\mathbb{Z}}.$$

Note that all numbers in C are well approximable by rational numbers.

Suppose $x \in R^n C \cap R^m C$; then $x - n\alpha, x - m\alpha \in C$. Therefore

$$(-n+m)\alpha \in C + (-C) := \Big\{ \sum_{k \geq 3} \frac{\varepsilon'_k}{2^{3^k}} \ \Big| \ \varepsilon'_k = 0 \text{ or } \pm 1 \Big\} \pmod{\mathbb{Z}}.$$

$(-n+m)\alpha$ is badly approximable, while $C + (-C)$ consists of well approximable numbers, which is a contradiction. Therefore, this Cantor set C satisfies the condition for Claim 2.1.

Fix a probability measure μ_0 on C without atoms such that $\mathrm{supp}(\mu_0) = C$. We also choose a sequence $(a_i)_{i \in \mathbb{Z}}$ of positive numbers satisfying $\sum_{i \in \mathbb{Z}} a_i = 1$. Now we can define a probability measure μ on S^1 by

$$\mu := \sum_{i \in \mathbb{Z}} a_i R_*^i \mu_0.$$

The Radon-Nikodým derivative $\frac{dR_*^{-1}\mu}{d\mu}$ is equal to $\frac{a_{i+1}}{a_i}$ on the set $R^i C$. Now we assume that $\frac{a_{i+1}}{a_i} \in [\frac{1}{D}, D]$ for some $D > 1$. Then it follows that $\frac{dR_*^{-1}\mu}{d\mu} \in L^\infty(S^1, \mu)$.

We define a homeomorphism h of S^1 by $h(0) = 0$ and $h(x) = y$ if and only if $Leb[0, x] = \mu[0, y]$, where Leb denotes the Lebesgue measure on S^1 or, more briefly, $h_* \, Leb = \mu$. Finally define a homeomorphism F of S^1 by $F := h^{-1} \circ R \circ h$. Then

$$\frac{dF_*^{-1} \, Leb}{d \, Leb} = \frac{dR_*^{-1}\mu}{d\mu} \circ h \in L^\infty(S^1, Leb);$$

i.e. the map F is a Lipschitz homeomorphism. The set $C' = h^{-1}C$ is a measurable fundamental domain of F.

3. Make it C^1

3.1. **What shall we do?** To prove the theorem, we are trying to make the Radon-Nikodým derivative $g = \frac{dR_*^{-1}\mu}{d\mu}$ continuous on S^1.

Fix an arbitrary point $x_0 \in C$. For a positive integer i,

$$\begin{aligned} a_i &= (a_i/a_{i-1}) \cdots (a_2/a_1)(a_1/a_0)a_0 \\ &= g(R^{i-1}x_0) \cdots g(Rx_0)g(x_0)a_0, \\ a_{-i} &= (a_{-i+1}/a_{-i})^{-1} \cdots (a_{-1}/a_{-2})^{-1}(a_0/a_{-1})^{-1}a_0 \\ &= g(R^{-i}x_0)^{-1} \cdots g(R^{-2}x_0)^{-1}g(R^{-1}x_0)^{-1}a_0. \end{aligned}$$

Set $\phi = \log g$ and define a map $\Phi\colon S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ by $\Phi(x, y) = (Rx, y + \phi(x))$. A simple calculation shows that $\Phi^n(x, y) = (R^n x, y + \phi^{(n)}(x))$, where

$$\begin{aligned} \phi^{(m)}(x) &= \sum_{i=0}^{m-1} \phi(R^i x) \qquad (m > 0), \\ \phi^{(-m)}(x) &= -\sum_{i=1}^{m} \phi(R^{-i} x) \qquad (m > 0), \\ \phi^{(0)}(x) &= 0. \end{aligned} \tag{3.1}$$

Therefore $a_i = \exp(\phi^{(i)}(x_0))a_0$. To satisfy $\sum_{i\in\mathbb{Z}} a_i = 1$, it suffices to find ϕ so that $\sum_{i\in\mathbb{Z}} \exp(\phi^{(i)}(x_0)) < \infty$.

3.2. **Construction step I.** Now we forget about the a_i's and the Cantor set C. As a first step, we construct a function $\phi \in C(S^1)$ satisfying $\sum_{i\in\mathbb{Z}} \exp(\phi^{(i)}(x_0)) < \infty$ for a single point x_0, where $\phi^{(i)}$ are defined by (3.1).

We will define continuous functions $\phi_n \in C(S^1)$ ($n \in \mathbb{N}$) in such a way that $\sum_{i=1}^\infty \|\phi_n\|_\infty < \infty$. Then $\phi = \sum_{i=1}^\infty \phi_n$ converges uniformly; thus ϕ is also continuous.

Fix an integer $n \in \mathbb{N}$. Choose a sufficiently small neighbourhood J of x_0 so that $R^{-2^n}J, \ldots, R^{-1}J, J, RJ, \ldots, R^{2^n-1}J$ are disjoint. Consider a bump function f on J so that $\operatorname{supp} f \subset J$, $f(x_0) = (3/4)^n$ and $0 \le f(x) < (3/4)^n$ on $J \setminus \{x_0\}$. Define $\phi_n\colon S^1 \to \mathbb{R}$ by

$$\phi_n(x) = \begin{cases} -f(R^{-i}x) & x \in R^iJ,\ i = 0, 1, \ldots, 2^n - 1, \\ f(R^{-i}x) & x \in R^iJ,\ i = -2^n, -2^n + 1, \ldots, -1, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.1. $\phi_n^{(i)}(x_0) \begin{cases} = -|i|(3/4)^n & \textit{for } -2^n \le i \le 2^n, \\ \le 0 & \textit{for any } i \in \mathbb{Z}. \end{cases}$

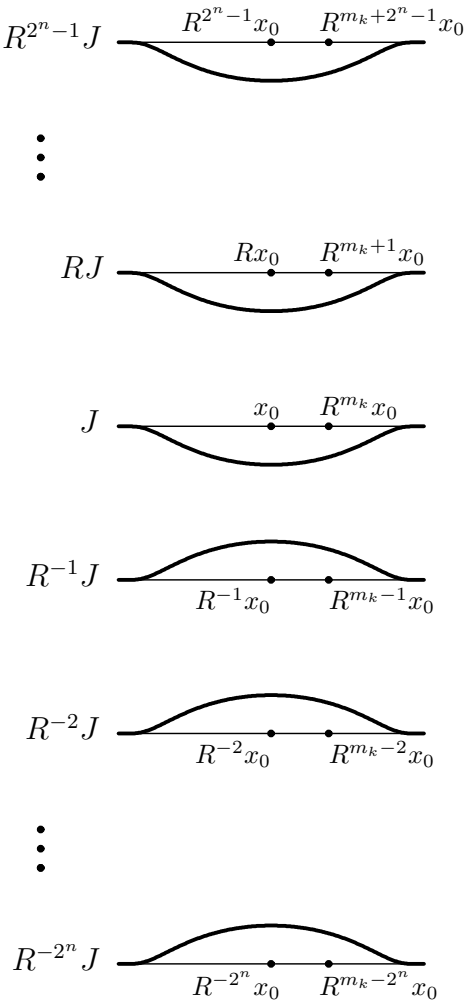

FIGURE 1. The graph of ϕ_n and the orbit of x_0

Proof. The equality for the first case is trivial. Define an increasing sequence $(m_k)_{k\in\mathbb{Z}}$ by $m_0 = 0$ and $\{m_k | k \in \mathbb{Z}\} = \{m \in \mathbb{Z} | R^m x_0 \in J\}$. Since $R^{-2^n}J, \dots, R^{2^n-1}J$ are disjoint, $m_{k+1} - m_k \geq 2^{n+1}$ for any $k \in \mathbb{Z}$. Using this sequence, $R^m x_0 \in R^i J$ if and only if $m = m_k + i$ for some k. Therefore,

$$\phi^{(i+1)}(x_0) = \begin{cases} \phi^{(i)}(x_0) & m_{k-1} + 2^n \leq i < m_k - 2^n, \\ \phi^{(i)}(x_0) + f(R^{m_k}x_0) & m_k - 2^n \leq i < m_k, \\ \phi^{(i)}(x_0) - f(R^{m_k}x_0) & m_k \leq i < m_k + 2^n \end{cases}$$

for some k; see also Figure 1. Induction for $|k|$ shows that

$$\phi^{(i)}(x_0) = \begin{cases} -2^n \cdot (3/4)^n & m_{k-1} + 2^n \leq i \leq m_k - 2^n, \\ -2^n \cdot (3/4)^n + (i - (m_k - 2^n)) f(R^{m_k}x_0) & m_k - 2^n \leq i \leq m_k, \\ -2^n \cdot (3/4)^n + ((m_k + 2^n) - i) f(R^{m_k}x_0) & m_k \leq i \leq m_k + 2^n. \end{cases}$$

Since $f(R^{m_k}x_0) \le (3/4)^n$, the inequality $\phi^{(i)}(x_0) \le 0$ also holds. $\square$

Therefore, if $2^n \le |i| < 2^{n+1}$, $\phi^{(i)}(x_0) \le \phi^{(i)}_{n+1}(x_0) = -|i|(3/4)^{n+1} \le -2^n \cdot (3/4)^{n+1} = -3/4 \cdot (3/2)^n$. Finally, $\sum_{i\in\mathbb{Z}} \exp(\phi^{(i)}(x_0)) \le 1 + \sum_{n=0}^{\infty} 2^{n+1} \exp(-3/4 \cdot (3/2)^n) = M < \infty$.

3.3. **Construction step II.** We will execute the same argument for the Cantor set C instead of the single point x_0. Since $R^{-2^n}C, \dots, C, \dots, R^{2^n-1}C$ are disjoint compact sets, there exists an ε-neighbourhood N of C such that $R^{-2^n}N, \dots, N, \dots, R^{2^n-1}N$ are disjoint. Define a bump function f so that $\operatorname{supp} f \subset N$, $f(x) = (3/4)^n$ on C and $0 \le f(x) < (3/4)^n$ on $N \setminus C$. Now we apply the same argument as in the previous subsection to obtain the function $\phi \in C(S^1)$ such that $\sum_{i\in\mathbb{Z}} \exp(\phi^{(i)}(x)) < M < \infty$ for any $x \in C$.

We define a finite measure $\tilde{\mu}$ on S^1 by

$$\tilde{\mu} := \sum_{i\in\mathbb{Z}} (\exp \circ \phi^{(i)} \circ R^{-i}) R^i_* \mu_0.$$

Normalize $\tilde{\mu}$ to obtain a probability measure μ, namely $\mu := \frac{\tilde{\mu}}{\int_{S^1} d\tilde{\mu}}$.

Define h and F as in section 2. Then

$$\frac{dF_*^{-1}\, Leb}{d\, Leb} = \frac{dR_*^{-1}\mu}{d\mu} \circ h = g \circ h$$

is a continuous function because $g(x) = \exp(\phi(x))$. We have proved Theorem 1.1 up to Claim 2.1.

4. Construction of Cantor set for general α

We are going to prove Claim 2.1 for a general irrational number α. For a real number x and a function $p\colon \mathbb{N} \to \mathbb{N}$, define the approximation constant $c_p(x)$ by

$$c_p(x) := \liminf_{q\to\infty} \left(p(q) \cdot \operatorname{dist}(x, \tfrac{1}{q}\mathbb{Z}) \right).$$

A real number x is said to be p-approximable if $c_p(x) = 0$. Note that x is well approximable if x is p-approximable for $p(q) = q^2$, so this is a generalization of well-approximability.

It is clear that $c_p(x) = 0$ if x is a rational number. On the other hand, for any irrational number x we can find a function p satisfying $c_p(x) > 0$. Moreover, we will show the following lemma.

Lemma 4.1. *For a given irrational number α, we can find a function p such that $c_p(m\alpha) \ge 1$ for any nonzero integer m.*

Proof. Since $c_p(-mx) = c_p(mx)$, it is enough to show the lemma for the case $m \in \mathbb{N}$. Let us start for any natural numbers n and q by taking a natural number $p_n(q)$ so that $p_n(q) \cdot \operatorname{dist}(n\alpha, \frac{1}{q}\mathbb{Z}) \ge 1$. Then define a function p by

$$p(q) = \max_{1\le n\le q} p_n(q).$$

By this construction $p(q) \geq p_m(q)$ for any $q \geq m$. Therefore

$$\begin{aligned} c_p(m\alpha) &= \liminf_{q\to\infty} \Big(p(q) \cdot \mathrm{dist}(m\alpha, \tfrac{1}{q}\mathbb{Z})\Big) \\ &\geq \liminf_{q\to\infty} \Big(p_m(q) \cdot \mathrm{dist}(m\alpha, \tfrac{1}{q}\mathbb{Z})\Big) \\ &\geq 1. \end{aligned}$$

□

For this function p, we inductively take an increasing sequence $q_0, q_1, \dots$ of natural numbers satisfying the following conditions: $q_0 = 1$, $q_n | q_{n+1}$, $q_n/q_{n+1} \leq 1/3$ and $p(q_n)/q_{n+1} \leq 2^{-n}$. Define a Cantor set C by

$$C := \Big\{\sum_{n=1}^{\infty} \frac{\varepsilon_n}{q_n} \;\Big|\; \varepsilon_n = 0 \text{ or } 1\Big\}.$$

This Cantor set C consists of p-approximable numbers. We can also show the following lemma.

Lemma 4.2. *For any $\beta \in C - C$, the approximation constant $c_p(\beta)$ is equal to 0, where*

$$C - C = \Big\{\sum_{n=1}^{\infty} \frac{\varepsilon'_n}{q_n} \;\Big|\; \varepsilon'_n = 0 \text{ or } \pm 1\Big\}.$$

Proof.

$$\begin{aligned} p(q_i) \cdot \mathrm{dist}(\beta, \tfrac{1}{q_i}\mathbb{Z}) &\leq p(q_i) \left| \sum_{n=i+1}^{\infty} \frac{\varepsilon'_n}{q_n} \right| \\ &\leq p(q_i) \sum_{n=i+1}^{\infty} \frac{1}{q_n} = \frac{p(q_i)}{q_{i+1}} \sum_{n=i+1}^{\infty} \frac{q_{i+1}}{q_n} \leq \frac{1}{2^i} \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k = \frac{3}{2^{i+1}}. \end{aligned}$$

Thus

$$\begin{aligned} c_p(\beta) &= \liminf_{q\to\infty} \Big(p(q) \cdot \mathrm{dist}(\beta, \tfrac{1}{q}\mathbb{Z})\Big) \\ &\leq \liminf_{i\to\infty} \Big(p(q_i) \cdot \mathrm{dist}(\beta, \tfrac{1}{q_i}\mathbb{Z})\Big) \\ &\leq \liminf_{i\to\infty} \frac{3}{2^{i+1}} \\ &= 0. \end{aligned}$$

Therefore $c_p(\beta) = 0$. □

Claim 2.1 follows from Lemma 4.1 and Lemma 4.2, so we have proved Theorem 1.1 for the general case.

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