A MARCINKIEWICZ MAXIMAL-MULTIPLIER THEOREM

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Abstract. For $r < 2$, we prove the boundedness of a maximal operator formed by applying all multipliers $m$ with $\|m\|_{V^r} \leq 1$ to a given function.

1. Introduction

Given an exponent $r$ and a function $f$ defined on $\mathbb{R}$, consider the $r$-variation norm

$$\|f\|_{V^r} = \|f\|_{L^\infty} + \sup_{N, \xi_0 < \cdots < \xi_N} \left( \sum_{i=1}^{N} |f(\xi_i) - f(\xi_{i-1})|^r \right)^{1/r},$$

where the supremum is over all strictly increasing finite length sequences of real numbers.

The classical Marcinkiewicz multiplier theorem states that if $r = 1$ and a function $m$ is of bounded 1-variation uniformly on dyadic shells, then $m$ is an $L^p$ multiplier for $1 < p < \infty$ and

$$\|(m \hat{f})\|_{L^p} \leq C_{p,1} \sup_{k \in \mathbb{Z}} \|1_{D_k} m\|_{V^1} \|f\|_{L^p},$$

where $D_k = [-2^{k+1}, -2^k) \cup (2^k, 2^{k+1}]$ and $\hat{\cdot}, \check{\cdot}$ denote the Fourier-transform and its inverse. Later, Coifman, Rubio de Francia, and Semmes [2] (see also [8]) showed that the requirement of bounded 1-variation can be relaxed to allow for functions of bounded 2-variation, and in fact (1.1) holds whenever $r \geq 2$ and $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{r}$.

The estimate [2] does not discriminate between multipliers of bounded 2-variation and those of bounded $r$-variation where $r < 2$, and so one might ask whether there is anything to be gained by controlling the variation norm of multipliers in the latter range of exponents.

Defining the maximal-multiplier operator

$$\mathcal{M}_r[f](x) = \sup_{m: \|m\|_{V^r} \leq 1} |(m \hat{f})^\cdot(x)|,$$

where the supremum is over all functions in the $V^r$ unit ball, we have

Theorem 1.1. Suppose $1 \leq r < 2$ and $r < p < \infty$. Then

$$\|\mathcal{M}_r[f]\|_{L^p} \leq C_{p,r} \|f\|_{L^p}.$$
The case $r = 1$ was observed independently by Lacey \[4\].

Note that in the definition of $M_r$, each $m$ is required to have finite $r$-variation on all of $\mathbb{R}$ rather than simply on each dyadic shell as in (1.1). This is necessary for boundedness, as can be seen from the counterexamples of Christ, Grafakos, Honzík and Seeger \[1\].

Although the maximal operator (1.2) would seem to be fairly strong, we do not yet know of an application for the bound above. We will, however, quickly illustrate a strategy for its use that falls an (important) $\epsilon$ short of success. Let $\Psi$ be (say) a Schwartz function, and for each $\xi, x \in \mathbb{R}$ and $k \in \mathbb{Z}$, consider the $2^k$-truncated partial Fourier integral

$$S_k[f](\xi, x) = \text{p.v.} \int f(x - t)e^{2\pi i \xi t} \Psi(2^{-k} t) \frac{1}{t} dt.$$

It was proven by Demeter, Lacey, Tao, and Thiele \[3\] that for $q = 2$ and $1 < p < \infty$,

$$\| \sup_{x \in \mathbb{R}} \| (S_k[f](\cdot, x) \hat{g})^\ast \|_{L^q} \|_{L^p_x} \leq C_{p,q} \| f \|_{L^p}.$$  \hspace{1cm} (1.4)

If we had the bound

$$\| S_k[f](\xi, x) \|_{L^p_x(V^r_x)} \leq C_{p,r} \| f \|_{L^p} \hspace{1cm} (2.1)$$

for some $r < 2$, then an application of Theorem 1.1 would give (1.4) for $q > r$ by rather different means than \[3\]. In fact, one can see by applying the method in Appendix D of \[6\] that (1.5) holds for $r > 2$ and $p > r'$. Unfortunately, it does fail for $r \leq 2$.

2. Proof of Theorem 1.1

The following lemma was proven in \[2\]; see also \[5\].

**Lemma 2.1.** Let $m$ be a compactly supported function on $\mathbb{R}$ of bounded $r$-variation for some $1 \leq r < \infty$. Then for each integer $j \geq 0$, one can find a collection $\Upsilon_j$ of pairwise disjoint subintervals of $\mathbb{R}$ and coefficients $\{b_v\}_{v \in \Upsilon_j} \subset \mathbb{R}$ so that $|\Upsilon_j| \leq 2^j$, $|b_v| \leq 2^{-j/r} \| m \|_{V^r}$, and

$$m = \sum_{j \geq 0} \sum_{v \in \Upsilon_j} b_v 1_v,$$

where the sum in $j$ converges uniformly.

The lemma above was applied in concert with Rubio de Francia’s square function estimate \[7\] to obtain (1.1). Here, we will argue similarly, exploiting the analogy between the Rubio de Francia square function estimate and the variation-norm Carleson theorem.

It was proven in \[7\] that for $p \geq 2$,

$$\sup_{I} \left\| \left( \sum_{I \in \mathcal{I}} |(1_I \hat{f})^\ast|^2 \right)^{1/2} \right\|_{L^p} \leq C_p \| f \|_{L^p},$$

where the supremum above is over all collections of pairwise disjoint subintervals of $\mathbb{R}$. Consider the partial Fourier integral

$$S[f](\xi, x) = (1_{(\infty, \xi]} \hat{f})^\ast(x).$$
It was proven in [6] that for \( s > 2 \) and \( p > s' \),
\[
\| S[f](\xi, x) \|_{L^p(V_s^\varepsilon)} \leq C_{p, s} \| f \|_{L^p}
\]
or, equivalently,
\[
(2.2) \quad \| \sup_I (\sum_{I \in I} |(1_I f)^\varepsilon|)^{1/s} \|_{L^p} \leq C_{p, s} \| f \|_{L^p}.
\]

Note that by standardizing limiting arguments, taking the supremum in (1.2) to be over all compactly supported \( m \) such that \( \| m \|_{V_r} \leq 1 \) does not change the definition of \( M_r \). Applying the decomposition (2.1), we see that for any compactly supported \( m \) with \( \| m \|_{V_r} \leq 1 \) we have
\[
| (m f)^\varepsilon(x) | \leq \sum_{j \geq 0} \sum_{v \in T_j} | b_v (1_{v} f)^\varepsilon(x) |
\leq \sum_{j \geq 0} \sup_{v \in T_j} | b_v|_\varepsilon \left( \sum_{v \in T_j} |(1_{v} f)^\varepsilon(x)|^s \right)^{1/s}
\leq C_{r, s} \sup_I \left( \sum_{I \in I} |(1_I f)^\varepsilon(x)|^s \right)^{1/s},
\]
where, for the last inequality, we require \( s < r' \).

Provided that \( r < 2 \) and \( p > r \) we can choose an \( s < r' \) with \( s > 2 \) and \( p > s' \), giving (2.2) and hence (1.3).

The argument of Lacey [4] for \( r = 1 \) follows a similar pattern, except with Marcinkiewicz’s method in place of [2] and the standard Carleson-Hunt theorem in place of [6].

References

4. Michael T. Lacey, personal communication.

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