

# INDEX OF REDUCIBILITY OF DISTINGUISHED PARAMETER IDEALS AND SEQUENTIALLY COHEN-MACAULAY MODULES

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ABSTRACT. It is shown that every sequentially Cohen-Macaulay module eventually has constant index of reducibility for distinguished parameter ideals.

## 1. INTRODUCTION

Throughout this paper let  $R$  be a commutative Noetherian local ring with maximal ideal  $\mathfrak{m}$ . Let  $M$  be a finitely generated  $R$ -module of dimension  $d > 0$ . Then we say that an  $R$ -submodule  $N$  of  $M$  is irreducible if  $N$  is not written as the intersection of two larger  $R$ -submodules of  $M$ . Every  $R$ -submodule  $N$  of  $M$  can be expressed as an irredundant intersection of irreducible  $R$ -submodules of  $M$ , and the number of irreducible  $R$ -submodules appearing in such an expression depends only on  $N$  and not on the expression. Let us call, for each parameter ideal  $\mathfrak{q}$  of  $M$ , the number  $N(\mathfrak{q}; M)$  of irreducible  $R$ -submodules of  $M$  that appear in an irredundant irreducible decomposition of  $\mathfrak{q}M$  the index of reducibility of  $M$  with respect to  $\mathfrak{q}$ . Remember that

$$N(\mathfrak{q}; M) = \ell_R([\mathfrak{q}M :_M \mathfrak{m}] / \mathfrak{q}M),$$

where  $\ell_R(*)$  stands for the length.

In 1957, D. G. Northcott [N, Theorem 3] proved that for parameter ideals  $\mathfrak{q}$  in a Cohen-Macaulay local ring  $R$ , the index  $N(\mathfrak{q}; R)$  of reducibility is constant and independent of the choice of  $\mathfrak{q}$ . However, this property of constant index of reducibility for parameter ideals does not characterize Cohen-Macaulay rings. The example of a non-Cohen-Macaulay local ring  $R$  with  $N(\mathfrak{q}; R) = 2$  for every parameter ideal  $\mathfrak{q}$  was firstly given in 1964 by S. Endo and M. Narita [EN]. In 1984 S. Goto and N. Suzuki [GS1] explored, for a given finitely generated  $R$ -module  $M$ , the supremum

$$\sup_{\mathfrak{q}} N(\mathfrak{q}; M),$$

where  $\mathfrak{q}$  runs through parameter ideals of  $M$ , and showed that the supremum is finite when  $M$  is a generalized Cohen-Macaulay module. Compared with the case of rings and modules with finite local cohomologies, the general case is much more complicated and difficult to treat. No standard induction techniques work. In fact, their striking examples [GS1, Example (3.9)] show that, in general, the supremum

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can be infinite. It is worth mentioning that in their examples the local rings are all sequentially Cohen-Macaulay. On the other hand, N. T. Cuong and H. L. Truong [CT, Theorem 1.1] showed that if  $M$  is a generalized Cohen-Macaulay module, then every parameter ideal  $\mathfrak{q}$  of  $M$  contained in some high power of the maximal ideal  $\mathfrak{m}$  has the same index  $N(\mathfrak{q}; M)$  of reducibility, whence  $M$  has, in the sense of M. Rogers [R], eventual constant index of reducibility for parameter ideals.

It now seems natural to ask whether sequentially Cohen-Macaulay modules have eventual constant index of reducibility for parameter ideals. This is, unfortunately, not true in general, as Rogers [R, Example 4.3] gave counterexamples. However, once we restrict our attention to certain special parameter ideals of  $M$ , the answer is affirmative, which we are eager to report in the present paper.

To state the main result, let us fix some notation. Let  $M$  be a finitely generated  $R$ -module of dimension  $d > 0$ . A filtration

$$\mathcal{D} : M = D_0 \supsetneq D_1 \supsetneq \cdots \supsetneq D_\ell = H_{\mathfrak{m}}^0(M)$$

of  $R$ -submodules of  $M$  is called the *dimension filtration* of  $M$ , if for all  $0 \leq i \leq \ell - 1$ ,  $D_{i+1}$  is the largest submodule of  $D_i$  with  $\dim_R D_{i+1} < \dim_R D_i$ , where  $\dim_R(0) = -\infty$  for convention. We say that  $M$  is *sequentially Cohen-Macaulay* if  $\mathcal{D}_i = D_i/D_{i+1}$  is Cohen-Macaulay for all  $0 \leq i \leq \ell - 1$ . Let  $\underline{x} = x_1, x_2, \dots, x_d$  be a system of parameters of  $M$ . Then  $\underline{x}$  is said to be *distinguished* if

$$(x_j \mid d_i < j \leq d)D_i = (0)$$

for all  $0 \leq i \leq \ell$ , where  $d_i = \dim_R D_i$  ([Sch, Definition 2.5]). A parameter ideal  $\mathfrak{q}$  of  $M$  is called *distinguished* if there exists a distinguished system  $x_1, x_2, \dots, x_d$  of parameters of  $M$  such that  $\mathfrak{q} = (x_1, x_2, \dots, x_d)$ .

With this notation the main result of this paper is stated as follows.

**Theorem 1.1.** *Let  $R$  be a Noetherian local ring with maximal ideal  $\mathfrak{m}$  and let  $M$  be a finitely generated  $R$ -module of dimension  $d > 0$ . If  $M$  is a sequentially Cohen-Macaulay  $R$ -module, then there is an integer  $n \gg 0$  such that for every distinguished parameter ideal  $\mathfrak{q}$  of  $M$  contained in  $\mathfrak{m}^n$ , one has the equality*

$$N(\mathfrak{q}; M) = \sum_{j=0}^d \ell_R((0) :_{H_{\mathfrak{m}}^j(M)} \mathfrak{m}).$$

*Hence the index of reducibility of  $M$  with respect to parameter ideals generated by distinguished systems of parameters is eventually constant.*

We shall prove Theorem 1.1 in Section 3. The notion of a sequentially Cohen-Macaulay module was introduced by R. Stanley [St] in the graded case, and the local case was studied in [Sch, CN]. A special type of sequentially Cohen-Macaulay rings called *approximately Cohen-Macaulay* rings was studied much earlier by Goto [G]. Our Theorem 1.1 partially covers his result [G, Proposition 3.1].

In our argument distinguished systems of parameters play an important role. In Section 2 we briefly note a characterization, Proposition 2.4, of distinguished systems of parameters.

2. DISTINGUISHED PARAMETER IDEALS

Let  $R$  be a Noetherian local ring with maximal ideal  $\mathfrak{m}$ . Let  $M$  be a finitely generated  $R$ -module with  $d = \dim_R M > 0$  and dimension filtration  $\mathcal{D} = \{D_i\}_{0 \leq i \leq \ell}$ . Let

$$\bigcap_{\mathfrak{p} \in \text{Ass}_R M} M(\mathfrak{p}) = (0)$$

be a primary decomposition of  $(0)$  in  $M$ , where  $M(\mathfrak{p})$  is a  $\mathfrak{p}$ -primary submodule of  $M$  for each  $\mathfrak{p} \in \text{Ass}_R M$ . We then have the following.

*Fact 2.1* ([CC]). (1)  $D_i = \bigcap_{\mathfrak{p} \in \text{Ass}_R M, \dim R/\mathfrak{p} \geq d_{i-1}} M(\mathfrak{p})$  for  $1 \leq i \leq \ell$ .

(2)  $\text{Ass}_R D_i = \{\mathfrak{p} \in \text{Ass}_R M \mid \dim R/\mathfrak{p} \leq d_i\}$  for  $0 \leq i \leq \ell$ .

(3)  $\text{Ass}_R D_{i-1}/D_i = \{\mathfrak{p} \in \text{Ass}_R M \mid \dim R/\mathfrak{p} = d_{i-1}\}$  for  $1 \leq i \leq \ell$ .

In particular

$$D_1 = \bigcap_{\mathfrak{p} \in \text{Assh}_R M} M(\mathfrak{p}),$$

where  $\text{Assh}_R M = \{\mathfrak{p} \in \text{Supp}_R M \mid \dim R/\mathfrak{p} = d\}$ ; hence  $D_1$  is the unmixed component of  $M$ . We have  $H_{\mathfrak{m}}^0(M) \subseteq D_1$ , and  $H_{\mathfrak{m}}^0(M) = D_1$  if  $\text{Ass}_R M \subseteq \text{Assh}_R M \cup \{\mathfrak{m}\}$ , because  $D_1$  is the largest submodule of  $M$  having dimension strictly smaller than  $d$ .

We put

$$N_i = \bigcap_{\mathfrak{p} \in \text{Ass}_R M, \dim R/\mathfrak{p} \leq d_i} M(\mathfrak{p})$$

for each  $0 \leq i \leq \ell$ ; hence  $D_i \cap N_i = (0)$ .

We note the following, which readily follows from the fact that  $\text{Ass}_R M/N_i = \text{Ass}_R D_i$  for all  $0 \leq i \leq \ell$ .

**Lemma 2.2.**

$$\sqrt{\text{Ann}_R D_i} = \sqrt{\text{Ann}_R M/N_i} = \bigcap_{\mathfrak{p} \in \text{Ass}_R M, \dim R/\mathfrak{p} \leq d_i} \mathfrak{p}.$$

We need the following.

**Lemma 2.3.** *Let  $\underline{x} = x_1, x_2, \dots, x_d$  be a distinguished system of parameters of  $M$ . Then*

$$D_i = (0) :_M x_j$$

*if  $1 \leq i \leq \ell$  and  $d_i < j \leq d_{i-1}$ . Hence  $(0) :_M x_j^n = (0) :_M x_j$  for all  $1 \leq j \leq d$  and  $n \geq 1$ .*

*Proof.* Let  $1 \leq i \leq \ell$ . Then since  $(x_j \mid d_i < j \leq d)D_i = (0)$ , we have  $D_i \subseteq (0) :_M x_j$  for all  $d_i < j \leq d$ . We will show that  $(0) :_M x_j \subseteq D_i$  if  $d_i < j \leq d_{i-1}$ . Assume that  $(0) :_M x_j \not\subseteq D_i$  with  $d_i < j \leq d_{i-1}$  and choose the integer  $1 \leq s \leq \ell$  as small as possible so that  $(0) :_M x_j \not\subseteq D_s$ . Then  $s \leq i$  and  $(0) :_M x_j \subseteq D_{s-1}$ . Let  $\varphi \in [(0) :_M x_j] \setminus D_s$ . Then  $x_j \cdot \overline{\varphi} = 0$  in  $D_{s-1}/D_s$  (here  $\overline{\varphi}$  stands for the image of  $\varphi$  in  $D_{s-1}/D_s$ ). Hence  $x_j$  is a zero-divisor for  $\mathcal{D}_{s-1}/\mathcal{D}_s$ . If  $j \leq d_{s-1}$ ,  $x_j$  is a parameter of  $D_{s-1}/D_s$ , whence  $x_j$  is a non-zero-divisor of  $D_{s-1}/D_s$ , because  $\text{Ass}_R D_{s-1}/D_s = \{\mathfrak{p} \in \text{Ass}_R M \mid \dim R/\mathfrak{p} = d_{s-1}\}$  (see Fact 2.1 (3)). This observation shows that  $d_{i-1} \geq j > d_{s-1}$ , so that we have  $i < s$ , which is impossible.

The second assertion now follows from the fact that the systems  $x_1^{n_1}, x_2^{n_2}, \dots, x_d^{n_d}$  of parameters of  $M$  are distinguished for all integers  $n_i \geq 1$  once the system  $x_1, x_2, \dots, x_d$  is distinguished. □

The following result gives a characterization of distinguished systems of parameters and the existence of this special kind of systems of parameters as well (cf. [Sch, Lemma 2.6]).

**Proposition 2.4.** *Let  $\underline{x} = x_1, x_2, \dots, x_d$  be a system of parameters of  $M$ . Then the following conditions are equivalent.*

- (1)  $\underline{x}$  is distinguished for  $M$ .
- (2) The following conditions are satisfied.
  - (i)  $(x_j \mid d_i < j \leq d) \subseteq \bigcap_{\substack{\mathfrak{p} \in \text{Ass}_R M, \\ \dim R/\mathfrak{p} \leq d_i}} \mathfrak{p}$  for all  $1 \leq i \leq \ell$ .
  - (ii)  $(0) :_M x_j = (0) :_M x_j^2$  for all  $1 \leq j \leq d$ .
- (3) The following conditions are satisfied.
  - (i) There is an integer  $n \geq 1$  such that  $\underline{x}(n) = x_1^n, x_2^n, \dots, x_d^n$  is a distinguished system of parameters of  $M$ .
  - (ii)  $(0) :_M x_j = (0) :_M x_j^2$  for all  $1 \leq j \leq d$ .

*Proof.* See Lemma 2.2 for the implication (1)  $\Rightarrow$  (2). Suppose condition (2) is satisfied. Then taking high powers of  $x_i$ , by Lemma 2.2 we may assume that  $(x_j^n \mid d_i < j \leq d)D_i = (0)$  for all  $0 \leq i \leq \ell$ , whence the system  $\underline{x}(n) = x_1^n, x_2^n, \dots, x_d^n$  is distinguished for  $M$ . Thus the implication (2)  $\Rightarrow$  (3) follows. We now consider the implication (3)  $\Rightarrow$  (1). Since  $\underline{x}(n)$  is a distinguished system of parameters of  $M$  and  $(0) :_M x_j = (0) :_M x_j^2$  for all  $1 \leq j \leq d$ , we get by Lemma 2.3 that  $D_i = (0) :_M x_j^n = (0) :_M x_j$  if  $d_i < j \leq d_{i-1}$  and  $1 \leq i \leq \ell$ . Hence  $(x_j \mid d_i < j \leq d)D_i = (0)$  for all  $1 \leq i \leq \ell$ . □

Let us note one of the simplest examples of distinguished systems of parameters.

**Example 2.5.** Let  $A = k[[X, Y, Z]]$  be the formal power series ring over a field  $k$  and put  $R = A/[(X) \cap (Y, Z)]$ . Then  $\dim R = 2$  and  $\text{depth } R = 1$ . Let  $x, y, z$  denote the images of  $X, Y, Z$  in  $R$ , respectively. Then the ring  $R$  has the dimension filtration

$$R \supseteq (x) \supseteq (0) = H_{\mathfrak{m}}^0(R),$$

and  $R$  is a sequentially Cohen-Macaulay ring, since  $(x) \cong A/(Y, Z)$ . The system  $\{x - y, z\}$  of parameters is distinguished, while  $\{x - y - z, x - y\}$  is not distinguished in any order.

### 3. PROOF OF THEOREM 1.1

Let  $M$  be a finitely generated  $R$ -module of dimension  $d > 0$  over a Noetherian local ring  $R$  with maximal ideal  $\mathfrak{m}$ . Let  $\mathcal{D} = \{D_i\}_{0 \leq i \leq \ell}$  denote the dimension filtration of  $M$  and put  $d_i = \dim_R D_i$  for each  $0 \leq i \leq \ell$ . Let

$$\Lambda(M) = \{0 < r \in \mathbb{Z} \mid M \text{ contains an } R\text{-submodule } N \text{ with } \dim_R N = r\}.$$

We put  $\mathcal{D}_i = D_i/D_{i+1}$  for  $0 \leq i \leq \ell - 1$ . Remember that our module  $M$  is sequentially Cohen-Macaulay if  $\mathcal{D}_i$  is Cohen-Macaulay for all  $0 \leq i \leq \ell - 1$ .

We now assume that  $M$  is a sequentially Cohen-Macaulay  $R$ -module. Hence for  $0 < j \in \mathbb{Z}$ ,  $H_{\mathfrak{m}}^j(M) \neq (0)$  if and only if  $j \in \Lambda(M)$ . Furthermore we have

$$H_{\mathfrak{m}}^{d_i}(M) \cong H_{\mathfrak{m}}^{d_i}(D_i) \cong H_{\mathfrak{m}}^{d_i}(\mathcal{D}_i)$$

for all  $0 \leq i \leq \ell - 1$ .

Let  $L$  be an arbitrary finitely generated  $R$ -module of dimension  $s \geq 0$ . We put

$$r_R(L) = \ell_R(\text{Ext}_R^s(R/\mathfrak{m}, L))$$

and call it the Cohen-Macaulay type of  $L$ . (Let us simply write  $r(R)$  for  $L = R$ .) We then have

$$N(\mathfrak{q}; L) = r_R(L/\mathfrak{q}L)$$

for a parameter ideal  $\mathfrak{q}$  of  $L$ . As is well known, if  $L$  is a Cohen-Macaulay  $R$ -module, then for every parameter ideal  $\mathfrak{q}$  of  $L$ , we have

$$N(\mathfrak{q}; L) = \ell_R(\text{Ext}_R^s(R/\mathfrak{m}, L)) = \ell_R((0) :_{\mathbb{H}_m^s(L)} \mathfrak{m}).$$

We are now ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* First of all, we choose, for each  $0 \leq i \leq \ell - 1$ , an integer  $n_i$  so that every system  $x_1, x_2, \dots, x_{d_i}$  of parameters for  $D_i$  contained in  $\mathfrak{m}^{n_i}$ , the canonical map

$$\phi_{D_i} : D_i/(x_1, x_2, \dots, x_{d_i})D_i \longrightarrow \mathbb{H}_m^{d_i}(D_i) = \lim_{q \rightarrow \infty} D_i/(x_1^q, x_2^q, \dots, x_{d_i}^q)D_i$$

is surjective on the socles ([GSa, Lemma 3.12]). Let  $n \geq \max\{n_i \mid 0 \leq i \leq \ell - 1\}$  be an integer. We put  $N = D_1$  and look at the exact sequence

$$0 \longrightarrow N \xrightarrow{\iota} M \xrightarrow{\epsilon} \mathcal{D}_0 \longrightarrow 0$$

of  $R$ -modules, where  $\iota$  (resp.  $\epsilon$ ) denotes the embedding (resp. the canonical epimorphism). Let  $\mathfrak{q} = (x_1, x_2, \dots, x_d)$  be a parameter ideal of  $M$  such that  $\mathfrak{q} \subseteq \mathfrak{m}^n$  and assume that  $x_1, x_2, \dots, x_d$  is distinguished for  $M$ . Then, since  $d > d_1 = \dim_R N$  and since  $x_1, x_2, \dots, x_d$  is a regular sequence for  $\mathcal{D}_0$ , we get the following commutative diagram:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & N/\mathfrak{q}N & \xrightarrow{\bar{\iota}} & M/\mathfrak{q}M & \xrightarrow{\bar{\epsilon}} & \mathcal{D}_0/\mathfrak{q}\mathcal{D}_0 \longrightarrow 0 \\ & & & & \downarrow \phi_M & & \downarrow \phi_{\mathcal{D}_0} \\ & & & & \mathbb{H}_m^d(M) & \xrightarrow{=} & \mathbb{H}_m^d(\mathcal{D}_0) \end{array}$$

with exact first row. Let  $x \in (0) :_{\mathcal{D}_0/\mathfrak{q}\mathcal{D}_0} \mathfrak{m}$ . Then, since  $\phi_M$  is surjective on the socles, we get an element  $y \in (0) :_{M/\mathfrak{q}M} \mathfrak{m}$  such that  $\phi_{\mathcal{D}_0}(x) = \phi_M(y)$ . Thus  $\bar{\epsilon}(y) = x$ , because the canonical map  $\phi_{\mathcal{D}_0}$  is injective, whence

$$N(\mathfrak{q}; M) = r_R(M/\mathfrak{q}M) = \begin{cases} r_R(N/\mathfrak{q}N) + r_R(\mathcal{D}_0) & \text{if } N \neq (0), \\ r_R(\mathcal{D}_0) & \text{if } N = (0). \end{cases}$$

We shall now show, by induction on the length  $\ell$  of the dimension filtration for  $M$ , that these numbers  $n \geq \max\{n_i \mid 0 \leq i \leq \ell - 1\}$  work well, as is predicted in Theorem 1.1.

If  $\ell = 1$  and  $N = H_m^0(M) = (0)$ , we have nothing to prove, since  $M$  is Cohen-Macaulay. If  $\ell = 1$  but  $N \neq (0)$ , we then have  $\mathfrak{q}N = 0$ , since the system  $x_1, x_2, \dots, x_d$  of parameters is distinguished for  $M$ , and so

$$\begin{aligned} N(\mathfrak{q}; M) &= r_R(M/\mathfrak{q}M) = r_R(N) + r_R(\mathcal{D}_0) \\ &= \sum_{j=0}^d \ell_R((0) :_{H_m^j(M)} \mathfrak{m}), \end{aligned}$$

because  $H_m^d(M) \cong H_m^d(\mathcal{D}_0)$ . Hence the result follows.

Suppose that  $\ell > 1$  and that our assertion holds true for  $\ell - 1$ . Since  $\ell > 1$ , we have  $d_1 > 0$ . The  $R$ -module  $N$  has the dimension filtration

$$\mathcal{D}_N : N = D_1 \supseteq D_2 \supseteq \dots \supseteq D_{\ell-1} \supseteq D_\ell = H_m^0(M) = H_m^0(N).$$

Therefore  $N$  is sequentially Cohen-Macaulay, and  $x_1, x_2, \dots, x_{d_1}$  is a distinguished system of parameters for  $N$ , since the system  $x_1, x_2, \dots, x_d$  of parameters is distinguished for  $M$ . Consequently, since  $\mathfrak{q}N = (x_1, x_2, \dots, x_{d_1})N$  and  $\mathfrak{q}_1 = (x_1, x_2, \dots, x_{d_1}) \subseteq \mathfrak{m}^n$  with  $n \geq \max\{n_i \mid 1 \leq i \leq \ell - 1\}$ , we get by the hypothesis of induction on  $\ell$  that

$$r_R(N/\mathfrak{q}N) = r_R(N/\mathfrak{q}_1N) = N(\mathfrak{q}_1; N) = \sum_{j=0}^{d_1} \ell_R((0) :_{H_m^j(N)} \mathfrak{m}).$$

Therefore, because  $H_m^j(N) \neq (0)$  and  $j > 0$  if and only if  $j \in \Lambda(M) \setminus \{d_0\} = \Lambda(N)$  and

$$H_m^{d_i}(N) \cong H_m^{d_i}(D_i) \cong H_m^{d_i}(M)$$

for all  $1 \leq i \leq \ell - 1$ , we have

$$\begin{aligned} N(\mathfrak{q}; M) = r_R(M/\mathfrak{q}M) &= r_R(N/\mathfrak{q}N) + r_R(\mathcal{D}_0) \\ &= \sum_{j=0}^{d_1} \ell_R((0) :_{H_m^j(N)} \mathfrak{m}) + \ell_R((0) :_{H_m^d(\mathcal{D}_0)} \mathfrak{m}) \\ &= \sum_{j=0}^{d_1} \ell_R((0) :_{H_m^j(M)} \mathfrak{m}) + \ell_R((0) :_{H_m^d(M)} \mathfrak{m}) \\ &= \sum_{j=0}^d \ell_R((0) :_{H_m^j(M)} \mathfrak{m}), \end{aligned}$$

as desired. □

Let us note a consequence of Theorem 1.1.

**Corollary 3.1** (cf. [G, Proposition 3.1]). *Let  $R$  be a Noetherian local ring of dimension  $d \geq 2$ . Let  $a$  be an element of  $R$  and assume that  $I = (0) : a = (0) : a^2 \neq (0)$ . If  $R/(a^2)$  is a Cohen-Macaulay ring of dimension  $d - 1$ , then  $R$  is a sequentially Cohen-Macaulay ring whose dimension filtration is given by*

$$R \supseteq I \supseteq (0) = H_m^0(R).$$

*When this is the case, the ring  $R/(a^n)$  is Cohen-Macaulay and*

$$r(R/(a^n)) = r_R(I) + r(R/I)$$

*for all integers  $n \gg 0$ .*

*Proof.* By [G, Lemma 2.1],  $R/I$  is a Cohen-Macaulay ring of dimension  $d$  and  $I$  is a Cohen-Macaulay  $R$ -module of dimension  $d - 1$ . Hence  $R$  is a sequentially Cohen-Macaulay ring with  $\Lambda(R) = \{d, d - 1\}$  and the dimension filtration of  $R$  is given by

$$R \supseteq I \supseteq (0) = H_m^0(R).$$

Let  $\underline{x} = x_1, x_2, \dots, x_d$  with  $x_d = a$  be a system of parameters of  $R$ . Then  $\underline{x}$  is distinguished for  $R$ , since  $aI = (0)$ . Therefore, because  $R/(a^n)$  is Cohen-Macaulay by [G, Lemma 2.2], for all integers  $n \gg 0$  we get by Theorem 1.1 that

$$r(R/(a^n)) = r(R/[(x_1^n, x_2^n, \dots, x_{d-1}^n) + (a^n)]) = r_R(I) + r(R/I),$$

as claimed.  $\square$

**Example 3.2** (cf. [G, Example 3.5 (5)]). Let  $A$  be a Cohen-Macaulay local ring of dimension  $d \geq 2$  and let  $M$  be a Cohen-Macaulay  $A$ -module of dimension  $d - 1$ . Let  $R = A \ltimes M$  denote the idealization of  $M$  over  $A$ . Then  $R$  is a sequentially Cohen-Macaulay ring. In fact, let  $a$  be a regular element of  $A$  such that  $aM = (0)$ . Then  $(0) \times M = (0) :_A a = (0) :_A a^2$  and  $R/a^2R = (A/(a^2)) \ltimes M$ . Hence  $R/a^2R$  is a Cohen-Macaulay ring of dimension  $d - 1$ . Thus, by Corollary 3.1,  $R$  is sequentially Cohen-Macaulay, and  $r(R/a^nR) = r(A) + r_A(M)$  for all integers  $n \gg 0$ .

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