ERRATUM TO “SOME GEOMETRIC PROPERTIES OF HYPERSURFACES WITH CONSTANT $r$-MEAN CURVATURE IN EUCLIDEAN SPACE”

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In computation (13) of our previous paper [3] there are some inaccuracies concerning the constants. The computation should read as follows:

\[
S_1S_{j+1} - (j+2)S_{j+2} = m \left( m \begin{pmatrix} m \\ j + 1 \end{pmatrix} H_1H_{j+1} - (j+2) \left( m \begin{pmatrix} m \\ j + 2 \end{pmatrix} H_{j+2} \right) \right)
\]

\[
= \left( m \begin{pmatrix} j + 1 \end{pmatrix} \right) \left( mH_1H_{j+1} - (m - j - 1)H_{j+2} \right)
\]

\[
\geq \left( m \begin{pmatrix} j + 1 \end{pmatrix} (j + 1)H_1H_{j+1} \right) \geq 0,
\]

where the inequality is a consequence of (12). Hence,

\[
A(r)v_j(r) \geq (j + 1) \left( m \begin{pmatrix} m \\ j + 1 \end{pmatrix} H_{j+1} \int_{\partial B_r} H_1 = \frac{(m-2)H_{j+1}}{m - j - 1} v_1(r). \right)
\]

Condition (2) of Theorem 1.1 should then read as follows:

\[(ii) \quad v_j(r)^{-1} \in L^1(+\infty) \quad \text{and} \quad \liminf_{r \to +\infty} \int_{r}^{+\infty} ds \frac{v_1(s)}{v_j(s)} \geq \frac{1}{2} \left( \frac{(m-2)H_{j+1}}{m - j - 1} \right)^{-1/2}. \]

Moreover, Remark 1.5 should be restated in this way:

As we will see later, condition $S_{j+1} \equiv 0$ together with rank$(A) > j$ at every point of $M$ implies the ellipticity of the operator $L_j$. Moreover, if we assume the additional hypothesis that there exists $p \in M$ such that $H_i(p) > 0$ for every $1 \leq i \leq j$, it can be proved that each $P_i$ is positive definite for every $1 \leq i \leq j$.
Similarly, Proposition 2.2(ii) has to be replaced by:

(ii) \( S_{j+1} \equiv 0, \text{rank}(A) > j \) at every point of \( M \), and there exists \( p \in M \) such that \( H_i(p) > 0 \) for every \( 1 \leq i \leq j \).

The next remark should be added after Proposition 2.2.

Remark. We stress that, by [10], when \( S_{j+1} \equiv 0 \), the sole condition \( \text{rank}(A) > j \) is equivalent to the requirement that \( L_j \) be elliptic.

Taking into account the previous observations, Theorem 1.4 has to be replaced by the following:

**Theorem 1.4.** Let \( f : M \to \mathbb{R}^{m+1} \) be a complete, connected orientable hypersurface with \( H_{j+1} \equiv 0 \), for some \( j \in \{0, \ldots, m-2\} \). If \( j \geq 1 \), assume that \( \text{rank}(A) > j \) at every point. Furthermore, if \( j \) is even, suppose that there exists \( p \in M \) such that \( H_j(p) > 0 \). Set

\[
v_j(r) = (m-j) \int_{\partial B_j} |S_j|, \quad v_{j+2}(r) = \int_{\partial B_j} |S_{j+2}|.
\]

If either

\[
(i) \quad |v_j(r)|^{-1} \notin L^1(+\infty) \quad \text{and} \quad H_{j+2} \notin L^1(M) \quad \text{or}
\]

\[
(ii) \quad |v_j(r)|^{-1} \in L^1(+\infty) \quad \text{and}
\]

\[
\liminf_{r \to +\infty} \sqrt{s_{j+2}(r)v_j(r)} \int_r^{+\infty} \frac{ds}{|v_j(s)|} > \frac{1}{2} \sqrt{\frac{1}{j+2}},
\]

then for every compact set \( K \subset M \) we have

\[
\bigcup_{p \in M \setminus K} T_p M \equiv \mathbb{R}^{m+1};
\]

that is, the tangent envelope of \( M \setminus K \) coincides with \( \mathbb{R}^{m+1} \).

**Proof.** We start by observing that we can assume that \( v_j \) is positive on \( \mathbb{R}^+ \). Indeed, in our assumptions, by the remark after Proposition 2.2 the operator \( L_j \) is elliptic; that is, \( P_j \) is either positive definite or negative definite everywhere. Thus, (3) of Lemma 2.1 implies that either \( H_j > 0 \) or \( H_j < 0 \) on \( M \). If \( j \) is odd, we can change the orientation of \( M \) in such a way that \( H_j \) is positive, whence \( v_j > 0 \) on \( \mathbb{R}^+ \). On the other hand, if \( j \) is even, this trick cannot be used and we have to rely on the existence of \( p \in M \) with \( H_j(p) > 0 \) to deduce that \( v_j > 0 \) on \( \mathbb{R}^+ \). Applying (5) of Lemma 2.1 we obtain

\[
0 < \text{Tr}(A^2 P_j) = -(j+2)S_{j+2};
\]

hence \( s_{j+2} < 0 \) on \( M \), and then \( v_{j+2} < 0 \) on \( \mathbb{R}^+ \). Now, suppose by contradiction that for some \( K \) the tangent envelope of \( M \setminus K \) does not coincide with \( \mathbb{R}^{m+1} \).

By choosing Cartesian coordinates appropriately, we can assume that the origin 0 satisfies

\[
0 \notin \bigcup_{p \in M \setminus K} T_p M.
\]
Then, the function $u = \langle f, \nu \rangle$ is nowhere vanishing and smooth on $M \setminus K$. Up to changing the sign of $u$ on each connected component, we can assume that $u > 0$ on $M \setminus K$. By Proposition 2.4, $T_j u = 0$ and hence $\lambda_1^{-T_j}(M \setminus K) \geq 0$. Note that here $H_{j+1} \equiv 0$ is essential. Defining

$$0 < A(r) = \frac{1}{v_j(r)} \int_{\partial B_r} \text{Tr}(A^2 P_j) = -(j + 2) \frac{1}{v_j(r)} \int_{\partial B_r} S_{j+2} = (j + 2) \frac{s_{j+2}(r)}{v_j(r)},$$

under assumptions (i) or (ii) the ODE $(v_j z ')'+Av_j z = 0$ is oscillatory. To show this fact, we rest upon the same oscillation criteria used in the proof of Theorem 1.1. The rest of the proof is identical to that of Theorem 1.1.

\[ \square \]

\textbf{Further references}

We would like to add the paper \cite{2} (for (ii) of Proposition 2.2) and two foundational works which have been of inspiration for the research on higher-order mean curvature hypersurfaces. The first one is the classic \cite{1}, which contains the original proof of Gårding’s inequality, and the second one, \cite{4}, characterizes hypersurfaces with $H_j$ constant in space forms from the variational point of view.

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\textbf{References}


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