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BLOW-UP PHENOMENA IN PARABOLIC PROBLEMS WITH TIME DEPENDENT COEFFICIENTS UNDER DIRICHLET BOUNDARY CONDITIONS

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ABSTRACT. A class of initial boundary value problems for the semilinear heat equation with time dependent coefficients is considered. Using a first order differential inequality technique, the influence of the data on the behaviour of the solutions (blow-up in finite or infinite time, global existence) is investigated. Lower and upper bounds are derived for the blow-up time when blow-up occurs.

1. INTRODUCTION

Blow-up phenomena of solutions to nonlinear parabolic equations have been assiduously investigated during the past decade. We refer the reader to the books of Straughan [20] and of Quittner and Souplet [19] as well as to the survey paper of Bandle and Brunner [2] for an account on this matter. Further contributions to the field are [1], [3]–[18], [22]–[24].

It is well known that the solutions may remain bounded for all time, or may blow up in finite or infinite time. When blow-up occurs at time $t^\star$, the evaluation of $t^\star$ is of great practical interest. Since $t^\star$ is usually not explicitly computable, we want to derive lower and upper bounds for $t^\star$.

The present paper investigates the blow-up phenomena of the solution $u(\mathbf{x}, t)$ of the following nonlinear parabolic problem:

$$(1.1) \qquad \begin{cases} u_t = \Delta u + k(t) f(u), \quad \mathbf{x} = (x_1, \ldots, x_N) \in \Omega, \quad t \in (0, t^\star), \\ u(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial\Omega, \quad t \in (0, t^\star), \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \end{cases}$$

where Ω is a bounded sufficiently smooth domain in $\mathbb{R}^N$, $N \geq 2$. The coefficient $k(t)$ is assumed nonnegative or strictly positive depending of the situation. The nonlinearity $f(u)$ is assumed to satisfy $f(0) = 0$, and $f(u) > 0$ for $u > 0$. The initial data $u_0(\mathbf{x})$ is assumed nonnegative so that the solution $u(\mathbf{x}, t)$ of problem (1.1) is nonnegative by the maximum principle. More specific assumptions on f and k will be made later. The particular case of $k =$ const. has already been investigated by L. E. Payne and P. W. Schaefer in [13]. This note may therefore be regarded as a complement of their paper.

In Section 2, we derive conditions on the data of problem (1.1) sufficient to instigate the blow-up of $u(\mathbf{x}, t)$ and derive under these conditions an upper bound

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for the blow-up time $t^\star$. In Section 3, we derive conditions on the data of problem (1.1) sufficient to insure the global existence of $u(\mathbf{x},t)$. In Section 4, we derive various lower bounds for $t^\star$, valid under appropriate assumptions on the data. In Sections 3 and 4, our argument makes use of the following Sobolev type inequality:

$$\left(\int_\Omega v^6 d\mathbf{x}\right)^{1/4} \le \Gamma\left(\int_\Omega |\nabla v|^2 d\mathbf{x}\right)^{3/4}, \quad \Gamma := \frac{2\cdot 3^{-3/4}}{\pi}, \tag{1.2}$$

valid in $\mathbb{R}^3$ for a nonnegative function v that vanishes on $\partial\Omega$. We refer to [21] for a proof of (1.2). The results derived in Sections 3 and 4 are therefore restricted to the 3-dimensional space. Some extensions are derived in Section 5. The case of problem (1.1) under Neumann boundary conditions will be investigated in a forthcoming paper.

2. Blow-up of $u(\mathbf{x},t)$ in finite time t^*

We first employ a method used by Kaplan [5] to obtain a condition which leads to blow-up at some finite time, and also leads to an upper bound for the blow-up time. Let λ_1 be the first eigenvalue and ϕ_1 be the associated eigenfunction of the Dirichlet-Laplace operator defined as

$$\Delta\phi_1 + \lambda_1\phi_1 = 0, \quad \phi_1 > 0, \quad \mathbf{x}\in\Omega; \quad \phi_1 = 0, \quad \mathbf{x}\in\partial\Omega, \tag{2.1}$$

$$\int_\Omega \phi_1 d\mathbf{x} = 1. \tag{2.2}$$

Let the auxiliary function $\sigma(t)$ be defined in $(0,t^\star)$ as

$$\sigma(t) := (k(t))^{1/(p-1)}\int_\Omega u\phi_1 d\mathbf{x}, \tag{2.3}$$

where $u(\mathbf{x},t)$ is the solution of (1.1). We assume that

$$f(0) = 0, \quad f(s) \ge s^p, \quad s > 0, \tag{2.4}$$

for some $p > 1$, and that $k(t) > 0$ satisfies the condition

$$\frac{k'}{k} \ge \beta, \tag{2.5}$$

for some constant β. With

$$\gamma := \lambda_1 - \frac{\beta}{p-1}, \tag{2.6}$$

we compute

$$\begin{aligned} \sigma'(t) \quad &\ge \quad \tfrac{\beta}{p-1}\sigma + k^{1/(p-1)}\int_\Omega \phi_1[(\Delta u + ku^p]d\mathbf{x} \\ &= \quad -\gamma\sigma + k^{p/(p-1)}\int_\Omega \phi_1 u^p d\mathbf{x}. \end{aligned} \tag{2.7}$$

Moreover, making use of Hölder's inequality, we have in view of (2.2),

$$\int_\Omega \phi_1 u d\mathbf{x} \le \left(\int_\Omega \phi_1 u^p d\mathbf{x}\right)^{1/p}. \tag{2.8}$$

Combining (2.8) and (2.7), we obtain the first order differential inequality

$$\sigma'(t) \ge -\gamma\sigma(t) + (\sigma(t))^p, \quad t\in(0,t^\star). \tag{2.9}$$

Integrating (2.9), we obtain

$$(\sigma(t))^{1-p} \leq \eta(t) := \begin{cases} \frac{1}{\gamma} + \delta e^{(p-1)\gamma t}, & \gamma \neq 0, \\ (\sigma(0))^{1-p} - (p-1)t, & \gamma = 0, \end{cases} \tag{2.10}$$

with

$$\delta := (\sigma(0))^{1-p} - \frac{1}{\gamma}, \quad \gamma \neq 0. \tag{2.11}$$

If $\eta(T_1) = 0$ for some $T_1 > 0$, then $\sigma(t)$ blows up at time $t^\star < T_1$. This leads to the following result.

Theorem 1. *Let $u(\mathbf{x}, t)$ be the solution of problem (1.1). Then the auxiliary function $\sigma(t)$ defined in (2.3) blows up at time $t^\star < T_1$ with*

$$T_1 := \begin{cases} \frac{1}{\gamma(p-1)} \log\left(-\frac{1}{\gamma\delta}\right) & \textit{if} \quad 0 < \gamma(\sigma(0))^{1-p} < 1, \\ \frac{1}{(p-1)(\sigma(0))^{p-1}} & \textit{if} \quad \gamma \leq 0. \end{cases} \tag{2.12}$$

Another upper bound for the blow-up time could be obtained by a variation of the concavity method of Levine [7]. For $k(t) = 1$, sufficient conditions leading to the blow-up of $u(\mathbf{x}, t)$ have been etablished by Philippin and Proytcheva in [17]. Their approach may be adapted when $k(t)$ is not constant. This leads to the following result.

Theorem 2. *Let $u(\mathbf{x}, t)$ be the solution of problem (1.1). Let $\psi(t)$ and $\zeta(t)$ be auxiliary functions defined as follows:*

$$\psi(t) := \int_\Omega u^2(\mathbf{x}, t) d\mathbf{x}, \tag{2.13}$$

$$\zeta(t) := \int_\Omega \{k(t)F(u) - \frac{1}{2}|\nabla u|^2\} d\mathbf{x}, \tag{2.14}$$

with

$$F(u) := \int_0^u f(s) ds. \tag{2.15}$$

Assume the following conditions on f and k:

$$k(0) > 0, \quad k'(t) \geq 0, \quad t > 0, \tag{2.16}$$

$$s f(s) \geq \frac{1}{2}(4 + \alpha) F(s), \tag{2.17}$$

where α is a positive parameter. Moreover we assume that the initial data satisfy the condition

$$\zeta(0) = \int_\Omega \{k(0)F(u_0) - \frac{1}{2}|\nabla u_0|^2\} d\mathbf{x} > 0. \tag{2.18}$$

Then we conclude that $u(\mathbf{x}, t)$ blows up at some finite time $t^\star < T_2$ with

$$T_2 := \frac{4}{\alpha(\alpha + 4)} \psi(0)(\zeta(0))^{-1}. \tag{2.19}$$

We refer to [17] for the proof of Theorem 2.

3. Conditions for global existence of $u(\mathbf{x},t)$

In this section, we assume that the data of problem (1.1) satisfy the following two conditions:

$$0 \le f(s) \le s^p, \quad s > 0, \quad p = const. > 1, \tag{3.1}$$

$$k(t) > 0, \quad \frac{k'(t)}{k(t)} \le \beta, \quad t \ge 0, \tag{3.2}$$

for some positive constant β and consider the auxiliary function $\Phi(t)$ defined as

$$\Phi(t) := M^{-1}(k(t))^{2n} \int_\Omega u^{2n(p-1)} d\mathbf{x}, \quad t \in (0, t^\star), \tag{3.3}$$

with

$$M := (k(0))^{2n} \int_\Omega u_0^{2n(p-1)} d\mathbf{x}. \tag{3.4}$$

In (3.3), (3.4), n is a parameter subject to the restrictions

$$n(p-1) \ge 1, \quad n > 3/4. \tag{3.5}$$

For instance we may choose $n = 1/(p-1)$ if $p \in (1,2)$, or $n = 1$ if $p \ge 2$. For convenience we set

$$v(\mathbf{x},t) := u^{n(p-1)} \tag{3.6}$$

and compute

$$\Phi'(t) = 2n\frac{k'}{k}\Phi + 2n(p-1)M^{-1}k^{2n} \int_\Omega u^{2n(p-1)-1}[\Delta u + k(t)f(u)]d\mathbf{x}. \tag{3.7}$$

With

$$\int_\Omega u^{2n(p-1)-1} \Delta u \, d\mathbf{x} = -\frac{2n(p-1)-1}{n^2(p-1)^2} \int_\Omega |\nabla v|^2 d\mathbf{x}, \tag{3.8}$$

we obtain, thanks to (3.1), (3.2),

$$\begin{aligned} \Phi'(t) \le 2n\beta\Phi \quad &+ \quad 2n(p-1)k^{2n+1}M^{-1} \int_\Omega v^{2+\frac{1}{n}} d\mathbf{x} \\ &- \quad \tfrac{2[2n(p-1)-1]}{n(p-1)} k^{2n} M^{-1} \int_\Omega |\nabla v|^2 d\mathbf{x}. \end{aligned} \tag{3.9}$$

Making use of Hölder's inequality

$$\int_\Omega v^{2+\frac{1}{n}} d\mathbf{x} \le \left(\int_\Omega v^2 d\mathbf{x}\right)^{(4n-1)/4n} \left(\int_\Omega v^6 d\mathbf{x}\right)^{1/4n}, \tag{3.10}$$

followed by the Sobolev type inequality (1.2), we obtain

$$\begin{aligned} k^{2n+1} \int_\Omega v^{2+\frac{1}{n}} d\mathbf{x} \quad &\le \quad k^{2n+1} \left(\int_\Omega v^2 d\mathbf{x}\right)^{(4n-1)/4n} \left(\int_\Omega |\nabla v|^2 d\mathbf{x}\right)^{3/4n} \Gamma^{1/n} \\ &= \quad \Gamma^{1/n} M^{(4n-1)/4n} \Phi^{(4n-1)/4n} \left(k^{2n} \int_\Omega |\nabla v|^2 d\mathbf{x}\right)^{3/4n}, \end{aligned} \tag{3.11}$$

where Γ is defined in (1.2). Combining (3.11) and (3.9), we obtain

(3.12)

$$\begin{aligned}
\Phi'(t) &\\
\leq 2n\beta\Phi &- \frac{2[2n(p-1)-1]}{n(p-1)} M^{-1}k^{2n}\int_\Omega |\nabla v|^2 d\mathbf{x}\\
&+ 2n(p-1)\Gamma^{1/n}M^{1/2n}\Phi^{(4n-1)/4n}\left(M^{-1}k^{2n}\int_\Omega |\nabla v|^2 d\mathbf{x}\right)^{3/4n}\\
&= 2n\beta\Phi + 2n\left(\lambda^{-1}k^{2n}M^{-1}\int_\Omega |\nabla v|^2 d\mathbf{x}\right)^{3/4n}\\
&\times\left\{\lambda^{3/4n}(p-1\Gamma^{1/n}\Phi^{\frac{(4n-1)}{4n}}M^{1/2n} - \frac{2n(p-1)-1}{n^2(p-1)}\lambda\left(M^{-1}k^{2n}\lambda^{-1}\int_\Omega |\nabla v|^2 d\mathbf{x}\right)^{\frac{(4n-3)}{4n}}\right\}
\end{aligned}$$

with arbitrary $\lambda \neq 0$. Choosing $\lambda := \lambda_1$, the first eigenvalue of problem (2.1), we have

$$\int_\Omega |\nabla v|^2 d\mathbf{x} \geq \lambda_1 \int_\Omega v^2 d\mathbf{x} \tag{3.13}$$

by the Rayleigh principle. Making use of (3.13) in the last factor of (3.12), we obtain the inequality

$$\begin{aligned}
\Phi' \quad &\leq \quad 2n\beta\Phi + 2n\left(\lambda_1^{-1}k^{2n}M^{-1}\int_\Omega |\nabla v|^2 d\mathbf{x}\right)^{3/4n}\Phi^{(4n-3)/4n}\\
&\times \quad \left\{\nu\Phi^{1/2n} - (\mu+\beta)\right\},
\end{aligned} \tag{3.14}$$

with

$$\nu := (p-1)\lambda_1^{3/4n}\Gamma^{1/n}M^{1/2n}, \qquad \mu := \frac{2n(p-1)-1}{n^2(p-1)}\lambda_1 - \beta. \tag{3.15}$$

Suppose that β is small enough to satisfy the condition

$$\mu > 0, \tag{3.16}$$

and that the initial data are small enough to satify the condition

$$\nu - \mu < 0. \tag{3.17}$$

Then either $\nu(\Phi(t))^{1/2n} - \mu$ remains negative for all time, or there exists a first time t_0 such that

$$\nu(\Phi(t_0))^{1/2n} - \mu = 0. \tag{3.18}$$

For $t \in (0, t_0)$, the last factor in (3.14) is then negative, and we may use again (3.13) in (3.14) to obtain the differential inequality

$$\Phi'(t) \leq 2n\Phi\{\nu\Phi^{1/2n} - \mu\} \leq 0, \quad t \in (0, t_0). \tag{3.19}$$

It follows from (3.19) that $\Phi(t)$ is nonincreasing on $(0, t_0)$, so that (3.18) cannot hold. We then conclude that (3.19) is valid for all time $t > 0$. Integrating this differential inequality, we obtain

$$\Phi(t) \leq \left\{\left(1 - \frac{\nu}{\mu}\right)e^{\mu t} + \frac{\nu}{\mu}\right\}^{-2n}, \quad t > 0. \tag{3.20}$$

This result is summarized in the next theorem.

Theorem 3. *Let Ω be a bounded domain in $\mathbb{R}^3$, and assume that the data of problem* (1.1) *satisfy the conditions* (3.1), (3.2), (3.16), (3.17). *Then the auxiliary function $\Phi(t)$ defined in* (3.3) *satisfies* (3.20), *and $u(\mathbf{x},t)$ exists for all time $t>0$.*

4. Lower bounds for $t^\star$

In this section some lower bounds for $t^\star$ will be derived for the possible blow-up time of the solution $u(\mathbf{x},t)$ of problem (1.1). Going back to the first inequality (3.12) and making use of the inequality

$$\begin{aligned}\Phi^{(4n-1)/4n}J^{3/4n} &= \left(c^{3/(4n-3)}\Phi^{(4n-1)/(4n-3)}\right)^{(4n-3)/4n}\left(c^{-1}J\right)^{3/4n}\\ &\le \tfrac{4n-3}{4n}c^{3/(4n-3)}\Phi^{(4n-1)/(4n-3)}+\tfrac{3}{4nc}J,\end{aligned}\tag{4.1}$$

with $J:=M^{-1}(k(t))^{2n}\int_\Omega|\nabla v|^2d\mathbf{x}$, valid for arbitrary $c>0$, we obtain

$$\begin{aligned}\Phi' &\le 2n\beta\Phi+\tfrac12(4n-3)(p-1)M^{1/2n}\Gamma^{1/n}c^{3/(4n-3)}\Phi^{(4n-1)/(4n-3)}\\ &+\left\{\tfrac{3(p-1)}{2c}M^{1/2n}\Gamma^{1/n}-\tfrac{2[2n(p-1)-1]}{n(p-1)}\right\}J.\end{aligned}\tag{4.2}$$

Selecting

$$c:=\frac{3n(p-1)^2}{4[2n(p-1)-1]}M^{1/2n}\Gamma^{1/n},\tag{4.3}$$

the last term in (4.2) vanishes and we obtain the differential inequality

$$\Phi'(t)\le 2n\beta\Phi+c_1\Phi^{(4n-1)/(4n-3)},\tag{4.4}$$

with

$$c_1:=c_0M^{2/(4n-3)},\tag{4.5}$$

$$c_0:=\frac12(4n-3)(p-1)\left(\frac{3n(p-1)^2}{4[2n(p-1)-1]}\right)^{3/(4n-3)}\Gamma^{4/(4n-3)}.\tag{4.6}$$

Integrating this differential inequality, we obtain

$$(\Phi(t))^{-2/(4n-3)}\ge\left(1+\frac{c_1}{2n\beta}\right)\exp\left(-\frac{4n\beta}{4n-3}t\right)-\frac{c_1}{2n\beta}.\tag{4.7}$$

We then conclude that if $\Phi(t)$ blows up at some time $t^\star$, we have

$$t^\star\ge t_1:=\frac{4n-3}{4n\beta}\log\left(1+\frac{2n\beta}{c_1}\right).\tag{4.8}$$

We note again that an appropriate choice of n is $n=1$ for $p\ge2$, or $n=(p-1)^{-1}$ for $p\in(1,2)$.

To establish a lower bound for $t^\star$ that does not require the condition (3.2), we define the auxiliary function

$$\Theta(t):=M_0^{-1}\int_\Omega u^{2n(p-1)}d\mathbf{x},\tag{4.9}$$

with

$$M_0:=\int_\Omega u_0^{2n(p-1)}d\mathbf{x}.\tag{4.10}$$

Of course we keep the restrictions (3.5) on the parameter n. Assuming (3.1), we compute

$$\Theta'(t) \leq 2n(p-1)M_0^{-1}k \int_\Omega v^{2+\frac{1}{n}} d\mathbf{x} - \frac{2[2n(p-1)-1]}{n(p-1)} M_0^{-1} \int_\Omega |\nabla v|^2 d\mathbf{x}, \tag{4.11}$$

where $v(\mathbf{x}, t)$ is defined in (3.6). Proceeding as before leads to the differential inequality

$$\Theta'(t) \leq c_2 (k(t))^{4n/(4n-3)} \Theta^{(4n-1)/(4n-3)}, \tag{4.12}$$

with

$$c_2 := c_0 M_0^{2/(4n-3)}, \tag{4.13}$$

where c_0 is defined in (4.6). Integrating (4.12), we obtain

$$(\Theta(t))^{-2/(4n-3)} \geq 1 - \frac{2c_2}{4n-3} \int_0^t (k(\tau))^{4n/(4n-3)} d\tau. \tag{4.14}$$

We then conclude that $\Theta(t)$ remains bounded for all time if the condition

$$\int_0^\infty (k(t))^{4n/(4n-3)} dt < \frac{4n-3}{2c_2} \tag{4.15}$$

is satisfied. On the other hand, if $\Theta(t)$ blows up at some finite time $t^\star$, then $t^\star > t_2$, with

$$\int_0^{t_2} (k(t))^{4n/(4n-3)} dt = \frac{4n-3}{2c_2}. \tag{4.16}$$

These results are summarized in the next theorem.

Theorem 4. *Let* $u(\mathbf{x}, t)$ *be the solution of problem* (1.1) *in a bounded domain* $\Omega \subset \mathbb{R}^3$.

a) Assume that f *satisfies* (3.1), (3.2). *Then the auxiliary function* $\Phi(t)$ *defined in* (3.3) *is bounded above by* (4.7), *and blows up at time* $t^\star > t_1$ *defined in* (4.8) *when blow-up occurs.*

b) Assume that f *satisfies* (3.1). *We then conclude that the auxiliary function* $\Theta(t)$ *defined in* (4.9) *remains bounded if the condition* (4.15) *is satisfied; otherwise* $\Theta(t)$ *blows up at time* $t^\star > t_2$ *defined in* (4.16) *when blow-up occurs.*

5. Some extensions

In this section we consider the more general problem

$$\begin{cases} \frac{1}{k_1(t)} u_t = k_2(t)\Delta u + k_3(t) f(u), \quad \mathbf{x} \in \Omega, \quad t \in (0, t^\star), \\ u(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial\Omega, \quad t \in (0, t^\star), \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}) \geq 0, \quad \mathbf{x} \in \Omega, \end{cases} \tag{5.1}$$

where the given differentiable functions k_1, k_2, k_3 are assumed positive. We consider the auxiliary function

$$\chi(t) := Q^{-1} K(t) \int_\Omega u^{2n(p-1)} d\mathbf{x}, \tag{5.2}$$

with

$$K(t) := (k_1(t))^{(4n-3)/2}\,(k_2(t))^{-3/2}\,(k_3(t))^{2n}\,, \tag{5.3}$$

$$Q := K(0)\int_\Omega u_0^{2n(p-1)}d\mathbf{x}, \tag{5.4}$$

where n is a parameter subject to the restrictions (3.5). Assuming (3.1) and

$$\left(\frac{1}{2n}\right)\frac{K'(t)}{K(t)} = \left(\frac{4n-3}{4n}\right)\frac{k_1'}{k_1} - \left(\frac{3}{4n}\right)\frac{k_2'}{k_2} + \frac{k_3'}{k_3} \le \beta \tag{5.5}$$

for some $\beta \in \mathbb{R}$, we have the following differential inequality:

$$\chi'(t) \le 2n\beta\chi + c_3\chi^{(4n-1)/(4n-3)}, \tag{5.6}$$

valid for $\Omega \in \mathbb{R}^3$, with

$$c_3 := c_0 Q^{2/(4n-3)}. \tag{5.7}$$

Integrating (5.6), we obtain

$$(\chi(t))^{-2/(4n-3)} \ge \epsilon(t) := \begin{cases} 1 - \frac{2c_3}{4n-3}t, & \beta = 0, \\ \left(1 + \frac{c_3}{2n\beta}\right)\exp\left(-\frac{4n\beta}{4n-3}t\right) - \frac{c_3}{2n\beta}, & \beta \neq 0. \end{cases} \tag{5.8}$$

We then conclude that if $\chi(t)$ blows up at some $t^\star$, we have

$$t^\star \ge t_3 := \begin{cases} \frac{4n-3}{2c_3}, & \beta = 0, \\ \frac{4n-3}{4n\beta}\log\left(1 + \frac{2n\beta}{c_3}\right), & \beta \neq 0. \end{cases} \tag{5.9}$$

For $\beta \neq 0$, the value of t_3 in (5.9) makes sense only in the following two cases: (i) $\beta > 0$, (ii) $\beta < 0$ and $1 + 2n\beta/c_3 > 0$. In all other cases the solution $u(\mathbf{x}, t)$ of (5.1) exists for all time $t > 0$.

To establish a lower bound for $t^\star$ that does not require the condition (5.5), we consider again the auxiliary function $\Theta(t)$ defined in (4.9), (4.10). Assuming (3.1), we obtain the differential inequality

$$\begin{aligned} \Theta'(t) &\le c_2 k_1(t)(k_2(t))^{-3/(4n-3)}(k_3(t))^{4n/(4n-3)}\Theta^{(4n-1)/(4n-3)} \\ &= c_2(K(t))^{2/(4n-3)}\Theta^{(4n-1)/(4n-3)}, \end{aligned} \tag{5.10}$$

with $K(t)$ defined in (5.3) and c_2 defined in (4.13). It then follows that

$$(\Theta(t))^{-2/(4n-3)} \ge 1 - \frac{2c_2}{4n-3}\int_0^t (K(\tau))^{2/(4n-3)}d\tau. \tag{5.11}$$

We then conclude that $\Theta(t)$ remains bounded for all time if the condition

$$\int_0^\infty (K(t))^{2/(4n-3)}dt < \frac{4n-3}{2c_2} \tag{5.12}$$

is satisfied. On the other hand, if $\Theta(t)$ blows up at some finite time $t^\star$, then $t^\star > t_4$ with

$$\int_0^{t_4} (K(t))^{2/(4n-3)}dt = \frac{4n-3}{2c_2}. \tag{5.13}$$

These results are summarized in the next theorem.

Theorem 5. *Let $u(\mathbf{x},t)$ be the solution of problem* (5.1) *in a bounded domain $\Omega \subset \mathbb{R}^3$.*

a) Assume (3.1), (5.5), (3.5). *Then the auxiliary function $\chi(t)$ defined in* (5.2) *blows up at time $t^\star$ bounded below by t_3 defined in* (5.9) *in the two cases (i) and (ii). Otherwise $u(\mathbf{x},t)$ exists for all time $t > 0$.*

b) Assume (3.1). *We then conclude that the auxiliary function $\Theta(t)$ defined in* (4.9), (4.10) *remains bounded if condition* (5.12) *is satisfied. Otherwise $\Theta(t)$ blows up at time $t^\star > t_4$ defined in* (5.13) *when blow-up occurs.*

To conclude this paper we show how problem (5.1) may be reduced to problem (1.1). To this end we replace the time variable by the new variable

$$z(t) := \int_0^t k_1(\tau)k_2(\tau)d\tau. \tag{5.14}$$

The differential equation in (5.1) then takes the following form:

$$u_z(\mathbf{x},z) = \Delta u + \kappa(z)f(u), \quad \mathbf{x} \in \Omega, \quad z \in (0,z^\star), \tag{5.15}$$

with $z^\star = z(t^\star)$ and with

$$\kappa(z) := \frac{k_3(t(z))}{k_2(t(z))}. \tag{5.16}$$

The results derived in Sections 2, 3, and 4 are therefore applicable to the more general problem (5.1).

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